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PROBLEMS AND SOLUTIONS

Submit problems and solutions separately, with the contributor's full name and affiliation on each submission. Submissions in LaTeX are strongly encouraged. All figures must be clearly labeled and suitable for high-quality reproduction. Proposals should be accompanied by solutions. An asterisk (*) marks a problem for which no solution has been received by the editors. Undergraduate and pre-college students are warmly encouraged to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the grade and school name. All submissions are subject to editorial review for correctness, originality, clarity, and mathematical interest. Correspondence regarding proposals and solutions: mathproblems-ks@hotmail.com.

*Solutions to the problems in this issue should arrive before
December 26, 2026*

Problems

180. *Proposed by Michel Bataille, Rouen, France.* Let x, y, z be real numbers such that $x+y+z = 0$. Prove that

$$\sqrt{\cos^2 x + \cos^2 y + \cos^2 z - \frac{3}{4}} \geq |\cos x| + |\cos y| + |\cos z| - \frac{3}{2}.$$

(The expression under the square root is nonnegative under the given constraint.)

181. *Proposed by Joe Santmyer, Las Cruces, New Mexico, USA.* Evaluate the integral

$$\int_0^1 \frac{1}{1+x^2} \ln\left(\frac{16(x^8 - 7x^6 + 16x^4 - 7x^2 + 1)}{(1+x^2)^2}\right) dx.$$

182. *Proposed by Armend Shabani, Department of Mathematics, University of Prishtina, Republic of Kosova.* A composition of a positive integer n is an ordered sequence of positive integers whose sum is n . It is called *parity-alternating* if consecutive parts have opposite parity. Let X_n be the number of parts in a parity-alternating composition of n chosen uniformly at random. Prove that the following limits exist and determine their values:

$$\lim_{n \rightarrow \infty} \frac{\mathbb{E}[X_n]}{n}, \quad \lim_{n \rightarrow \infty} \frac{\text{Var}(X_n)}{n}.$$

183. *Proposed by Paolo Perfetti, Department of Mathematics, University of Rome Tor Vergata, Rome, Italy.* Evaluate

$$\sum_{n=1}^{\infty} \left[(2n-1) \left(\frac{1}{n^2} + \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \cdots \right) - 2 \right]^2.$$

184. *Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania.* Prove that 3 is the largest positive value of the constant k such that

$$\frac{1}{a+k} + \frac{1}{b+k} + \frac{1}{c+k} + \frac{1}{d+k} + \frac{1}{e+k} \geq \frac{5}{1+k}$$

whenever $a \geq b \geq 1 \geq c \geq d \geq e \geq 0$ with $ab + bc + cd + de + ea = 5$.

185. *Proposed by Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Romania.* Let $n \geq 3$ be an integer. Find the least constant K_n for which

$$(n-2) \sum_{i=1}^n a_i + K_n \prod_{i=1}^n a_i \geq 2 \sum_{1 \leq i < j \leq n} a_i a_j$$

holds for all nonnegative real numbers $a_1, \dots, a_n$ satisfying

$$\sum_{i=1}^n a_i^2 = n-1.$$

186. *Proposed by Valmir Krasniqi, Institute of Science, Technology, Engineering and Mathematics, Prishtina, Republic of Kosova.* Let $\mathbb{R}$ denote the set of real numbers. Determine all functions $f : \mathbb{R} \rightarrow \mathbb{R}$ such that

$$f(x^3 + y^3) = f(x+y) \left((f(x-y))^2 + f(xy) \right)$$

for all $x, y \in \mathbb{R}$.

LET'S CHALLENGE AI WITH PROBLEMS

This section presents mathematical problems intended to challenge both human solvers and artificial intelligence (AI). All readers are welcome to submit problem proposals.

Complete solutions and partial results will be published in the next issue. At the end of each solution or report of partial results, authors must state whether AI was used and, if so, identify the AI system and model used.

*Solutions and partial results for the AI challenges in this issue should arrive before
December 26, 2026*

1. *Proposed by the Editorial Board.* Find all positive integers n such that

$$2^n - 1 \mid n^2(3^n - 1).$$

No problem is ever permanently closed. We will be very pleased to consider publishing new solutions or comments on previous problems.

Solutions

173. Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania. Find all 2×2 matrices with real entries such that $e^A = A - A^T$, where A^T denotes the transpose of A and e^A denotes matrix exponentiation.

Solution 1 by Michel Bataille, Rouen, France. Let O, I , and J denote the zero matrix, the unit matrix, and the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, respectively. We claim that the solutions are the matrices

$$[\ln((4m+1)\pi)]I \pm \left(\frac{\pi}{2} + 2m\pi\right)J$$

for any nonnegative integer m .

To prove this, we shall use the following result $\mathcal{R}$ (whose proof is postponed to the end): we have $e^A = pA + qI$ where $p = \frac{e^\mu - e^\lambda}{\mu - \lambda}$, $q = \frac{\mu e^\lambda - \lambda e^\mu}{\mu - \lambda}$ if the eigenvalues of A are the distinct complex numbers λ, μ and $p = e^\lambda$, $q = e^\lambda(1 - \lambda)$ if A has an eigenvalue λ of algebraic multiplicity 2.

Let A be a solution. First note that A cannot be scalar: if $A = \alpha I$, then $e^A = e^\alpha I \neq O = A - A^T$. Second, we must have $A - A^T = pA + qI$, hence $A^T = (1 - p)A - qI$ and therefore $A = (1 - p)[(1 - p)A - qI] - qI$. We deduce that $(p - 2)(pA + qI) = O$. Since $pA + qI = e^A$ is invertible and hence nonzero, we conclude that $p = 2$.

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The equation now writes as $A - A^T = 2A + qI$, that is,

$$\begin{pmatrix} 0 & b - c \\ c - b & 0 \end{pmatrix} = \begin{pmatrix} 2a + q & 2b \\ 2c & 2d + q \end{pmatrix},$$

from which we readily deduce that A is of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ (a, b real with $b \neq 0$).

The eigenvalues of A are $\lambda = a + ib, \mu = a - ib$ and from $\mathcal{R}$, we obtain $\frac{e^\mu - e^\lambda}{\mu - \lambda} = e^a \frac{\sin b}{b} = 2$ and $q = \frac{\mu e^\lambda - \lambda e^\mu}{\mu - \lambda} = e^a \cos b - 2a$ and therefore $\cos b = 0$ (since $2a + q = 0$ from $A - A^T = 2A + qI$). It follows that $b = \frac{\pi}{2} + k\pi$ for some integer k and then $e^a = (-1)^k(\pi + 2k\pi)$ (since $e^a \sin b = 2b$). Distinguishing k even and k odd and because $e^a > 0$, we deduce $a = \ln((4m+1)\pi)$, $b = \pm(\frac{\pi}{2} + 2m\pi)$ for some nonnegative integer m , as expected.

Conversely, it is easily checked that any matrix A of the form $[\ln((4m+1)\pi)]I \pm (\frac{\pi}{2} + 2m\pi)J$ ($m = 0, 1, \dots$) is indeed a solution.

Proof of $\mathcal{R}$. Suppose that A has two distinct eigenvalues λ, μ . By the Hamilton-Cayley theorem, we have $(A - \lambda I)(A - \mu I) = O$, hence $(A - \lambda I)^2 + (\lambda - \mu)(A - \lambda I) = O$.

By an easy induction we deduce that $(A - \lambda I)^k = (\mu - \lambda)^{k-1}(A - \lambda I)$ for any positive integer k . Now, since λI and $A - \lambda I$ commute, we have

$$e^A = e^{\lambda I} e^{A - \lambda I} = e^{\lambda I} \left(I + \sum_{k=1}^{\infty} \frac{(\mu - \lambda)^{k-1}}{k!} (A - \lambda I) \right) = e^{\lambda I} \left(I + \frac{e^{\mu - \lambda} - 1}{\mu - \lambda} (A - \lambda I) \right)$$

and the result follows after a short calculation. If A has an eigenvalue λ of algebraic multiplicity 2, we use a similar calculation or alternatively, we consider a passage to the limit as $\mu \rightarrow \lambda$ in the previous result.

Solution 2 by Ulrich Abel and Vitaliy Kushnirevych, Technische Hochschule Mittelhessen, Germany.

Editorial note. The parametrization and the corresponding factors in the submitted solution have been corrected below.

We show that all solutions are given by

$$A = \begin{pmatrix} \ln(2\beta_m) & \mp\beta_m \\ \pm\beta_m & \ln(2\beta_m) \end{pmatrix}, \quad \beta_m = \left(2m + \frac{1}{2}\right)\pi, \quad m = 0, 1, 2, \dots$$

Denote $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and assume that

$$e^A = A - A^\top = (c - b) \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Since $\det(e^A) = e^{\text{tr}(A)} \neq 0$, we infer that $c \neq b$. Let $\begin{pmatrix} v \\ w \end{pmatrix}$ be a nontrivial eigenvector of A associated with an eigenvalue λ . Then

$$e^A \begin{pmatrix} v \\ w \end{pmatrix} = e^\lambda \begin{pmatrix} v \\ w \end{pmatrix} = (c - b) \begin{pmatrix} -w \\ v \end{pmatrix}.$$

Equating the components gives

$$e^{2\lambda}v = -(c - b)e^\lambda w = -(c - b)^2v.$$

Since $v = 0$ implies $w = 0$, we have $v \neq 0$ and therefore $e^{2\lambda} = -(c - b)^2$. As b, c are real, it follows that

$$\lambda = \ln|c - b| + i\left(n + \frac{1}{2}\right)\pi, \quad n \in \mathbb{Z}.$$

Write the two eigenvalues as

$$\lambda_1 = \frac{a + d - i\omega}{2}, \quad \lambda_2 = \frac{a + d + i\omega}{2}, \quad \omega = \sqrt{4\det A - (a + d)^2}.$$

Since $\text{Im } \lambda \neq 0$, we have

$$4\det A - (a + d)^2 = -4bc - (a - d)^2 > 0.$$

Thus $\omega > 0$, $bc < 0$, and in particular $c \neq 0$. Also $\text{Re } \lambda = \ln|c - b|$. Direct calculation confirms that $A = SDS^{-1}$, where

$$D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad S = \frac{1}{2c} \begin{pmatrix} a - d - i\omega & a - d + i\omega \\ 2c & 2c \end{pmatrix},$$

and

$$S^{-1} = \frac{1}{2i\omega} \begin{pmatrix} -2c & a - d + i\omega \\ 2c & -(a - d) + i\omega \end{pmatrix}.$$

Consequently, if $E_j = e^{\lambda_j}$, then

$$e^A = Se^D S^{-1} = \frac{1}{2i\omega} \begin{pmatrix} (E_2 - E_1)(a - d) + (E_1 + E_2)i\omega & 2(E_2 - E_1)b \\ 2(E_2 - E_1)c & -(E_2 - E_1)(a - d) + (E_1 + E_2)i\omega \end{pmatrix}.$$

Since $\text{tr}(e^A) = 0$, we have $E_1 + E_2 = 0$ and hence

$$e^A = \frac{E_2 - E_1}{2i\omega} \begin{pmatrix} a-d & 2b \\ 2c & -(a-d) \end{pmatrix}.$$

Moreover, $E_1 \neq 0$ and $E_2 = -E_1$, so $E_2 - E_1 \neq 0$. Comparison with $A - A^\top$ yields $a = d$ and $b = -c$. The relation $(a+d)/2 = \ln|c-b|$ now gives $a = \ln|2c|$.

Noting that

$$\exp \begin{pmatrix} a & -c \\ c & a \end{pmatrix} = e^a \begin{pmatrix} \cos c & -\sin c \\ \sin c & \cos c \end{pmatrix},$$

we obtain $\cos c = 0$ and $e^a \sin c = 2c$. Since $e^a = 2|c|$, the latter equality requires $\sin c$ to have the sign of c . Therefore

$$c = \pm \left(2m + \frac{1}{2}\right)\pi, \quad a = \ln((4m+1)\pi), \quad m = 0, 1, 2, \dots$$

Conversely, these values satisfy $e^A = A - A^\top$, as the displayed exponential formula shows.

Appendix: a different approach. The real algebra of matrices of the form $\begin{pmatrix} x & -y \\ y & x \end{pmatrix}$ is isomorphic to $\mathbb{C}$ via $\begin{pmatrix} x & -y \\ y & x \end{pmatrix} \mapsto x + iy$. After the reduction to $a = d$ and $b = -c$, the equation becomes

$$e^{a+ic} = 2ci.$$

For $c > 0$, this gives

$$a + ic = \ln(2c) + i\left(\frac{\pi}{2} + 2m\pi\right), \quad m = 0, 1, 2, \dots$$

The case $c < 0$ follows by complex conjugation and leads to the same family of matrices.

Solution 3 by Albert Stadler, Herrliberg, Switzerland. Let

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \tau = \frac{a+d}{2}, \quad \Delta = (a-d)^2 + 4bc, \quad r = \sqrt{\Delta}.$$

For $\Delta < 0$, we take $\sqrt{\Delta} = i\sqrt{-\Delta}$. Put

$$C = \cosh\left(\frac{r}{2}\right), \quad S = \begin{cases} \frac{\sinh(r/2)}{r}, & r \neq 0, \\ \frac{1}{2}, & r = 0. \end{cases}$$

Both $\cosh z$ and $\sinh z/z$ are even functions, so the choice of the square root does not affect these expressions.

The Mathematica command

```
A = {{a, b}, {c, d}};
FullSimplify[MatrixExp[A] - A + Transpose[A]]
```

gives an output that can be written as

$$e^A - A + A^\top = \begin{pmatrix} e^\tau(C + (a-d)S) & -b + c + 2be^\tau S \\ b - c + 2ce^\tau S & e^\tau(C - (a-d)S) \end{pmatrix}.$$

The same formula follows directly from $(A - \tau I)^2 = \frac{r^2}{4}I$. We also note that the eigenvalues of A are $\tau \pm \frac{r}{2}$.

Thus $e^A = A - A^\top$ is equivalent to

$$\begin{aligned} C &= 0, & \text{(i)} \\ (a - d)S &= 0, & \text{(ii)} \\ -b + c + 2be^\tau S &= 0, & \text{(iii)} \\ b - c + 2ce^\tau S &= 0. & \text{(iv)} \end{aligned}$$

Condition (i) implies that $\Delta < 0$ and

$$\frac{r}{2} = i\frac{\pi}{2}(2k+1)$$

for some nonnegative integer k . In particular, $r \neq 0$, and

$$S = \frac{\sinh(i\pi(2k+1)/2)}{i\pi(2k+1)} = \frac{(-1)^k}{\pi(2k+1)} \neq 0.$$

It follows from (ii) that $a = d$.

Multiplying (iii) by c and subtracting b times (iv) gives $c^2 - b^2 = 0$. Since $\Delta = 4bc < 0$, the numbers b and c have opposite signs and equal moduli. Therefore $b = -c \neq 0$. Condition (iii) now reduces to $e^a S = 1$, that is,

$$\frac{e^a(-1)^k}{\pi(2k+1)} = 1.$$

Since $e^a > 0$, the integer k is even. Writing $k = 2m$, we obtain

$$a = d = \ln(\pi(4m+1)).$$

Finally, $r/2 = i|b|$ yields

$$b = -c = \pm \frac{\pi}{2}(4m+1).$$

Hence all solutions belong to the family

$$A = \begin{pmatrix} \ln(\pi(4m+1)) & \pm \frac{\pi}{2}(4m+1) \\ \mp \frac{\pi}{2}(4m+1) & \ln(\pi(4m+1)) \end{pmatrix}, \quad m = 0, 1, 2, \dots$$

Conversely, for any matrix in this family, $\cos b = 0$ and $e^a \sin b = 2b$, so

$$e^A = e^a \begin{pmatrix} \cos b & \sin b \\ -\sin b & \cos b \end{pmatrix} = \begin{pmatrix} 0 & 2b \\ -2b & 0 \end{pmatrix} = A - A^\top.$$

Editorial observation. All three solutions give the same family

$$A = \ln((4m+1)\pi)I \pm \frac{(4m+1)\pi}{2}J, \quad m = 0, 1, 2, \dots, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Also solved by the proposer.

174. *Proposed by Vasile Cîrtoaje, Petroleum-Gas, University of Ploiesti, Romania.* Prove that 2 is the least positive value of the constant k such that

$$\left(\frac{ka_1 + a_2 + \cdots + a_7}{k+6} \right)^2 \geq \frac{a_1a_2 + a_2a_3 + \cdots + a_8a_1}{8}$$

whenever $a_1 \geq a_2 \geq \cdots \geq a_8 \geq 0$.

Solution 1 by the editors. For necessity, take $a_1 = a_2 = a_3 = 1$ and $a_4 = \cdots = a_8 = 0$. The required inequality becomes

$$\left(\frac{k+2}{k+6} \right)^2 \geq \frac{1}{4}.$$

Since $k > 0$, taking square roots gives $2(k+2) \geq k+6$, hence $k \geq 2$.

We prove sufficiency for $k = 2$. Let $h = a_8$, $b_i = a_i - h$ for $1 \leq i \leq 7$, and put

$$T = b_2 + \cdots + b_7, \quad C = \sum_{i=1}^6 b_i b_{i+1}.$$

All b_i are nonnegative, and $b_i \leq b_1$. Consequently,

$$C \leq b_1 \sum_{i=1}^6 b_{i+1} = b_1 T.$$

Writing $a_9 = a_1$, we also have

$$\begin{aligned} 2a_1 + a_2 + \cdots + a_7 &= 8h + 2b_1 + T, \\ \sum_{i=1}^8 a_i a_{i+1} &= 8h^2 + 2h(b_1 + T) + C. \end{aligned}$$

Therefore,

$$\begin{aligned} (2a_1 + a_2 + \cdots + a_7)^2 - 8 \sum_{i=1}^8 a_i a_{i+1} \\ = 16hb_1 + (2b_1 - T)^2 + 8(b_1 T - C) \geq 0. \end{aligned}$$

Dividing by 64 proves the desired inequality for $k = 2$. Thus the least positive value is 2. The test tuple $a_1 = a_2 = a_3 = 1$ and $a_4 = \cdots = a_8 = 0$ attains equality at $k = 2$.

Solution 2 by Paolo Perfetti, Department of Mathematics, University of Rome Tor Vergata, Rome, Italy. To see necessity, take $a_1 = a_2 = a_3 = 1$ and $a_4 = \cdots = a_8 = 0$. Then $((k+2)/(k+6))^2 \geq 1/4$, which forces $k \geq 2$ for $k > 0$. We now prove that the inequality holds for $k = 2$.

Define $a_1 = a$, $a_2 = b$, $a_3 = c$, $a_4 = d$, $a_5 = e$, $a_6 = f$, $a_7 = g$, and $a_8 = h$. We can set

$$\begin{aligned} g &= h + x, & f &= g + y, & e &= f + z, & d &= e + t, \\ c &= d + u, & b &= c + v, & a &= b + w, \end{aligned}$$

where h, x, y, z, t, u, v, w are all nonnegative. After multiplying by 64, the desired inequality is

$$(2a + b + c + d + e + f + g)^2 - 8(ab + bc + cd + de + ef + fg + gh + ha) \geq 0.$$

Its left-hand side expands to

$$\begin{aligned}
& 16h(x + y + z + t + u + v + w) \\
& + 16x^2 + 9y^2 + 4z^2 + t^2 + v^2 + 4w^2 \\
& + 24x(y + z + t + u + v + w) \\
& + 12yz + 14yt + 16yu + 18yv + 20yw \\
& + 4zt + 8zu + 12zv + 16zw \\
& + 6tv + 12tw + 8uw + 4vw,
\end{aligned}$$

which is nonnegative. Therefore the least positive constant is $k = 2$.

Also solved by the proposer.

175. *Proposed by Michel Bataille, Rouen, France.* Let m and n be coprime positive integers such that $mn + 1$ is a square-free divisor of $m^4 + n^4 + 2$. Prove that $m = n = 1$.

Solution by Thammadol Tansrivorarat and Artem Borisov, University of Bristol, United Kingdom. The condition $mn + 1 \mid m^4 + n^4 + 2$ implies

$$mn + 1 \mid m^4 + n^4 + 2 + 2(mn + 1)(mn - 1) = (m^2 + n^2)^2.$$

Since $mn + 1$ is square-free, it follows that $mn + 1 \mid m^2 + n^2$. We prove the remaining part by Vieta jumping (compare the classical descent method in IMO 1988, Problem 6). Let

$$k = \frac{m^2 + n^2}{mn + 1},$$

where k is a positive integer. This means that m is a root of

$$X^2 - knX + n^2 - k = 0.$$

By Vieta's formula, the other root is

$$r = kn - m = \frac{n^2 - k}{m},$$

which is also an integer.

If $n = 1$, then $m + 1 \mid m^2 + 1$ gives $m + 1 \mid 2$, so $m = 1$. If $m = n$, coprimality immediately gives $m = n = 1$. Suppose the reduced equation $m^2 + n^2 = k(mn + 1)$ has a coprime positive solution other than $(m, n) = (1, 1)$. For this fixed k , choose such a solution with the smallest sum $m + n$. By symmetry, assume $m \geq n \geq 2$.

We claim that r cannot be negative. If $r < 0$, then $-rkn > k$, whence

$$r^2 - rkn + n^2 - k > r^2 + k + n^2 - k > 0,$$

contrary to the fact that r is a root. If $r = 0$, then $k = n^2$, and $n^2 \mid m^2 + n^2$ implies $n^2 \mid m^2$. Since m and n are coprime, this forces $n = 1$, again a contradiction. Thus $r > 0$.

Moreover,

$$\gcd(r, n) = \gcd(kn - m, n) = \gcd(m, n) = 1,$$

and the quadratic equation gives

$$r^2 + n^2 = k(rn + 1).$$

Hence (r, n) is another positive coprime solution of the reduced equation. However,

$$r = \frac{n^2 - k}{m} < m,$$

because $n \leq m$ and $k > 0$. It follows that $r + n < m + n$, contradicting minimality. Therefore $m = n = 1$.

Also solved by the proposer.

176. *Proposed by Stănescu Florin, Șerban Cioculescu School, Găești, Romania.* Let $(a_n)_{n \geq 1}$ be a sequence of real numbers defined recursively by

$$a_1 \in (0, 1], \quad a_{n+1} = \frac{1 - e^{-n^2 a_n}}{(n+1)^2}.$$

Compute the limit

$$\lim_{n \rightarrow \infty} \frac{n^4}{\ln n} ((n^2 - 1)(1 - e^{a_n}) - n(e^{-n a_n} - 1)).$$

Solution 1 by Michel Bataille, Rouen, France. We show that the required limit is $\frac{2}{3}$.

It seems simpler to work with the sequence (u_n) defined by $u_n = n^2 a_n$ (so that $u_{n+1} = 1 - e^{-u_n}$ for all $n \geq 1$). We then have to evaluate the limit of $\frac{n^4}{\ln n} \cdot X_n$ where

$$X_n = (n^2 - 1)(1 - e^{\frac{u_n}{n^2}}) - n(e^{-\frac{u_n}{n}} - 1).$$

Since $e^x \geq x + 1$ for all real x (with equality if and only if $x = 0$), we have $u_n - u_{n+1} \geq 0$ for all n . In addition, by an immediate induction, we see that $u_n > 0$ for all n . The sequence (u_n) being bounded below and non-increasing, is convergent and its limit ℓ satisfies $\ell = 1 - e^{-\ell}$. Thus $\ell = 0$.

We deduce that $u_{n+1} \sim u_n$ as $n \rightarrow \infty$ (since $\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \frac{1 - e^{-u_n}}{u_n} = 1$) and that

$$\begin{aligned} u_{n+1}^{-1} &= (1 - e^{-u_n})^{-1} = \left(u_n - \frac{u_n^2}{2} + \frac{u_n^3}{6} + o(u_n^3) \right)^{-1} = u_n^{-1} \left(1 - \frac{u_n}{2} + \frac{u_n^2}{6} + o(u_n^2) \right)^{-1} \\ &= u_n^{-1} \left(1 + \frac{u_n}{2} - \frac{u_n^2}{6} + \frac{u_n^2}{4} + o(u_n^2) \right) = u_n^{-1} + \frac{1}{2} + \frac{u_n}{12} + o(u_n). \end{aligned}$$

From the Stolz-Cesaro theorem, we first obtain $\sum_{k=1}^n (u_{k+1}^{-1} - u_k^{-1}) \sim \frac{n}{2}$ (since $u_{n+1}^{-1} - u_n^{-1} \sim \frac{1}{2}$), hence $u_n^{-1} \sim \frac{n}{2}$ and $u_n \sim \frac{2}{n}$.

Second, from

$$\left(u_{n+1}^{-1} - \frac{n+1}{2} \right) - \left(u_n^{-1} - \frac{n}{2} \right) \sim \frac{u_n}{12} \sim \frac{1}{6n}$$

we are similarly led to $u_n^{-1} - \frac{n}{2} \sim \frac{\ln n}{6}$ and then to

$$u_n = \frac{2}{n} \left(1 + \frac{\ln n}{3n} + o\left(\frac{\ln n}{n}\right) \right)^{-1} = \frac{2}{n} - \frac{2 \ln n}{3n^2} + o\left(\frac{\ln n}{n^2}\right).$$

On the other hand, we have

$$(n^2 - 1)(1 - e^{\frac{u_n}{n^2}}) = (n^2 - 1) \left(-\frac{u_n}{n^2} - \frac{u_n^2}{2n^4} + o(1/n^6) \right) = -u_n + \frac{u_n}{n^2} - \frac{u_n^2}{2n^2} + o(1/n^4)$$

and

$$n(e^{-\frac{u_n}{n}} - 1) = n \left(-\frac{u_n}{n} + \frac{u_n^2}{2n^2} - \frac{u_n^3}{6n^3} + o(1/n^6) \right) = -u_n + \frac{u_n^2}{2n} + o(1/n^4)$$

so that

$$\begin{aligned} X_n &= (n^2 - 1)(1 - e^{u_n/n^2}) - n(e^{-u_n/n} - 1) \\ &= \frac{u_n}{2n} \left(\frac{2}{n} - u_n - \frac{u_n}{n} + o(1/n^2) \right) \\ &= \frac{u_n}{2n} \left(\frac{2 \ln n}{3n^2} + o\left(\frac{\ln n}{n^2}\right) \right), \end{aligned}$$

since $u_n/n \sim 2/n^2 = o(\ln n/n^2)$. Thus, $X_n \sim \frac{u_n}{2n} \left(\frac{2 \ln n}{3n^2} \right) \sim \frac{2 \ln n}{3n^4}$ and the result follows.

Solution 2 by Paolo Perfetti, Department of Mathematics, University of Rome Tor Vergata, Rome, Italy.

Editorial note. The asymptotic remainder estimates in the submitted proof have been corrected.

Let $b_n = n^2 a_n$. Then

$$b_{n+1} = 1 - e^{-b_n} \leq 1 - (1 - b_n) = b_n.$$

Thus (b_n) is positive and decreasing, and converges to a value L satisfying $L = 1 - e^{-L}$. Therefore $L = 0$.

Expanding at zero, we have

$$\begin{aligned} \frac{1}{b_{n+1}} &= \frac{1}{1 - e^{-b_n}} \\ &= \frac{1}{b_n(1 - b_n/2 + O(b_n^2))} \\ &= \frac{1}{b_n} + \frac{1}{2} + O(b_n). \end{aligned}$$

Summing from $n = 1$ to N , and using $b_n \rightarrow 0$, gives

$$\frac{1}{b_{N+1}} - \frac{1}{b_1} = \frac{N}{2} + \sum_{n=1}^N O(b_n) = \frac{N}{2} + o(N).$$

Consequently, $b_n \sim 2/n$ and $b_n = O(1/n)$. Returning to the same sum, we obtain

$$\frac{1}{b_{N+1}} = \frac{N}{2} + O(\ln N),$$

so

$$b_n = \frac{2}{n} + O\left(\frac{\ln n}{n^2}\right).$$

It follows that

$$\sum_{n=1}^N b_n = 2 \ln N + O(1), \quad \sum_{n=1}^N b_n^2 = O(1).$$

For a more precise estimate, use

$$\frac{1}{1 - e^{-t}} = \frac{1}{t} + \frac{1}{2} + \frac{t}{12} + O(t^3) \quad (t \rightarrow 0).$$

Hence

$$\begin{aligned} \frac{1}{b_{N+1}} - \frac{1}{b_1} &= \sum_{n=1}^N \left(\frac{1}{2} + \frac{b_n}{12} + O(b_n^3) \right) \\ &= \frac{N}{2} + \frac{\ln N}{6} + O(1), \end{aligned}$$

because $\sum_{n \geq 1} b_n^3$ converges. Inverting this estimate and replacing $N + 1$ by n yields

$$b_n = \frac{2}{n} - \frac{2 \ln n}{3n^2} + O(1/n^2).$$

We have therefore obtained

$$a_n = \frac{2}{n^3} - \frac{2 \ln n}{3n^4} + O(1/n^4).$$

Using $e^x = 1 + x + x^2/2 + O(x^3)$, we obtain

$$\begin{aligned} & (n^2 - 1)(1 - e^{a_n}) - n(e^{-na_n} - 1) \\ &= a_n - \frac{n^3 a_n^2}{2} - \frac{n^2 a_n^2}{2} + \frac{a_n^2}{2} + O(n^4 a_n^3) \\ &= a_n - \frac{n^3 a_n^2}{2} + O(1/n^4). \end{aligned}$$

Moreover,

$$\begin{aligned} a_n^2 &= \frac{4}{n^6} - \frac{8 \ln n}{3n^7} + O(1/n^7), \\ a_n - \frac{n^3 a_n^2}{2} &= \frac{2 \ln n}{3n^4} + O(1/n^4). \end{aligned}$$

Thus

$$\frac{n^4}{\ln n} ((n^2 - 1)(1 - e^{a_n}) - n(e^{-na_n} - 1)) = \frac{2}{3} + O(1/\ln n),$$

and the required limit is $2/3$.

Solution 3 by Albert Stadler, Herrliberg, Switzerland. Put $b_n = n^2 a_n$. Then $b_1 \in (0, 1]$ and

$$b_{n+1} = 1 - e^{-b_n},$$

so $0 < b_n \leq 1$ for all $n \geq 1$. Moreover,

$$b_n - b_{n+1} = \sum_{k=1}^{\infty} \frac{b_n^{2k}}{(2k)!} \left(1 - \frac{b_n}{2k+1}\right) > 0.$$

Thus (b_n) is decreasing and has a limit $b \geq 0$. The equation $b = 1 - e^{-b}$ forces $b = 0$. Consequently,

$$\sum_{j=1}^{N-1} b_j = o(N).$$

The generating function of the Bernoulli numbers is

$$\frac{t}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}, \quad |t| < 2\pi.$$

It gives

$$\begin{aligned} \frac{1}{b_{n+1}} - \frac{1}{b_n} &= \frac{1}{b_n} \left(\frac{b_n}{1 - e^{-b_n}} - 1 \right) \\ &= \sum_{k=1}^{\infty} (-1)^k B_k \frac{b_n^{k-1}}{k!} \\ &= \frac{1}{2} + \frac{b_n}{12} + O(b_n^3). \end{aligned} \tag{*}$$

Summing from $n = 1$ to $n = N - 1$, we obtain

$$\frac{1}{b_N} - \frac{1}{b_1} = \frac{N-1}{2} + \frac{1}{12} \sum_{j=1}^{N-1} b_j + O\left(\sum_{j=1}^{N-1} b_j^3\right). \tag{**}$$

Since $b_j^3 \leq b_j$ and $\sum_{j < N} b_j = o(N)$, this implies

$$\frac{1}{b_n} = \frac{n}{2}(1 + o(1)), \quad b_n = \frac{2}{n}(1 + o(1)).$$

In particular, $b_n = O(1/n)$, so $\sum_{j < N} b_j = O(\ln N)$ and $\sum_{j < N} b_j^3 = O(1)$. Substitution into (**) gives

$$\frac{1}{b_n} = \frac{n}{2} + O(\ln n), \quad b_n = \frac{2}{n} + O\left(\frac{\ln n}{n^2}\right).$$

Because $\sum_{j \geq 2} (\ln j)/j^2$ converges, we now have

$$\sum_{j=1}^{n-1} b_j = 2 \sum_{j=1}^{n-1} \frac{1}{j} + O(1) = 2 \ln n + O(1).$$

Using (**) once more, we obtain

$$\frac{1}{b_n} = \frac{n}{2} + \frac{1}{6} \ln n + O(1) = \frac{n}{2} \left(1 + \frac{\ln n}{3n} + O\left(\frac{1}{n}\right)\right).$$

Therefore

$$b_n = \frac{2}{n} - \frac{2 \ln n}{3n^2} + O\left(\frac{1}{n^2}\right), \quad a_n = \frac{2}{n^3} - \frac{2 \ln n}{3n^4} + O\left(\frac{1}{n^4}\right).$$

Finally, Taylor expansion gives

$$\begin{aligned} & (n^2 - 1)(1 - e^{a_n}) - n(e^{-na_n} - 1) \\ &= a_n - \frac{n^3 + n^2 - 1}{2} a_n^2 + O(n^4 a_n^3) \\ &= a_n - \frac{1}{2} n^3 a_n^2 + O\left(\frac{1}{n^4}\right) \\ &= \frac{2 \ln n}{3n^4} + O\left(\frac{1}{n^4}\right). \end{aligned}$$

Hence the required limit is $2/3$.

Also solved by the proposer.

177. *Proposed by Dion Aliu, University of York, United Kingdom.* Let $S(n)$ be the biggest square divisor of n . Let $\{a_n\}_{n=1}^{\infty}$ be defined as follows:

$$a_1 = a, \quad a_2 = b$$

and

$$a_{n+1} = S(a_n) + S(a_{n-1}).$$

Prove that for every choice of $a, b \in \mathbb{N}$, the sequence is eventually constant.

Solution by the editors. For a positive integer m , put $Q(m) = \sqrt{S(m)}$. The prime factorization of m shows that

$$Q(c^2 m) = cQ(m) \quad (c, m \in \mathbb{N}).$$

Set $x_n = Q(a_n)$ and $d = \gcd(x_1, x_2)$. The prime factorization also gives $d = Q(\gcd(a, b))$: for every prime p , the exponent of p in d equals $\min\{\lfloor v_p(a)/2 \rfloor, \lfloor v_p(b)/2 \rfloor\} = \lfloor \min\{v_p(a), v_p(b)\}/2 \rfloor$. The given recurrence implies

$$x_{n+1} = Q(x_n^2 + x_{n-1}^2) \quad (n \geq 2).$$

Consequently, every x_n is divisible by d , and the positive integers $u_n = x_n/d$ satisfy

$$u_{n+1} = Q(u_n^2 + u_{n-1}^2), \quad \gcd(u_1, u_2) = 1.$$

If $\gcd(u_{n-1}, u_n) = 1$, then $\gcd(u_n, u_{n+1}) = 1$ as well. Indeed, a prime dividing both u_n and u_{n+1} would divide $u_n^2 + u_{n-1}^2$, and hence also u_{n-1} , a contradiction. Thus every two consecutive terms of (u_n) are coprime. Their sum of squares is either odd or congruent to 2 modulo 4, so its greatest square divisor is odd. It follows that u_n is odd for every $n \geq 3$.

For $n \geq 4$, therefore, $u_n^2 + u_{n-1}^2 \equiv 2 \pmod{8}$. The quotient of this number by its greatest square divisor is an even positive integer, and is consequently at least 2. Hence

$$u_{n+1}^2 \leq \frac{u_n^2 + u_{n-1}^2}{2}.$$

Let $M_n = \max(u_{n-1}, u_n)$ for $n \geq 4$. The preceding inequality gives $M_{n+1} \leq M_n$. If $M_n > 1$, coprimality ensures that $u_{n-1} \neq u_n$, and hence $u_{n+1} < M_n$. Applying the inequality once more gives $u_{n+2} < M_n$, so

$$M_{n+2} < M_n \quad \text{whenever } M_n > 1.$$

Since the M_n are positive integers, this strict decrease forces $M_n = 1$ eventually. Thereafter $u_n = 1$ and $x_n = d$ for all sufficiently large n . Finally,

$$a_{n+1} = x_n^2 + x_{n-1}^2 = 2d^2$$

for all sufficiently large n , proving the assertion. In fact, the eventual constant is

$$2d^2 = 2S(\gcd(a, b)).$$

Also solved by the proposer.

178. *Proposed by Paolo Perfetti, Department of Mathematics, University of Rome Tor Vergata, Rome, Italy.* Compute the integral

$$\int_0^\infty \frac{\ln(1-x+x^2)}{1+x^2} dx.$$

Solution 1 by Michel Bataille, Rouen, France. The proposed integral I is the sum $J + K$ where

$$J = \int_0^1 \frac{\ln(1-x+x^2)}{1+x^2} dx, \quad K = \int_1^\infty \frac{\ln(1-x+x^2)}{1+x^2} dx$$

and the change of variables $x = \frac{1}{u}$ shows that

$$K = \int_0^1 \frac{\ln(1-u+u^2)}{1+u^2} du - 2 \int_0^1 \frac{\ln(u)}{1+u^2} du = J + 2G$$

where G denotes the Catalan constant $G = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{(2n-1)^2}$. Thus, $I = 2J + 2G$ and it remains to calculate J . To this aim, we use the method and some of the results developed by Kee-Wai Lau in the solution to a recent problem ([1]).

For $a \in [1/2, \infty)$, let $J(a) = \int_0^1 \frac{\ln(a-x+ax^2)}{1+x^2} dx$. First, we calculate $J(1/2)$:

$$J(1/2) = 2 \int_0^1 \frac{\ln(1-x)}{1+x^2} dx - (\ln 2) \int_0^1 \frac{dx}{1+x^2} = 2 \left(\frac{\pi}{8} \ln 2 - G \right) - \frac{\pi}{4} \ln 2 = -2G$$

where we use

$$\begin{aligned}
 \int_0^1 \frac{\ln(1-x)}{1+x^2} dx &= \int_0^1 \frac{\ln(1+x)}{1+x^2} dx + \int_0^1 \frac{\ln \frac{1-x}{1+x}}{1+x^2} dx \\
 &= \frac{\pi}{8} \ln 2 + \int_0^{\pi/4} \ln \left(\frac{1-\tan \theta}{1+\tan \theta} \right) d\theta \quad ([1]) \\
 &= \frac{\pi}{8} \ln 2 + \int_0^{\pi/4} \ln \tan \left(\frac{\pi}{4} - \theta \right) d\theta = \frac{\pi}{8} \ln 2 + \int_0^{\pi/4} \ln(\tan u) du \\
 &= \frac{\pi}{8} \ln 2 + \int_0^1 \frac{\ln x}{1+x^2} dx = \frac{\pi}{8} \ln 2 - G
 \end{aligned}$$

For $a > 1/2$, we have

$$\begin{aligned}
 J'(a) &= \int_0^1 \frac{dx}{ax^2 - x + a} = \frac{2}{\sqrt{4a^2 - 1}} \left[\arctan \left(\frac{2ax - 1}{\sqrt{4a^2 - 1}} \right) \right]_0^1 \\
 &= \frac{2}{\sqrt{4a^2 - 1}} \arctan \left(\sqrt{\frac{2a+1}{2a-1}} \right) = \frac{\pi}{\sqrt{4a^2 - 1}} - \frac{2}{\sqrt{4a^2 - 1}} \arctan \left(\sqrt{\frac{2a-1}{2a+1}} \right) \\
 &= \frac{\pi}{\sqrt{4a^2 - 1}} - I'(a)
 \end{aligned}$$

where $I(a) = \int_0^1 \frac{\ln(a+x+ax^2)}{1+x^2} dx$ (see [1]).

At the lower endpoint, the substitution $a = 1/2$ gives

$$I(1/2) = 2 \int_0^1 \frac{\ln(1+x)}{1+x^2} dx - \frac{\pi}{4} \ln 2 = 0.$$

Since $\int_{1/2}^1 \frac{1}{\sqrt{4a^2-1}} da = \frac{\ln(2+\sqrt{3})}{2}$ (change of variables $a = \frac{1}{2} \cosh(t)$) and $\int_{1/2}^1 I'(a) da = I(1) = \frac{\pi}{6} \ln(2+\sqrt{3}) - \frac{G}{3}$ (see [1]), we obtain

$$J = J(1) = J(1/2) + \int_{1/2}^1 J'(a) da = -2G + \frac{\pi}{2} \ln(2+\sqrt{3}) - \frac{\pi}{6} \ln(2+\sqrt{3}) + \frac{G}{3} = \frac{\pi}{3} \ln(2+\sqrt{3}) - \frac{5G}{3}$$

and finally

$$I = 2J + 2G = \frac{2\pi}{3} \ln(2+\sqrt{3}) - \frac{4G}{3}.$$

[1] Kee-Wai Lau, Solution to 2176, *Mathematics Magazine*, Vol. 97, No 4, October 2024, p. 435-6).

Solution 2 by Kee-Wai Lau, Hong Kong, China. Denote the given integral by I , and let

$$C = \frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{7^2} + \cdots$$

be the Catalan constant. We show that

$$I = \frac{2(\pi \ln(2+\sqrt{3}) - 2C)}{3}. \quad (1)$$

We first introduce four integrals. Let

$$I_1 = \int_0^1 \frac{\ln x}{1+x^2} dx, \quad I_2 = \int_0^1 \frac{\ln(1-x)}{1+x^2} dx,$$

$$I_3 = \int_0^{\pi/3} x \sec x \, dx, \quad I_4 = \int_{1/2}^1 \frac{\arctan\left(\sqrt{\frac{2x+1}{2x-1}}\right)}{\sqrt{4x^2-1}} \, dx.$$

It is known (I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products*, Elsevier Inc., 2007, entry 4.231(12), p. 536 and entry 4.291(10), p. 556) that

$$I_1 = -C, \tag{2}$$

and

$$I_2 = \frac{\pi \ln 2}{8} - C. \tag{3}$$

It is shown (Solution to problem 2176, *Mathematics Magazine*, **97** (2024), 435–436) that

$$I_3 = \frac{\pi \ln(2 + \sqrt{3}) - 2C}{3}. \tag{4}$$

By the substitution $x = \frac{\sec \theta}{2}$, we obtain

$$\begin{aligned} I_4 &= \frac{1}{2} \int_0^{\pi/3} \arctan\left(\frac{1}{\tan(\theta/2)}\right) \sec \theta \, d\theta \\ &= \frac{1}{4} \int_0^{\pi/3} (\pi - \theta) \sec \theta \, d\theta \\ &= \frac{\pi \ln(2 + \sqrt{3}) - I_3}{4}. \end{aligned}$$

By (4), we obtain

$$I_4 = \frac{\pi \ln(2 + \sqrt{3}) + C}{6}. \tag{5}$$

We now come to the proof of (1). For $t \geq \frac{1}{2}$, let

$$f(t) = \int_0^1 \frac{\ln(t - x + tx^2)}{1 + x^2} \, dx.$$

By the substitution $x = \frac{1}{y}$, we have

$$\int_1^\infty \frac{\ln(1 - x + x^2)}{1 + x^2} \, dx = \int_0^1 \frac{\ln(1 - y + y^2)}{1 + y^2} \, dy - 2I_1,$$

so that

$$I = 2 \left(\int_0^1 \frac{\ln(1 - x + x^2)}{1 + x^2} \, dx - I_1 \right),$$

or

$$f(1) = \frac{I}{2} + I_1. \tag{6}$$

For $t > \frac{1}{2}$, we have

$$\begin{aligned}\frac{df(t)}{dt} &= \int_0^1 \frac{1}{t-x+tx^2} dx \\ &= \left[\frac{2 \arctan\left(\frac{2tx-1}{\sqrt{4t^2-1}}\right)}{\sqrt{4t^2-1}} \right]_0^1 \\ &= \frac{2 \arctan\left(\sqrt{\frac{2t+1}{2t-1}}\right)}{\sqrt{4t^2-1}}.\end{aligned}$$

Thus $f(1) - f(1/2) = 2I_4$. Note that

$$f(1/2) = 2I_2 - \frac{\pi \ln 2}{4},$$

and so

$$f(1) = 2I_4 + 2I_2 - \frac{\pi \ln 2}{4}. \quad (7)$$

By (6) and (7), we deduce that

$$I = 4I_2 + 4I_4 - 2I_1 - \frac{\pi \ln 2}{2}.$$

Now (1) follows from (2), (3) and (5).

Solution 3 by Albert Stadler, Herrliberg, Switzerland. We prove that the integral equals

$$\frac{2\pi}{3} \ln(2 + \sqrt{3}) - \frac{4G}{3},$$

where G is Catalan's constant. We use the following two integral evaluations (the first is also justified below via the Clausen function):

$$\int_0^{\pi/6} \frac{u}{\sin u} du = \frac{4G}{3} - \frac{\pi}{6} \operatorname{arccosh}(2) = \frac{4G}{3} - \frac{\pi}{6} \ln(2 + \sqrt{3}),$$

and

$$\int_0^\infty \frac{\ln(1+x^2)}{1+x^2} dx = \pi \ln 2.$$

For the first evaluation, define the Clausen function by $\operatorname{Cl}_2(t) = \sum_{n \geq 1} \sin(nt)/n^2$. Integration by parts together with the Fourier series for $\ln \tan u$ gives

$$\int_0^{\pi/6} \frac{u}{\sin u} du = \frac{\pi}{6} \ln \tan \frac{\pi}{12} + \operatorname{Cl}_2\left(\frac{\pi}{6}\right) + \operatorname{Cl}_2\left(\frac{5\pi}{6}\right).$$

Grouping the terms with odd indices in the two Clausen series yields $\operatorname{Cl}_2(\pi/6) + \operatorname{Cl}_2(5\pi/6) = 4G/3$. Since $\tan(\pi/12) = 2 - \sqrt{3}$, this recovers the stated evaluation.

For $0 \leq t \leq 1$, put

$$F(t) = \int_0^\infty \frac{\ln(1-tx+x^2)}{1+x^2} dx.$$

Since $1-tx+x^2 \geq (1+x^2)/2$ for $x \geq 0$, differentiation under the integral sign is justified by the integrable bound $2x/(1+x^2)^2$. Thus

$$f(t) := F'(t) = - \int_0^\infty \frac{x}{(1-tx+x^2)(1+x^2)} dx.$$

For $0 < t \leq 1$, the partial-fraction identity

$$\frac{-x}{(1-tx+x^2)(1+x^2)} = \frac{1}{t(1+x^2)} - \frac{1}{t(1-tx+x^2)}$$

gives

$$\begin{aligned} f(t) &= \frac{1}{t} \int_0^\infty \frac{dx}{1+x^2} - \frac{1}{t} \int_0^\infty \frac{dx}{1-tx+x^2} \\ &= \frac{\pi}{2t} - \frac{\pi + 2 \arctan(t/\sqrt{4-t^2})}{t\sqrt{4-t^2}}. \end{aligned}$$

The apparent singularity at $t = 0$ is removable; the integral definition gives $f(0) = -1/2$.

Substituting $t = 2 \sin u$, we obtain

$$\begin{aligned} \int_0^1 f(t) dt &= \int_0^{\pi/6} \left(\frac{\pi \cos u}{2 \sin u} - \frac{\pi}{2 \sin u} - \frac{u}{\sin u} \right) du \\ &= -\frac{\pi}{2} \int_0^{\pi/6} \tan\left(\frac{u}{2}\right) du - \int_0^{\pi/6} \frac{u}{\sin u} du. \end{aligned}$$

Here

$$-\frac{\pi}{2} \int_0^{\pi/6} \tan\left(\frac{u}{2}\right) du = \pi \ln\left(\cos \frac{\pi}{12}\right) = \frac{\pi}{2} \ln\left(\frac{2+\sqrt{3}}{4}\right).$$

Consequently,

$$\begin{aligned} \int_0^1 f(t) dt &= \frac{\pi}{2} \ln\left(\frac{2+\sqrt{3}}{4}\right) - \frac{4G}{3} + \frac{\pi}{6} \ln(2+\sqrt{3}) \\ &= -\pi \ln 2 + \frac{2\pi}{3} \ln(2+\sqrt{3}) - \frac{4G}{3}. \end{aligned}$$

Since $F(0) = \pi \ln 2$, the fundamental theorem of calculus now gives

$$\int_0^\infty \frac{\ln(1-x+x^2)}{1+x^2} dx = F(1) = F(0) + \int_0^1 f(t) dt = \frac{2\pi}{3} \ln(2+\sqrt{3}) - \frac{4G}{3}.$$

Also solved by Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; and Joe Santmyer, Las Cruces, New Mexico, USA.

Also solved by the proposer.

179. *Proposed by Joe Santmyer, Las Cruces, New Mexico, USA.* If $a \geq 0$ show that

$$\int_{-\pi/2}^{\pi/2} \frac{d\theta}{1+ae^{-\pi \tan(\theta)}} = \pi \left(\frac{1-a+\ln(a)}{(1-a)\ln(a)} \right).$$

Solution 1 by Albert Stadler, Herrliberg, Switzerland. The substitution $x = \tan \theta$ gives

$$I(a) := \int_{-\pi/2}^{\pi/2} \frac{d\theta}{1+ae^{-\pi \tan \theta}} = \int_{-\infty}^{\infty} \frac{dx}{(1+ae^{-\pi x})(1+x^2)}.$$

For $a = 0$,

$$I(0) = \pi = \lim_{a \downarrow 0} \pi \frac{1-a+\ln a}{(1-a)\ln a}.$$

For $a = 1$, the change of variables $x \mapsto -x$ shows that

$$\begin{aligned} 2I(1) &= \int_{-\infty}^{\infty} \left(\frac{1}{1+e^{-\pi x}} + \frac{1}{1+e^{\pi x}} \right) \frac{dx}{1+x^2} \\ &= \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi. \end{aligned}$$

Hence

$$I(1) = \frac{\pi}{2} = \lim_{a \rightarrow 1} \pi \frac{1-a+\ln a}{(1-a)\ln a}.$$

Now assume $a > 0$ and $a \neq 1$, and write $\xi = (\ln a)/\pi$. The function

$$H(z) = \frac{1}{(1+ae^{-\pi z})(1+z^2)}$$

is meromorphic, with simple poles at $z = \pm i$ and

$$z = \xi + (2k+1)i, \quad k \in \mathbb{Z}.$$

Let P_N be the positively oriented rectangular contour with successive vertices $-2N$, $2N$, $2N+2Ni$, and $-2N+2Ni$. For all sufficiently large integers N , so that $2N > |\xi|$, the residue theorem yields

$$\frac{1}{2\pi i} \int_{P_N} H(z) dz = \frac{1}{2i(1-a)} + \frac{1}{\pi} \sum_{k=1}^N \frac{1}{1+(\xi+(2k-1)i)^2}.$$

Indeed, the residues at the upper-half-plane poles are

$$\begin{aligned} \operatorname{Res}_{z=i} H(z) &= \frac{1}{2i(1-a)}, \\ \operatorname{Res}_{z=\xi+(2k-1)i} H(z) &= \frac{1}{\pi(1+(\xi+(2k-1)i)^2)}, \quad k = 1, 2, \dots \end{aligned}$$

On each of the three nonreal sides, $|1+z^2| \geq 4N^2-1$. On the top side, $e^{-\pi z}$ is positive real, and hence $|1+ae^{-\pi z}| \geq 1$. For the right, top, and left sides, respectively, we therefore have

$$\begin{aligned} \left| \frac{1}{2\pi i} \int_{\text{right}} H(z) dz \right| &\leq \frac{2N}{2\pi|1-ae^{-2\pi N}|(4N^2-1)} = O(N^{-1}), \\ \left| \frac{1}{2\pi i} \int_{\text{top}} H(z) dz \right| &\leq \frac{4N}{2\pi(4N^2-1)} = O(N^{-1}), \\ \left| \frac{1}{2\pi i} \int_{\text{left}} H(z) dz \right| &\leq \frac{2N}{2\pi|ae^{2\pi N}-1|(4N^2-1)} = O(N^{-1}). \end{aligned}$$

Letting $N \rightarrow \infty$ gives

$$I(a) = \frac{\pi}{1-a} + \sum_{k=1}^{\infty} \frac{2i}{1+(\xi+(2k-1)i)^2}.$$

Put $\rho = (\ln a)/(\pi i)$. Then

$$\frac{2i}{1+(\xi+(2k-1)i)^2} = i \left(\frac{1}{\rho+2k} - \frac{1}{\rho+2k-2} \right).$$

The series telescopes:

$$\begin{aligned} \sum_{k=1}^{\infty} \frac{2i}{1+(\xi+(2k-1)i)^2} &= i \lim_{N \rightarrow \infty} \left(\frac{1}{\rho+2N} - \frac{1}{\rho} \right) \\ &= -\frac{i}{\rho} = \frac{\pi}{\ln a}. \end{aligned}$$

Thus

$$I(a) = \frac{\pi}{1-a} + \frac{\pi}{\ln a} = \pi \frac{1-a+\ln a}{(1-a)\ln a},$$

with the values at $a = 0$ and $a = 1$ understood by continuous extension.

Solution 2 by Paolo Perfetti, Department of Mathematics, University of Rome Tor Vergata, Rome, Italy. With $\theta = \arctan t$, followed by $t \mapsto -t$, the integral becomes

$$I(a) = \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)(1+ae^{-\pi t})} = \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)(1+ae^{\pi t})}.$$

For $a = 0$, the integral is trivially

$$\int_{-\pi/2}^{\pi/2} d\theta = \pi.$$

For $a = 1$, we have

$$\begin{aligned} I(1) &= \frac{1}{2} \left[\int_{-\infty}^{\infty} \frac{dt}{(1+t^2)(1+e^{-\pi t})} + \int_{-\infty}^{\infty} \frac{dt}{(1+t^2)(1+e^{\pi t})} \right] \\ &= \frac{1}{2} \int_{-\infty}^{\infty} \frac{dt}{1+t^2} = \frac{\pi}{2}. \end{aligned}$$

Now suppose $a > 0$ and $a \neq 1$. Consider the complex function

$$f(z) = \frac{1}{(1+z^2)(1+ae^{\pi z})}.$$

Put $\xi = -\ln a/\pi$. The poles in the upper half-plane are $z = i$ and

$$z_k = \xi + i(2k+1), \quad k = 0, 1, 2, \dots$$

They are distinct because $a \neq 1$. Close the contour with the rectangle having vertices $-N, N, N + 2Ni, -N + 2Ni$. For sufficiently large N , it contains i and $z_0, \dots, z_{N-1}$. On its top edge, $|1+ae^{\pi z}| \geq 1$, and $|1+z^2| \geq (2N-1)(2N+1)$. On the vertical edges, $|1+z^2| = |z-i||z+i| \geq N^2$, because $\operatorname{Re} z = \pm N$ and both $|z-i|$ and $|z+i|$ are at least N ; the exponential factor is uniformly bounded away from zero on the left and grows exponentially on the right. Thus the integrals over these three edges tend to zero.

The residue theorem consequently gives

$$I(a) = 2\pi i \left[\operatorname{Res}_{z=i} f + \sum_{k=0}^{\infty} \operatorname{Res}_{z=z_k} f \right].$$

Since

$$\operatorname{Res}_{z=i} f = \frac{1}{2i(1-a)}, \quad \operatorname{Res}_{z=z_k} f = -\frac{1}{\pi(1+z_k^2)},$$

we obtain

$$I(a) = \frac{\pi}{1-a} - \sum_{k=0}^{\infty} \frac{2i}{1+(\xi+i(2k+1))^2}.$$

The series telescopes:

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{2i}{1+(\xi+i(2k+1))^2} &= \sum_{k=0}^{\infty} \left(\frac{1}{\xi+2ki} - \frac{1}{\xi+2(k+1)i} \right) \\ &= \frac{1}{\xi} - \lim_{N \rightarrow \infty} \frac{1}{\xi+2(N+1)i} \\ &= -\frac{\pi}{\ln a}. \end{aligned}$$

Therefore

$$I(a) = \frac{\pi}{1-a} + \frac{\pi}{\ln a} = \pi \frac{1-a+\ln a}{(1-a)\ln a}.$$

At $a = 0$ and $a = 1$, the right-hand side is understood by continuous extension, giving the values already computed above.

Also solved by the proposer.

MATHCONTEST SECTION

This section presents interesting and elegant problems selected from university mathematics competitions around the world, particularly those suitable for undergraduate olympiad training. Proposals are always welcome. The sources of the problems will be given when their solutions are published.

*Solutions to the problems in this issue should arrive before
December 26, 2026*

Proposals

125. Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be twice continuously differentiable, with

$$\int_{-1}^1 f(x) dx = 0, \quad f(-1) = f(1) = 1.$$

Prove that

$$\int_{-1}^1 (f''(x))^2 dx \geq 15,$$

and determine all functions for which equality holds.

126. Let $a \geq 2$ be an integer. Prove that there is a positive integer b such that, for every positive integer n , some prime $p = p(a, b, n)$ divides $a^n + b$ but p^2 does not divide $a^n + b$.

127. Call a function $f : [0, \infty) \rightarrow [0, \infty)$ *max-polynomial* if there exist finitely many real polynomials $p_1, \dots, p_m$, each nonnegative on $[0, \infty)$, such that

$$f(x) = \max\{p_1(x), \dots, p_m(x)\} \quad (x \geq 0).$$

Is the function $f(x) = \min\{x, x^2\}$ max-polynomial?

128. Let $A \in M_n(\mathbb{C})$ satisfy

$$A^* A^2 = A A^*,$$

where A^* is the conjugate transpose of A . Prove that $A^2 = A$.

129. Alice and Bob play a game with a string of n digits, each in $\{0, 1, 2\}$. Initially all digits are 0. A move consists of adding or subtracting 1 from a single digit, provided the resulting string has not appeared earlier in the game. The players alternate moves, with Alice playing first; the first player unable to move loses. For each integer $n \geq 1$, determine which player has a winning strategy.

Solutions

120. Let $g : \mathbb{R} \rightarrow \mathbb{R}$ be continuous and such that $g(0) = 0$ and $g(x)g(-x) > 0$ for any $x > 0$. Find all solutions $f : \mathbb{R} \rightarrow \mathbb{R}$ to the functional equation

$$g(f(x+y)) = g(f(x)) + g(f(y)), \quad \text{for all } x, y \in \mathbb{R}.$$

(Imperial Mathematics Competition 2018/19, Round One, Problem 1(i))

Solution by Michel Bataille, Rouen, France. First we examine some consequences of the hypotheses on g . Clearly, we must also have $g(x)g(-x) > 0$ if $x < 0$ and therefore $g(x) = 0$ holds if and only if $x = 0$. Since g is continuous and $g(x) \neq 0$ when $x > 0$, g has a constant sign on $(0, \infty)$ (a consequence of the Intermediate Value Theorem). Similarly, g has a constant sign on $(-\infty, 0)$. Since $g(x)g(-x) > 0$ for $x \neq 0$, either $g(x) > 0$ for all nonzero real x or $g(x) < 0$ for all nonzero real x . We assume that the former holds (otherwise, we replace g by $-g$, which does not change the problem).

Taking $x = y = 0$ gives $g(f(0)) = 0$. Taking $y = -x$ then gives $g(f(x)) + g(f(-x)) = 0$. Both summands are nonnegative, so $g(f(x)) = 0$ for every x . Since g vanishes only at 0, we obtain $f(x) = 0$ for all x .

Conversely, the zero function is obviously a solution.

In conclusion the unique solution to the given equation is the zero function f_0 defined by $f_0(x) = 0$ for all x .

121. Let $A = \{1, 2, \dots, n\}$. Prove that if $n > 11$ then there is a bijective map $f : A \rightarrow A$ with the property that, for every $a \in A$, exactly one of $f(f(f(f(a)))) = a$ and $f(f(f(f(f(a)))))) = a$ holds.

(Imperial Mathematics Competition 2018/19, Round One, Problem 2(ii))

Solution by the editors. Write $n = 4q + r$, where $q \geq 3$ and $0 \leq r \leq 3$. Then

$$n = 4(q - r) + 5r,$$

and $q - r \geq 0$. Partition A into $q - r$ sets of size 4 and r sets of size 5. On each set, define f to cyclically permute its elements. The resulting map is a bijection.

If a belongs to a cycle of length 4, then $f^4(a) = a$ and $f^5(a) = f(a) \neq a$. If a belongs to a cycle of length 5, then $f^5(a) = a$ and $f^4(a) \neq a$. Thus exactly one of the two required equalities holds for every $a \in A$.

Also solved by Stephen C. Locke, Department of Mathematics and Statistics, Florida Atlantic University.

122. A magic square of size n is an $n \times n$ array of real numbers such that all the rows, all the columns and the two main diagonals have the same sum. Determine the dimension, over $\mathbb{R}$, of the vector space of $n \times n$ magic squares.

(Imperial Mathematics Competition 2018/19, Round One, Problem 3)

Solution 1 by Michel Bataille, Rouen, France. The dimension is 1 if $n = 1$ and $n = 2$. If $n \geq 3$, the dimension is $n^2 - 2n$. This is a well-known result and various proofs can be found in the following articles:

J.E. Ward, Vector Spaces of Magic Squares, *Math. Magazine*, Vol. 53, No 2, March 1980, p. 108

N.J. Lord, The dimension of the space of magic squares, *Math. Gazette*, Vol. 66, No 437, Oct. 1982, p. 199

R.B. Coates, The dimension of the space of magic squares, *Math. Gazette*, Vol. 67, No 442, Dec. 1983, p. 303

A.F. Beardon, The dimension of the space of magic squares, *Math. Gazette*, Vol. 87, No 508, March 2003, p. 112

Solution 2 by Henry Ricardo, Westchester Area Math Circle, New York, USA. For $n \geq 3$, this is a well-known problem whose solution is $n^2 - 2n = n(n - 2)$. The proof of this fact as well as other information about magic squares can be found in "Vector Spaces of Magic Squares" by James E. Ward III (*Mathematics Magazine* **53** (1980), 108-111).

Solution 3 by the editors. For $n = 1$ the space has dimension 1. For $n = 2$ the row, column, and diagonal conditions force all four entries to be equal, so the dimension is again 1. Now let $n \geq 3$, and denote by J_n the all-ones matrix. Every magic square has a unique decomposition

$$A = \frac{s}{n}J_n + B,$$

where s is its common line sum and B has every row sum and every column sum equal to 0. The space U of matrices B with zero row and column sums has dimension $(n - 1)^2$: its top-left $(n - 1) \times (n - 1)$ block can be chosen freely, and the remaining entries are then forced. The requirement that the two diagonal sums of B vanish imposes two further independent linear conditions on U . To see their independence, write E_{ij} for the matrix unit and set

$$W_1 = E_{11} + E_{22} - E_{12} - E_{21} \in U.$$

For $n \geq 4$, the main- and secondary-diagonal sums of W_1 are $(2, 0)$, while those of

$$W_2 = E_{1,n-1} + E_{2n} - E_{1n} - E_{2,n-1} \in U$$

are $(0, -2)$. For $n = 3$, take instead $W_2 = E_{12} + E_{23} - E_{13} - E_{22}$; the corresponding pairs of diagonal sums are $(2, 1)$ and $(-1, -2)$, which are linearly independent. Hence the two diagonal conditions have rank 2, and the dimension of the space of magic squares is

$$1 + (n - 1)^2 - 2 = n^2 - 2n \quad (n \geq 3).$$

123. For $u, v \in \mathbb{R}^4$, let $\langle u, v \rangle$ denote the usual dot product. Define a vector field to be a map $\omega : \mathbb{R}^4 \rightarrow \mathbb{R}^4$ such that $\langle \omega(z), z \rangle = 0$ for all $z \in \mathbb{R}^4$. Find a maximal collection of vector fields $\{\omega_1, \dots, \omega_k\}$ such that the map Ω sending z to $\lambda_1\omega_1(z) + \dots + \lambda_k\omega_k(z)$, with $\lambda_1, \dots, \lambda_k \in \mathbb{R}$, is nonzero on $\mathbb{R}^4 \setminus \{0\}$ unless $\lambda_1 = \dots = \lambda_k = 0$.

(Imperial Mathematics Competition 2018/19, Round One, Problem 4)

Solution by the editors. The largest possible collection has $k = 3$. Indeed, fix $z_0 \neq 0$. Each vector $\omega_j(z_0)$ lies in the three-dimensional space $z_0^\perp$. The stated property requires these vectors to be linearly independent: any nontrivial linear dependence would produce a constant linear combination that vanishes at z_0 . Consequently, $k \leq 3$.

To attain this bound, for $z = (a, b, c, d)$ define

$$\omega_1(z) = (-b, a, -d, c),$$

$$\omega_2(z) = (-c, d, a, -b),$$

$$\omega_3(z) = (-d, -c, b, a).$$

Direct calculation gives $\langle \omega_j(z), z \rangle = 0$ for $j = 1, 2, 3$, together with

$$\begin{aligned}\langle \omega_1(z), \omega_2(z) \rangle &= bc + ad - ad - bc = 0, \\ \langle \omega_1(z), \omega_3(z) \rangle &= bd - ac - bd + ac = 0, \\ \langle \omega_2(z), \omega_3(z) \rangle &= cd - cd + ab - ab = 0.\end{aligned}$$

Moreover, $\|\omega_j(z)\|^2 = \|z\|^2$ for each j . Therefore, for all real constants $\lambda_1, \lambda_2, \lambda_3$,

$$\|\lambda_1\omega_1(z) + \lambda_2\omega_2(z) + \lambda_3\omega_3(z)\|^2 = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\|z\|^2.$$

This expression is positive whenever $z \neq 0$ and the coefficients are not all zero, proving that $k = 3$ is attained.

124. For a continuously differentiable function $f : [0, 1] \rightarrow \mathbb{R}$ with $f(1/2) = 0$, show that

$$\left(\int_0^1 f(x) dx \right)^2 \leq \frac{1}{4} \int_0^1 (f'(x))^2 dx.$$

(Imperial Mathematics Competition 2018/19, Round One, Problem 5)

Solution 1 by Michel Bataille, Rouen, France. Let $I_0 = \int_0^{1/2} (f'(x))^2 dx$ and $I_1 = \int_{1/2}^1 (f'(x))^2 dx$.

For $0 \leq t \leq \frac{1}{2}$, we have

$$(f(t))^2 = (f(t) - f(1/2))^2 = \left(\int_t^{1/2} f'(u) du \right)^2 \leq \left(\int_t^{1/2} 1^2 du \right) \left(\int_t^{1/2} (f'(u))^2 du \right)$$

(by the Cauchy-Schwarz inequality) and therefore

$$(f(t))^2 \leq \left(\frac{1}{2} - t \right) I_0.$$

It follows that

$$\int_0^{1/2} (f(t))^2 dt \leq I_0 \int_0^{1/2} \left(\frac{1}{2} - t \right) dt = \frac{I_0}{8}. \quad (1)$$

Similarly, for $\frac{1}{2} \leq t \leq 1$, we have

$$(f(t))^2 \leq \left(\int_{1/2}^t f'(u) du \right)^2 \leq \left(t - \frac{1}{2} \right) I_1,$$

hence

$$\int_{1/2}^1 (f(t))^2 dt \leq I_1 \int_{1/2}^1 \left(t - \frac{1}{2} \right) dt = \frac{I_1}{8}. \quad (2)$$

From (1) and (2) we now deduce that

$$\int_0^1 (f(t))^2 dt \leq \frac{I_0 + I_1}{8} = \frac{1}{8} \int_0^1 (f'(x))^2 dx.$$

Using the Cauchy-Schwarz inequality again, we obtain

$$\left(\int_0^1 f(x) dx \right)^2 \leq \left(\int_0^1 1^2 dt \right) \left(\int_0^1 (f(t))^2 dt \right) = \int_0^1 (f(t))^2 dt$$

and consequently

$$\left(\int_0^1 f(x) dx \right)^2 \leq \frac{1}{8} \int_0^1 (f'(x))^2 dx,$$

which is better than the required inequality.

Solution 2 by Paolo Perfetti, Department of Mathematics, University of Rome Tor Vergata, Rome, Italy. We prove the stronger inequality with the sharp constant $1/12$. Integration by parts, using $f(1/2) = 0$, gives

$$\begin{aligned} I &:= \int_0^1 f(x) dx = \int_0^{1/2} f(x) dx + \int_{1/2}^1 f(x) dx \\ &= [xf(x)]_0^{1/2} - \int_0^{1/2} xf'(x) dx \\ &\quad + [(x-1)f(x)]_{1/2}^1 - \int_{1/2}^1 (x-1)f'(x) dx \\ &= - \int_0^{1/2} xf'(x) dx - \int_{1/2}^1 (x-1)f'(x) dx. \end{aligned}$$

Put

$$A = \int_0^{1/2} (f'(x))^2 dx, \quad B = \int_{1/2}^1 (f'(x))^2 dx.$$

Since

$$\int_0^{1/2} x^2 dx = \int_{1/2}^1 (x-1)^2 dx = \frac{1}{24},$$

the Cauchy–Schwarz inequality yields

$$\begin{aligned} |I| &\leq \left(\int_0^{1/2} x^2 dx \right)^{1/2} \sqrt{A} + \left(\int_{1/2}^1 (x-1)^2 dx \right)^{1/2} \sqrt{B} \\ &\leq \frac{1}{\sqrt{24}} (\sqrt{A} + \sqrt{B}). \end{aligned}$$

Using $(\sqrt{A} + \sqrt{B})^2 \leq 2(A + B)$, we obtain

$$\begin{aligned} \left(\int_0^1 f(x) dx \right)^2 &\leq \frac{2(A + B)}{24} \\ &= \frac{1}{12} \int_0^1 (f'(x))^2 dx \\ &\leq \frac{1}{4} \int_0^1 (f'(x))^2 dx. \end{aligned}$$

Solution 3 by Henry Ricardo, Westchester Area Math Circle, New York, USA. We improve the upper bound by reducing the constant to $1/12$.

For $x \in [0, 1]$, we can write

$$f(x) = \int_0^1 f(t) dt + \int_0^x tf'(t) dt + \int_x^1 (t-1)f'(t) dt.$$

Setting $x = 1/2$ gives us

$$- \int_0^1 f(t) dt = \int_0^{1/2} tf'(t) dt + \int_{1/2}^1 (t-1)f'(t) dt.$$

Squaring and then applying the Quadratic Mean (QM) and C-B-S inequalities successively, we see that

$$\begin{aligned}
 \left(\int_0^1 f(x) dx \right)^2 &= \left(\int_0^{1/2} t f'(t) dt + \int_{1/2}^1 (t-1) f'(t) dt \right)^2 \\
 &\stackrel{QM}{\leq} 2 \left[\left(\int_0^{1/2} t f'(t) dt \right)^2 + \left(\int_{1/2}^1 (t-1) f'(t) dt \right)^2 \right] \\
 &\stackrel{C-B-S}{\leq} 2 \left[\int_0^{1/2} t^2 dt \cdot \int_0^{1/2} (f'(t))^2 dt + \int_{1/2}^1 (1-t)^2 dt \cdot \int_{1/2}^1 (f'(t))^2 dt \right] \\
 &= 2 \left[\frac{1}{24} \int_0^1 (f'(t))^2 dt \right] = \frac{1}{12} \int_0^1 (f'(t))^2 dt.
 \end{aligned}$$

Editorial remark on sharpness. The constant $1/12$ in the preceding solution is optimal even in the class of continuously differentiable functions, although a nonzero extremizer in that class does not exist. Indeed, the condition $f(1/2) = 0$ gives

$$\int_0^1 f(x) dx = \int_0^1 k(x) f'(x) dx, \quad k(x) = \begin{cases} -x, & 0 \leq x < 1/2, \\ 1-x, & 1/2 < x \leq 1. \end{cases}$$

Since $\int_0^1 k(x)^2 dx = 1/12$, Cauchy–Schwarz proves the estimate with that constant. For optimality, choose continuous functions k_ε converging to k in $L^2(0, 1)$, and let f_ε be a continuously differentiable antiderivative of k_ε normalized by $f_\varepsilon(1/2) = 0$. Then

$$\lim_{\varepsilon \downarrow 0} \frac{\left(\int_0^1 f_\varepsilon(x) dx \right)^2}{\int_0^1 (f'_\varepsilon(x))^2 dx} = \frac{1}{12}.$$

MATHNOTES SECTION

Functional extensions of a cyclic radical inequality

VALMIR KRASNIQI

Abstract. We give a complete solution to Junior Problem 80, proposed by Valmir Krasniqi and recorded as open in Volume 7, Issue 1 of *MathProblems*. We then study sums of positive ratios under a multiplicative constraint. A classical Hölder allocation inequality, with an exact minimum, reduces these inequalities to a comparison of scalar means. We use this reduction to construct a family associated with any positive function that is convex in the logarithmic variable, and determine the exact admissible range of its denominator parameter. An explicit polynomial certificate then establishes a sharp cubic radical inequality for arbitrary positive denominator allocations with a prescribed total budget. This inequality is extended through convex numerator transformations, concave denominator transformations, and balanced matrix redistributions. Power and nonpower examples are given, together with sharp constants and equality conditions. A local expansion and an explicit rational allocation show failure throughout $0 < \theta \leq 2/9$ for the unrestricted cubic exponent problem.

2020 Mathematics Subject Classification. Primary 26D15; Secondary 26D07, 26A51.

Keywords. Hölder inequality, Jensen inequality, product constraint, positive functions, sharp constants, cyclic inequalities.

The starting point is Junior Problem 80, proposed by Valmir Krasniqi and recorded in *MathProblems*, Volume 7, Issue 1, p. 698 [1]. The editorial comment in that issue listed the problem as open. It asks us to prove that, for $x, y, z > 0$ with $xyz = 1$,

$$\sqrt{\frac{x^3 + 3}{x^2 + y + 2z}} + \sqrt{\frac{y^3 + 3}{y^2 + z + 2x}} + \sqrt{\frac{z^3 + 3}{z^2 + x + 2y}} \geq 3. \quad (1)$$

Corollary 12 below gives a complete proof of this problem and shows that equality holds precisely at $x = y = z = 1$. The constant 3 is therefore best possible.

Many cyclic inequalities contain denominators whose individual terms are difficult to compare, although their sum has a simple symmetric form. Here the sum of the denominators in (1) is $x^2 + y^2 + z^2 + 3(x + y + z)$. We extend the published problem to the following cyclic family. If $x, y, z > 0$, $xyz = 1$, and $0 \leq \alpha \leq 3$, then

$$\sqrt{\frac{x^3 + 3}{x^2 + \alpha y + (3 - \alpha)z}} + \sqrt{\frac{y^3 + 3}{y^2 + \alpha z + (3 - \alpha)x}} + \sqrt{\frac{z^3 + 3}{z^2 + \alpha x + (3 - \alpha)y}} \geq 3. \quad (2)$$

The constant is attained precisely at $x = y = z = 1$. The sum of the denominators in (2) is independent of α . This observation suggests treating the total denominator budget before addressing its distribution among the three terms.

The allocation inequality used below is a classical consequence of Hölder's inequality. Hölder, Jensen, AM–GM, and Schur are established tools in the theory of inequalities; see [2, 3]. Supporting lines and convexity arguments are treated systematically in [4, Chapter 3]. We include complete proofs of the reduction and its consequences to make the equality conditions and the roles of the hypotheses explicit.

The paper develops two routes to functional extensions. The first applies to an arbitrary number of variables and uses a positive function that is convex in the logarithmic variable. A logarithmic

correction in the denominator disappears after summation under the product constraint, leading to an exactly determined parameter interval. The second starts from a specific ternary cubic estimate, proved by an explicit rational minorant and a nonnegative polynomial decomposition. It yields the original cyclic inequality, unrestricted denominator allocations, and nonlinear changes of both numerator and denominator.

Throughout, positive functions take finite values in $(0, \infty)$. In the weighted statements, $w_i > 0$ and $\sum_{i=1}^n w_i = 1$. The weighted product condition means

$$\prod_{i=1}^n x_i^{w_i} = 1, \quad \text{equivalently} \quad \sum_{i=1}^n w_i \ln x_i = 0. \quad (3)$$

Lemma 1. *Let $\theta > 0$, $q = \theta/(1 + \theta)$, and $A_i, d_i > 0$. Put $B = \sum_i w_i d_i$. Then*

$$\sum_{i=1}^n w_i \left(\frac{A_i}{d_i} \right)^\theta \geq \frac{(\sum_{i=1}^n w_i A_i^q)^{1+\theta}}{B^\theta}. \quad (4)$$

For fixed A_i and $B > 0$, the right-hand side is the exact minimum over positive d_i with weighted sum B . The unique minimizing allocation is

$$d_i = \frac{B A_i^q}{\sum_j w_j A_j^q}. \quad (5)$$

Proof. Set $R_i = (A_i/d_i)^\theta$. Hölder's inequality with conjugate exponents $1 + \theta$ and $(1 + \theta)/\theta$ gives

$$\sum_i w_i A_i^q = \sum_i w_i R_i^{1/(1+\theta)} d_i^{\theta/(1+\theta)} \leq \left(\sum_i w_i R_i \right)^{1/(1+\theta)} B^{\theta/(1+\theta)}.$$

Raising both sides to $1 + \theta$ proves (4). Because every weight and every term is positive, equality in Hölder holds exactly when $R_i = c d_i$ for one constant $c > 0$. Equivalently, d_i is proportional to A_i^q . The budget condition determines the proportionality constant uniquely and gives (5). Substitution verifies that this allocation attains the bound. For equal weights, the same inequality has the particularly useful unweighted form

$$\sum_{i=1}^n \left(\frac{A_i}{d_i} \right)^\theta \geq \frac{(\sum_{i=1}^n A_i^q)^{1+\theta}}{(\sum_{i=1}^n d_i)^\theta}. \quad (6)$$

□

Proposition 2. *Let $f, g : (0, \infty) \rightarrow (0, \infty)$ and fix a positive tuple $(x_1, \dots, x_n)$. The inequality*

$$\sum_i w_i \left(\frac{f(x_i)}{d_i} \right)^\theta \geq 1$$

holds for every positive allocation satisfying $\sum_i w_i d_i \leq \sum_i w_i g(x_i)$ if and only if

$$\left(\sum_i w_i f(x_i)^q \right)^{1/q} \geq \sum_i w_i g(x_i). \quad (7)$$

Proof. The minimum in Lemma 1 decreases as the permitted budget increases. Its smallest value under the stated constraint is therefore attained at $B = \sum_i w_i g(x_i)$. That value is at least 1 precisely when (7) holds. □

Proposition 2 reduces the allocation problem to a scalar comparison. The next section supplies a checkable class of functions for which that comparison follows from a supporting line.

Remark 3. Suppose a matrix $M = (M_{ij})$ has nonnegative entries, each row sums to 1, and

$$\sum_i w_i M_{ij} = w_j \quad (1 \leq j \leq n).$$

Then $d_i = \sum_j M_{ij} g(x_j)$ is positive and preserves the weighted denominator budget:

$$\sum_i w_i d_i = \sum_j \left(\sum_i w_i M_{ij} \right) g(x_j) = \sum_j w_j g(x_j).$$

For equal weights, these conditions say that M is doubly stochastic.

Consequently, an allocation inequality valid for every positive allocation under this budget also holds for these matrix-mixed denominators. If, in addition, $f(1) = g(1) = 1$ and the allocation inequality has lower bound 1, then the resulting matrix-mixed inequality is sharp. Indeed, at $x_1 = \cdots = x_n = 1$, the row-sum condition gives $d_1 = \cdots = d_n = 1$, and hence

$$\sum_i w_i \left(\frac{f(x_i)}{d_i} \right)^\theta = \sum_i w_i = 1.$$

Theorem 4. Let $n \geq 2$, $\theta > 0$, and

$$r = \frac{1 + \theta}{\theta} > 1.$$

Let $h : (0, \infty) \rightarrow (0, \infty)$ satisfy $h(1) = 1$, and suppose

$$J(s) = h(e^s)$$

is convex on $\mathbb{R}$. Choose a supporting slope m at 0, so that

$$h(t) \geq 1 + m \ln t \quad (t > 0). \quad (8)$$

For $a \geq 0$, define

$$f(t) = h(t)^r, \quad g_a(t) = 1 + a(h(t) - 1 - m \ln t). \quad (9)$$

If $0 \leq a \leq r$, every tuple satisfying (3) and every positive allocation with

$$\sum_i w_i d_i \leq \sum_i w_i g_a(x_i)$$

satisfy

$$\sum_i w_i \left(\frac{h(x_i)^r}{d_i} \right)^\theta \geq 1. \quad (10)$$

Equality holds if and only if $\sum_i w_i h(x_i) = 1$ and $d_i = h(x_i)$ for every i . If J is strictly convex, these conditions are equivalent to all $x_i = 1$ and all $d_i = 1$. Under this strict convexity hypothesis, the interval $0 \leq a \leq r$ is the exact admissible interval among $a \geq 0$ for the unrestricted allocation class.

Proof. A finite convex function on $\mathbb{R}$ has a supporting line at every point, so the choice of m is possible. Equation (8) also shows that $g_a(t) \geq 1$, hence all the functions in (9) are positive.

Put $T = \sum_i w_i h(x_i)$. Jensen's inequality and (3) give

$$T = \sum_i w_i J(\ln x_i) \geq J\left(\sum_i w_i \ln x_i\right) = 1.$$

The logarithmic term in (9) disappears after weighted summation, giving

$$\sum_i w_i g_a(x_i) = 1 + a(T - 1).$$

For $T \geq 1$ and $0 \leq a \leq r$, the supporting line to $T \mapsto T^r$ at 1 yields

$$1 + a(T - 1) \leq 1 + r(T - 1) \leq T^r. \quad (11)$$

Since $rq = 1$, Lemma 1 now gives

$$\sum_i w_i \left(\frac{h(x_i)^r}{d_i} \right)^\theta \geq \frac{T^{1+\theta}}{[1 + a(T - 1)]^\theta} \geq \frac{T^{1+\theta}}{T^{r\theta}} = 1.$$

Because $r > 1$, the second inequality in (11) is strict whenever $T > 1$. Equality therefore requires $T = 1$. The permitted budget is then at most 1, and equality in Lemma 1 requires budget 1 and $d_i = h(x_i)$ for every i . Conversely, these conditions give equality by direct substitution. If J is strictly convex, strict Jensen shows that $T = 1$ forces all $\ln x_i$ to coincide. The product constraint then gives $x_i = 1$ and consequently $d_i = 1$ for every i .

Finally, suppose $a > r$. Strict convexity and continuity provide nonunit product-constrained tuples with $T > 1$ arbitrarily close to 1. For example, use $\ln x_1 = s$, $\ln x_2 = -w_1 s / w_2$, and all other logarithms zero, then let $s \rightarrow 0$. For such a tuple choose

$$d_i = \frac{[1 + a(T - 1)]h(x_i)}{T}.$$

This is the optimizing allocation and has the required total budget. Moreover,

$$\lim_{T \downarrow 1} \frac{T^r - 1}{T - 1} = r < a.$$

Hence $T^r < 1 + a(T - 1)$ for all sufficiently small positive $T - 1$, and the corresponding value in (10) is less than 1. Thus every $a > r$ fails for some admissible tuple and allocation. $\square$

Corollary 5. *Under the hypotheses of Theorem 4, let positive functions $\tilde{f}, \tilde{g}$ satisfy*

$$\tilde{f}(t) \geq h(t)^r, \quad \tilde{g}(t) \leq g_a(t), \quad \tilde{f}(1) = \tilde{g}(1) = 1.$$

For $0 \leq a \leq r$ and positive allocations with $\sum_i w_i d_i \leq \sum_i w_i \tilde{g}(x_i)$, one has

$$\sum_i w_i \left(\frac{\tilde{f}(x_i)}{d_i} \right)^\theta \geq 1.$$

If J is strictly convex, equality again holds exactly at $x_i = d_i = 1$ for all i .

Proof. The permitted budget is no larger than that in Theorem 4, and each numerator is no smaller. Strictness off the unit tuple follows from that theorem; at the unit tuple the normalization gives the stated equality case. $\square$

Remark 6. Convexity of $s \mapsto h(e^s)$ means convexity in the logarithmic variable. It is different from the conventional condition that $\ln h$ is convex. The sharpness of $a \leq r$ concerns all positive allocations under the budget constraint; it does not assert necessity for every fixed matrix in Remark 3.

Example 7. Each of the following choices satisfies the strict convexity hypotheses:

$$\begin{aligned} h(t) &= t^\beta, & m &= \beta, & g_a(t) &= 1 + a(t^\beta - 1 - \beta \ln t), & \beta \neq 0; \\ h(t) &= \cosh(\beta \ln t), & m &= 0, & g_a(t) &= 1 + a(\cosh(\beta \ln t) - 1), & \beta > 0; \\ h(t) &= 1 + (\ln t)^2, & m &= 0, & g_a(t) &= 1 + a(\ln t)^2. \end{aligned}$$

In every case, use $f = h^r$ and $0 \leq a \leq r$. The first family follows from strict convexity of the exponential, the second from strict convexity of the hyperbolic cosine, and the third from strict convexity of a quadratic. Their displayed denominator functions are positive by their respective supporting-line inequalities. For equal weights and $\theta = 1/2$, the resulting sharp inequality is

$$\sum_{i=1}^n \sqrt{\frac{h(x_i)^3}{d_i}} \geq n, \quad \prod_{i=1}^n x_i = 1, \quad \sum_i d_i \leq \sum_i g_a(x_i), \quad 0 \leq a \leq 3.$$

For $h(t) = t^\beta$ with $\beta > 0$, the numerator tends to zero as $t \downarrow 0$, while $g_a(t) \geq 1$. Thus this family is not obtained from a pointwise comparison of numerator and denominator.

For the remainder of the paper, $x, y, z > 0$ and $xyz = 1$. Define

$$F = \sum_{\text{cyc}} \sqrt[3]{x^3 + 3}, \quad G = x^2 + y^2 + z^2 + 3(x + y + z). \quad (12)$$

AM–GM gives $G \geq 12$. We prove the symmetric estimate that will supply the scalar condition in Proposition 2.

Lemma 8. For $t > 0$, let

$$h_0(t) = \frac{19t^3 + 3t^2 + 17t + 25}{32(t^2 + 1)}.$$

Then

$$\sqrt[3]{t^3 + 3} \geq \sqrt[3]{4} h_0(t), \quad (13)$$

with equality if and only if $t = 1$.

Proof. Put $m_0(t) = 19t^3 + 3t^2 + 17t + 25$. Direct expansion gives the exact identity

$$\begin{aligned} & 32^3(t^2 + 1)^3(t^3 + 3) - 4m_0(t)^3 \\ &= 4(t - 1)^4(1333t^5 + 2083t^4 + 5986t^3 + 8438t^2 + 3929t + 8951). \end{aligned}$$

The polynomial in parentheses is positive for $t > 0$. Division by the positive denominator and taking cube roots prove (13) and its equality condition. $\square$

The minorant has a simple local motivation. With $s = t - 1$, both $h_0(t)$ and $((t^3 + 3)/4)^{1/3}$ have the expansion

$$1 + \frac{s}{4} + \frac{3s^2}{16} - \frac{s^3}{64} + O(s^4).$$

Thus the rational function matches the value and the first three derivatives of the normalized cubic root at the equality point. This local matching motivates the choice, while the displayed factorization proves the required global direction.

Lemma 9. *Let $H = h_0(x) + h_0(y) + h_0(z)$. Then*

$$4H^3 \geq 9G, \quad (14)$$

with equality if and only if $x = y = z = 1$.

Proof. Set $p = x + y + z$, $\rho = xy + yz + zx$, and

$$D = (p-1)^2 + (\rho-1)^2 = (1+x^2)(1+y^2)(1+z^2) > 0.$$

The identity follows by expanding the product and using $xyz = 1$. In these variables, $G = p^2 - 2\rho + 3p$. Putting the three rational terms over a common denominator gives

$$H = \frac{N}{32D}, \quad N = 19p^3 + 19p\rho^2 + 15p^2 - 40p\rho + 31\rho^2 - 26p - 108\rho + 90.$$

All values of h_0 are positive, so $H > 0$.

For $b = (2p+3)/3 > 0$, the identity

$$H^3 - b^3 - 3b^2(H-b) = (H-b)^2(H+2b) \geq 0$$

gives

$$4H^3 - 9G \geq \frac{K}{8D}, \quad K = 3b^2N - (64b^3 + 72G)D. \quad (15)$$

Put $u = p-3$, $v = \rho-3$, and $S = p^3 + 9 - 4p\rho$. AM-GM gives $u, v \geq 0$, and Schur's inequality gives $S \geq 0$. An exact expansion is

$$\begin{aligned} K = & 93S + 9u + 327(u-v)^2 + 2u^2 + 33v^2 \\ & + 120v(u-v)^2 + 9uv^2 + 24v^3 + \mathcal{R}(u, v), \end{aligned} \quad (16)$$

where

$$\mathcal{R}(u, v) = \frac{u^3}{27}(172u^2 + 1948u + 172v^2 + 616v + 6092) + \frac{52}{3}u^2v^2. \quad (17)$$

Every term on the right of (16) is nonnegative. Therefore $K \geq 9u \geq 0$, and (15) proves (14). If $p > 3$, then $u > 0$ and the inequality is strict. If $p = 3$, equality in AM-GM forces $x = y = z = 1$. At this triple, $H = 3$ and $G = 12$, which verifies equality. $\square$

Theorem 10. *For the quantities in (12),*

$$F^3 \geq 9G. \quad (18)$$

Equality holds precisely at $x = y = z = 1$. More quantitatively, with D, K as in Lemma 9,

$$F^3 \geq 9G + \frac{K}{8D}, \quad K \geq 9(x+y+z-3). \quad (19)$$

Proof. Lemma 8 gives $F \geq \sqrt[3]{4H}$, hence $F^3 \geq 4H^3$. Apply (15) and (16). The equality condition follows from Lemma 9, and direct substitution verifies the converse. $\square$

Theorem 11. If $d_x, d_y, d_z > 0$ satisfy $d_x + d_y + d_z \leq G$, then

$$\sqrt{\frac{x^3 + 3}{d_x}} + \sqrt{\frac{y^3 + 3}{d_y}} + \sqrt{\frac{z^3 + 3}{d_z}} \geq 3. \quad (20)$$

Equality holds if and only if $x = y = z = 1$ and $d_x = d_y = d_z = 4$. In particular, the lower bound 3 is best possible for this allocation class.

Proof. Use Lemma 1 with equal weights and $\theta = 1/2$, or its unweighted version, to obtain

$$\left(\sum_{\text{cyc}} \sqrt{\frac{x^3 + 3}{d_x}} \right)^2 \geq \frac{F^3}{d_x + d_y + d_z} \geq \frac{F^3}{G} \geq 9.$$

Equality requires equality in Theorem 10, hence the unit tuple. There the numerator values are all 4. The minimizing allocation in Lemma 1, together with the full budget 12, gives $d_x = d_y = d_z = 4$. These values do attain equality. $\square$

Corollary 12. Let $x, y, z > 0$ and $xyz = 1$. Then the inequality stated in Junior Problem 80 of MathProblems, Volume 7, Issue 1 (2024), p. 698 [1], holds:

$$\sqrt{\frac{x^3 + 3}{x^2 + y + 2z}} + \sqrt{\frac{y^3 + 3}{y^2 + z + 2x}} + \sqrt{\frac{z^3 + 3}{z^2 + x + 2y}} \geq 3.$$

The constant 3 is best possible, with equality if and only if $x = y = z = 1$.

Proof. Set

$$d_x = x^2 + y + 2z, \quad d_y = y^2 + z + 2x, \quad d_z = z^2 + x + 2y.$$

These denominators are positive and satisfy

$$d_x + d_y + d_z = x^2 + y^2 + z^2 + 3(x + y + z) = G.$$

Let L denote the sum of the three radicals in the statement. By the Hölder allocation inequality of Lemma 1,

$$L^2 \geq \frac{F^3}{d_x + d_y + d_z} = \frac{F^3}{G} \geq 9,$$

where the last inequality is Theorem 10. Thus $L \geq 3$. Equality in the last inequality forces $x = y = z = 1$. At this tuple, every denominator is 4 and every radical equals 1, so equality does hold. This proves the published problem and its exact equality case. $\square$

The following cyclic family extends Problem 80, which is its case $\alpha = 1$.

Corollary 13. Inequality (2) holds for every $\alpha \in [0, 3]$, with equality precisely at the unit tuple.

Proof. Take $d_x = x^2 + \alpha y + (3 - \alpha)z$ and cyclically define d_y, d_z . They are positive, including at both endpoints of the parameter interval, and their sum equals G . Theorem 11 applies. At the unit tuple every denominator is 4, so its equality condition gives the assertion. $\square$

Corollary 14. Let P, Q be nonnegative 3×3 matrices whose columns each sum to 1. Writing $(x_1, x_2, x_3) = (x, y, z)$, define

$$d_i = \sum_{j=1}^3 P_{ij}x_j^2 + 3 \sum_{j=1}^3 Q_{ij}x_j.$$

Assume that no row of the block matrix $(P \mid Q)$ is identically zero (equivalently, every d_i is positive). Then (20) holds with these denominators. For fixed P, Q , equality holds if and only if the tuple is the unit tuple and

$$\sum_j P_{ij} + 3 \sum_j Q_{ij} = 4 \quad (i = 1, 2, 3).$$

In particular, every pair of doubly stochastic matrices gives a sharp inequality with equality at the unit tuple.

Proof. The column sums give $\sum_i d_i = G$. The conclusion and its equality condition follow from Theorem 11. $\square$

Corollary 15. Every allocation in Theorem 11 satisfies

$$\sum_{\text{cyc}} \sqrt{\frac{x^3 + 3}{d_x}} \geq 3 \sqrt{1 + \frac{K}{72DG}} \geq 3 \sqrt{1 + \frac{x + y + z - 3}{8DG}}.$$

Proof. Combine the first inequality in the proof of Theorem 11 with (19), and then use $K \geq 9(x + y + z - 3)$. $\square$

We next replace the cubic numerator and the separate denominator components by general functions. The hypotheses below are aggregate conditions obtained from convexity and concavity; pointwise comparison with the original terms is not required.

Theorem 16. Define

$$\eta(t) = \left(\frac{t^3 + 3}{4} \right)^{1/3}, \quad I = ((3/4)^{1/3}, \infty).$$

Let $\Phi : I \rightarrow (0, \infty)$ be convex, with $\Phi(1) = 1$ and $\Phi(s) \geq s$ for $s \geq 1$. Let $U, V : (0, \infty) \rightarrow (0, \infty)$ be concave, with

$$U(1) = V(1) = 1, \quad U(s) \leq s, \quad V(s) \leq s \quad (s \geq 1).$$

For $0 \leq \alpha \leq 3$, put

$$\delta_x = U(x^2) + \alpha V(y) + (3 - \alpha)V(z)$$

and define δ_y, δ_z cyclically. For every $\theta \geq 1/2$,

$$\sum_{\text{cyc}} \left(\frac{4\Phi(\eta(x))^3}{\delta_x} \right)^\theta \geq 3. \quad (21)$$

Equality holds if and only if $x = y = z = 1$, so the constant 3 is best possible.

Proof. By Theorem 10 and $G \geq 12$, the mean $A = \frac{1}{3} \sum_{\text{cyc}} \eta(x)$ is at least 1. Convexity gives

$$\sum_{\text{cyc}} \Phi(\eta(x)) \geq 3\Phi(A) \geq 3A = \sum_{\text{cyc}} \eta(x).$$

AM–GM gives $\frac{1}{3} \sum x^2 \geq 1$ and $\frac{1}{3} \sum x \geq 1$. Concavity and the hypotheses above 1 therefore imply

$$\sum_{\text{cyc}} U(x^2) \leq 3U\left(\frac{x^2 + y^2 + z^2}{3}\right) \leq x^2 + y^2 + z^2,$$

and, similarly, $\sum_{\text{cyc}} V(x) \leq x + y + z$. Consequently,

$$\delta_x + \delta_y + \delta_z \leq G.$$

Set $R_x = \sqrt{4\Phi(\eta(x))^3/\delta_x}$, with cyclic definitions. Lemma 1 yields

$$(R_x + R_y + R_z)^2 \geq \frac{4(\sum_{\text{cyc}} \Phi(\eta(x)))^3}{\delta_x + \delta_y + \delta_z} \geq \frac{F^3}{G} \geq 9.$$

For $s = 2\theta \geq 1$, convexity of $t \mapsto t^s$ gives

$$\sum_{\text{cyc}} R_x^s \geq 3 \left(\frac{R_x + R_y + R_z}{3} \right)^s \geq 3.$$

Off the unit tuple, the strict cubic scalar estimate makes the lower bound strict. At the unit tuple, normalization gives $\delta_x = \delta_y = \delta_z = 4$ and all ratios equal to 1. $\square$

Example 17. Choose $\Phi(s) = s^p$, $U(s) = s^u$, and $V(s) = s^v$, where

$$p \geq 1, \quad 0 \leq u, v \leq 1.$$

These functions meet every hypothesis of Theorem 16. Thus, for $\theta \geq 1/2$,

$$\sum_{\text{cyc}} \left(\frac{4((x^3 + 3)/4)^p}{x^{2u} + \alpha y^v + (3 - \alpha)z^v} \right)^\theta \geq 3. \quad (22)$$

The original inequality is the case $p = u = v = 1$ and $\theta = 1/2$. The displayed parameter ranges are sufficient; no optimality of all these ranges is asserted. For $p > 1$ and $t < 1$, the new numerator $4((t^3 + 3)/4)^p$ is smaller than $t^3 + 3$. For $0 < u < 1$ and $0 < s < 1$, one has $s^u > s$. Thus the result allows changes in directions that are not favorable term by term.

Example 18. For $c \geq 1$ and $b > 0$, take

$$\Phi(s) = \exp(c(s - 1)), \quad U(s) = V(s) = \frac{\ln(1 + bs)}{\ln(1 + b)}.$$

The first function is positive and convex, and $\exp(c(s - 1)) \geq 1 + c(s - 1) \geq s$ for $s \geq 1$. The other functions are positive and concave on $(0, \infty)$, and are normalized at 1. Their derivative there is

$$U'(1) = \frac{b}{(1 + b)\ln(1 + b)} < 1,$$

because $\ln(1 + b) > b/(1 + b)$. Their supporting tangent at 1 therefore gives $U(s) = V(s) \leq s$ for $s \geq 1$. Substitution in (21) supplies a nonpower functional inequality, with equality precisely at the unit tuple.

The exponent restriction in Theorem 16 is sufficient. The following local calculation and explicit allocation concern the different problem in which the original cubic numerator is retained and every positive allocation with total budget G is permitted.

Proposition 19. For each $0 < \theta \leq 2/9$, there exist $x, y, z > 0$ with $xyz = 1$ and positive d_x, d_y, d_z with sum G such that

$$\sum_{\text{cyc}} \left(\frac{x^3 + 3}{d_x} \right)^\theta < 3.$$

For every $\theta \geq 1/2$, the reverse inequality with lower bound 3 holds for all allocations with sum at most G .

Proof. The last assertion follows from Theorem 11 and the power mean inequality. For the obstruction, put $q = \theta/(1 + \theta)$ and use the product-constrained path

$$x = e^{2s}, \quad y = z = e^{-s}.$$

Let $A(s) = (x^3 + 3)^q + (y^3 + 3)^q + (z^3 + 3)^q$. By Lemma 1, the exact unweighted minimum over allocations with sum $G(s)$ is

$$L_\theta(s) = \frac{A(s)^{1+\theta}}{G(s)^\theta},$$

attained at $d_i = G(s)(x_i^3 + 3)^q/A(s)$. Taylor expansion at $s = 0$ gives

$$\begin{aligned} \frac{A(s)}{3 \cdot 4^q} &= 1 + \frac{9q(q+3)}{16}s^2 + \frac{9q(q^2+9q+6)}{64}s^3 + O(s^4), \\ \frac{G(s)}{12} &= 1 + \frac{7}{4}s^2 + \frac{11}{12}s^3 + O(s^4). \end{aligned}$$

Taking logarithms and substituting $q = \theta/(1 + \theta)$ yields

$$\ln \frac{L_\theta(s)}{3} = \frac{\theta(8\theta - 1)}{16(1 + \theta)}s^2 + \frac{\theta(256\theta^2 + 215\theta - 14)}{192(1 + \theta)^2}s^3 + O(s^4). \quad (23)$$

For $0 < \theta < 1/8$, the quadratic coefficient is negative, giving $L_\theta(s) < 3$ for sufficiently small nonzero s . At $\theta = 1/8$, the quadratic coefficient vanishes and the cubic coefficient is $5/576$. Taking sufficiently small negative s again gives $L_\theta(s) < 3$. The optimizing denominators are positive, so these are admissible counterexamples.

We can extend the failure interval using a fully explicit allocation. Take

$$x = y = \frac{1}{3}, \quad z = 9, \quad d_x = d_y = 24, \quad d_z = \frac{560}{9}.$$

These values satisfy $xyz = 1$ and

$$d_x + d_y + d_z = \frac{992}{9} = x^2 + y^2 + z^2 + 3(x + y + z) = G.$$

For $\theta_0 = 2/9$, the sum on the left of the proposed inequality is

$$2 \left(\frac{41}{324} \right)^{2/9} + \left(\frac{1647}{140} \right)^{2/9}.$$

By raising the following rational inequalities to the ninth power, one verifies

$$\left(\frac{41}{324} \right)^{2/9} < \frac{79}{125}, \quad \left(\frac{1647}{140} \right)^{2/9} < \frac{173}{100}.$$

Therefore the sum is smaller than

$$2 \cdot \frac{79}{125} + \frac{173}{100} = \frac{1497}{500} < 3.$$

Finally, if $0 < \theta < \theta_0$, put $r = \theta/\theta_0 \in (0, 1)$ and $u_i = ((x_i^3 + 3)/d_i)^{\theta_0}$ for this same allocation. Concavity of $t \mapsto t^r$ gives

$$\sum_{i=1}^3 u_i^r \leq 3^{1-r} \left(\sum_{i=1}^3 u_i \right)^r < 3.$$

Thus the explicit counterexample works for every $0 < \theta \leq 2/9$. $\square$

Remark 20. The allocations in Proposition 19 are unrestricted positive allocations with total budget G ; they are not required to have the structured cyclic form in (2). They therefore do not establish failure for that cyclic family. The exact exponent threshold for the unrestricted cubic allocation problem is not determined here. Numerical observations communicated by Mohammed Aassila suggest a possible transition near $\theta_* \approx 0.227397$, with a candidate contact configuration $(x, y, z) \approx (0.3611, 0.3611, 7.6681)$. This is numerical evidence only, not a proof of a global threshold. In particular, it does not change the sufficient range $\theta \geq 1/2$ for the general functional statements. Sharpness of a lower-bound constant, such as 3, must be distinguished from optimality of an exponent interval or a functional hypothesis.

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- [2] G. H. Hardy, J. E. Littlewood, and G. Pólya, *Inequalities*, 2nd ed., Cambridge University Press, Cambridge, 1952.
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JUNIOR PROBLEMS

Solutions to the problems in this issue should arrive before
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Proposals

86. *Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam.* Without using a calculator, evaluate the following expression:

$$A = \frac{\cos 160^\circ}{\cos 80^\circ} + \frac{\cos 80^\circ}{\cos 40^\circ} + \frac{\cos 40^\circ}{\cos 160^\circ}.$$

87. *Proposed by Mihaly Bencze, Braşov, Romania, and Neculai Stanciu, Buzău, Romania.* Let $n \geq 3$ and $a_k > 0$ for $k = 1, 2, \dots, n$. With indices understood modulo n , prove that

$$\sum_{i=1}^n \left(\frac{a_{i+1} + a_{i+2}}{a_i^2} + \frac{a_i + a_{i+2}}{a_{i+1}^2} + \frac{a_i + a_{i+1}}{a_{i+2}^2} \right) \geq \frac{6n^2}{\sum_{k=1}^n a_k}.$$

88. *Proposed by Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Solve for real numbers the following equation:

$$x^2 - 5x - 2\sqrt{x-2} + 7 + \log_2 \frac{x^2 - 5x + 8}{\sqrt{x-2}} + \log_3 \frac{x^2 - 5x + 8}{2\sqrt{x-2}} = 0.$$

89. *Proposed by Valmir Krasniqi, Institute of Science, Technology, Engineering and Mathematics, Prishtina, Republic of Kosova.* Let $\mathbb{Q}^+$ denote the positive rational numbers, and let $\mathbb{N} = \{1, 2, 3, \dots\}$. Find all functions $f : \mathbb{Q}^+ \rightarrow \mathbb{N}$ such that

$$\gcd\left(f(x+y), f\left(\frac{xy}{x+y}\right)\right)^2 = f(x)f(y)$$

for all $x, y \in \mathbb{Q}^+$.

90. *Proposed by Valmir Krasniqi, Institute of Science, Technology, Engineering and Mathematics, Prishtina, Republic of Kosova.* Find all functions $f : (0, \infty) \rightarrow (0, \infty)$ such that, for every $x > 0$,

$$f\left(\frac{x^3+1}{f(x)}\right) = x^2 + x + 1, \quad \frac{x^2}{2} \leq f(x) \leq x^2 + 1.$$

No problem is ever permanently closed. We will be very pleased to consider publishing new solutions or comments on previous problems.

Solutions

For the solution of Junior Problem 80, see the MathNotes section, p. 726.

81. Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, Buzău, Romania. Let x, y and z be positive real numbers. Show that

$$\sum_{\text{cyc}} \frac{2}{x+y} \leq \sum_{\text{cyc}} \frac{8x}{3y^2 + 2yz + 3z^2} \leq \frac{1}{xyz} \sum_{\text{cyc}} x^2.$$

Solution 1 by Michel Bataille, Rouen, France. From the weighted arithmetic mean-geometric mean inequality, we have

$$\frac{3}{8}y^2 + \frac{2}{8}yz + \frac{3}{8}z^2 \geq (y^2)^{3/8}(yz)^{2/8}(z^2)^{3/8} = yz$$

hence $\frac{8x}{3y^2 + 2yz + 3z^2} \leq \frac{x}{yz} = \frac{x^2}{xyz}$ and therefore

$$\sum_{\text{cyc}} \frac{8x}{3y^2 + 2yz + 3z^2} \leq \sum_{\text{cyc}} \frac{x}{yz} = \frac{1}{xyz} \sum_{\text{cyc}} x^2,$$

which proves the right inequality.

To show the left inequality, we can suppose $x + y + z = 1$ (due to homogeneity). The inequality then rewrites as

$$\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z} \leq \sum_{\text{cyc}} \frac{4x}{3y^2 + 2yz + 3z^2}.$$

From the convexity of $t \mapsto \frac{1}{t}$ on $(0, \infty)$, we deduce that

$$\begin{aligned} \sum_{\text{cyc}} \frac{4x}{3y^2 + 2yz + 3z^2} &\geq \frac{4}{3xy^2 + 2xyz + 3xz^2 + 3yz^2 + 2xyz + 3yx^2 + 3zx^2 + 2xyz + 3zy^2} \\ &= \frac{4}{6p + 3S}. \end{aligned}$$

where $p = xyz$ and $S = \sum_{\text{cyc}} (xy^2 + yx^2)$. Setting $m = xy + yz + zx$, we have

$$\frac{1}{1-x} + \frac{1}{1-y} + \frac{1}{1-z} = \frac{m+1}{m-p} \quad \text{and} \quad S = (xy + yz + zx)(x + y + z) - 3xyz = m - 3p,$$

so it suffices to show that

$$\frac{m+1}{m-p} \leq \frac{4}{3m-3p}. \tag{1}$$

Now we have

$$m = (xy + yz + zx)(x + y + z) \geq 3\sqrt[3]{x^2y^2z^2} \cdot 3\sqrt[3]{xyz} = 9p,$$

hence $m - p > 0$ and

$$1 = (x + y + z)^2 = x^2 + y^2 + z^2 + 2m \geq 3m,$$

hence $m + 1 \leq \frac{4}{3}$ and (1) holds.

Solution 2 by Paolo Perfetti, Department of Mathematics, University of Rome Tor Vergata, Rome, Italy. For the right inequality, $3y^2 + 3z^2 \geq 6yz$, so

$$\sum_{\text{cyc}} \frac{8x}{3y^2 + 2yz + 3z^2} \leq \sum_{\text{cyc}} \frac{8x}{8yz} = \frac{1}{xyz} \sum_{\text{cyc}} x^2.$$

For the left inequality, the assertion rewrites as

$$\sum_{\text{cyc}} \frac{8x^2}{3xy^2 + 2xyz + 3xz^2} \geq \sum_{\text{cyc}} \frac{2}{x+y}.$$

The Cauchy–Schwarz inequality yields

$$\sum_{\text{cyc}} \frac{8x^2}{3xy^2 + 2xyz + 3xz^2} \geq \frac{8(x+y+z)^2}{3(x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2) + 6xyz}.$$

Hence it suffices to prove that

$$\frac{8(x+y+z)^2}{3(x^2y + xy^2 + y^2z + yz^2 + z^2x + zx^2) + 6xyz} \geq \sum_{\text{cyc}} \frac{2}{x+y}.$$

Putting the two sides over a common denominator reduces this to

$$\frac{2x^2 + y^2 + z^2 - xy - xz - yz}{3(z+x)(y+z)(x+y)} \geq 0.$$

This is evidently true, since the denominator is positive and

$$x^2 + y^2 + z^2 - xy - xz - yz = \frac{1}{2}((x-y)^2 + (y-z)^2 + (z-x)^2) \geq 0.$$

Solution 3 by Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania.

Denote

$$x + y + z = p, \quad xy + yz + zx = q, \quad xyz = r.$$

Using the Cauchy–Schwarz inequality, we have

$$\begin{aligned} \sum_{\text{cyc}} \frac{8x}{3y^2 + 2yz + 3z^2} &= 8 \sum_{\text{cyc}} \frac{x^2}{x(3y^2 + 2yz + 3z^2)} \\ &\geq \frac{8 \left(\sum_{\text{cyc}} x \right)^2}{3 \sum_{\text{cyc}} xy(x+y) + 6xyz} \\ &= \frac{8p^2}{3pq - 3r}. \end{aligned} \tag{1}$$

Therefore, it suffices to prove that

$$\frac{8p^2}{3pq - 3r} \geq \sum_{\text{cyc}} \frac{2}{x+y} = \frac{2(p^2 + q)}{pq - r}. \tag{2}$$

We have $pq \geq 9r$, so $pq - r > 0$. Consequently, (2) is equivalent to

$$\frac{4p^2}{3pq - 3r} \geq \frac{p^2 + q}{pq - r},$$

or to $(p^2 - 3q)(pq - r) \geq 0$. This holds because

$$p^2 - 3q = \frac{1}{2}((x - y)^2 + (y - z)^2 + (z - x)^2) \geq 0.$$

For the right inequality, note that

$$3y^2 + 2yz + 3z^2 - 8yz = 3(y - z)^2 \geq 0.$$

Thus

$$\sum_{\text{cyc}} \frac{8x}{3y^2 + 2yz + 3z^2} \leq \sum_{\text{cyc}} \frac{8x}{8yz} = \frac{1}{xyz} \sum_{\text{cyc}} x^2.$$

Also solved by Henry Ricardo, Westchester Area Math Circle, New York, USA.

Also solved by the proposers.

82. *Proposed by George-Florin Șerban, National Pedagogical College “D.P.Perpessiciu”, Braila, Romania.* Find all prime numbers p and q such that $p^3 + q^3 - 7pq = 2023$.

Solution 1 by Titu Zvonaru, Comănești, Romania. By symmetry, assume $p \leq q$. Since

$$2q^3 \geq p^3 + q^3 = 2023 + 7pq > 2023,$$

we have $q > 10$ and therefore $q \geq 11$. The arithmetic-geometric mean inequality gives

$$\begin{aligned} 2023 + 11^3 &= p^3 + q^3 + 11^3 - 7pq \\ &\geq 3 \cdot 11pq - 7pq = 26pq. \end{aligned}$$

Hence

$$pq \leq \frac{2023 + 1331}{26} = 129.$$

It follows that $p \leq \sqrt{129} < 12$, so

$$p \in \{2, 3, 5, 7, 11\}.$$

In each case, the equation becomes $q(q^2 - 7p) = 2023 - p^3$.

- If $p = 2$, then $q(q^2 - 14) = 5 \cdot 13 \cdot 31$. Since q is prime and $q \geq 11$, we have $q \in \{13, 31\}$. Only $q = 13$ satisfies the equation.
- If $p = 3$, then $q(q^2 - 21) = 2^2 \cdot 499$. The only prime divisor at least 11 is 499, which is excluded by $pq \leq 129$.
- If $p = 5$, then $q(q^2 - 35) = 2 \cdot 13 \cdot 73$. The bound $pq \leq 129$ leaves only $q = 13$, which does not satisfy the equation.
- If $p = 7$, then $q(q^2 - 49) = 2^4 \cdot 3 \cdot 5 \cdot 7$. There is no prime divisor $q \geq 11$.
- If $p = 11$, then $q(q^2 - 77) = 2^2 \cdot 173$. The only prime divisor at least 11 is 173, again excluded by $pq \leq 129$.

Restoring symmetry, the complete set of solutions is

$$(p, q) = (2, 13), \quad (p, q) = (13, 2).$$

Solution 2 by Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania.

Editorial note. The candidate check below has been completed for positive primes, excluding the negative and nonprime pairs in the submitted list.

Let $p + q = s$ and $pq = u$. The equation becomes

$$s^3 - 3us - 7u = 2023,$$

which is equivalent to

$$u = \frac{s^3 - 2023}{3s + 7}.$$

Since u is an integer, $27u$ is an integer, and polynomial division gives

$$\begin{aligned} 27u &= 9s^2 - 21s + 49 - \frac{7^3 + 27 \cdot 2023}{3s + 7} \\ &= 9s^2 - 21s + 49 - \frac{4 \cdot 7 \cdot 13 \cdot 151}{3s + 7}. \end{aligned}$$

Hence $3s + 7$ divides $4 \cdot 7 \cdot 13 \cdot 151$.

Because p and q are positive primes, $u > 0$, so $s^3 > 2023$ and $s \geq 13$. Also $u = pq \leq s^2/4$. It follows that

$$2023 = s^3 - (3s + 7)u \geq s^3 - \frac{(3s + 7)s^2}{4} = \frac{s^2(s - 7)}{4}.$$

If $s \geq 23$, the last expression is at least

$$\frac{23^2(23 - 7)}{4} = 2116 > 2023,$$

a contradiction. Thus $13 \leq s \leq 22$, and consequently

$$46 \leq 3s + 7 \leq 73.$$

The only divisor of $4 \cdot 7 \cdot 13 \cdot 151$ in this interval is 52. Indeed, the factor 151 is too large, and the divisors of $4 \cdot 7 \cdot 13 = 364$ are

$$1, 2, 4, 7, 13, 14, 26, 28, 52, 91, 182, 364.$$

Therefore $3s + 7 = 52$, so $s = 15$ and

$$u = \frac{15^3 - 2023}{52} = 26.$$

The numbers p, q are the roots of

$$X^2 - 15X + 26 = (X - 2)(X - 13) = 0.$$

Both 2 and 13 are prime, and

$$2^3 + 13^3 - 7 \cdot 2 \cdot 13 = 2023.$$

Hence the solutions are $(p, q) = (2, 13)$ and $(13, 2)$.

Also solved by the proposer.

83. *Proposed by Fiton Hoxha, Technical University of Munich, Germany.* A permutation of the numbers $1, 2, \dots, n$ is given. In each step, if the first element of the permutation is $k \neq 1$, we form a new permutation by moving k to the k -th position. We say that the process terminates if eventually the first element becomes 1.

- (a) Show that no matter what initial permutation we start with, the process will always terminate.
- (b) Find the maximal number of steps that the process could take before terminating.

Example: For $n = 6$, the process could look like this:

$$361245 \longrightarrow 613245 \longrightarrow 132456.$$

Solution by Daniel Văcaru, Pitești, Romania.

Editorial note. The submitted termination argument is incomplete, and the claimed maximum n is incorrect. The editors supply the following corrected argument.

A move with first entry $k > 1$ replaces

$$(k, x_2, \dots, x_k, x_{k+1}, \dots, x_n)$$

by

$$(x_2, \dots, x_k, k, x_{k+1}, \dots, x_n).$$

In particular, it places k in position k . The case $n = 1$ is immediate, so assume $n \geq 2$.

(a) Consider any finite initial segment of the process, and let M_k be the number of moves that place the value k , for $2 \leq k \leq n$. After a move places k in position k , a move of a value smaller than k leaves that position unchanged. Thus, between two successive moves of k , there must be at least one move of a value larger than k . The intervening intervals are disjoint, so

$$M_k \leq 1 + \sum_{j=k+1}^n M_j.$$

In particular, $M_n \leq 1$. Descending induction now gives

$$M_k \leq 2^{n-k}, \quad 2 \leq k \leq n,$$

because

$$1 + \sum_{j=k+1}^n 2^{n-j} = 2^{n-k}.$$

Therefore every such finite initial segment has at most

$$\sum_{k=2}^n M_k \leq \sum_{k=2}^n 2^{n-k} = 2^{n-1} - 1$$

moves. The process must consequently terminate, and termination occurs when its first entry is 1.

(b) The bound is attained by

$$\pi_n = (2, 3, \dots, n, 1).$$

We prove inductively that this initial permutation reaches $(1, 2, \dots, n)$ in exactly $T_n = 2^{n-1} - 1$ moves. For $n = 1$, take $\pi_1 = (1)$ and $T_1 = 0$. Suppose the assertion is true for $n - 1$. During the first stage for π_n , the last entry 1 is unchanged. The first $n - 1$ entries evolve exactly as the process for π_{n-1} with its value 1 replaced by n , until n reaches the first position. By induction, this takes T_{n-1} moves and gives

$$(n, 2, 3, \dots, n-1, 1).$$

The next move places n last and leaves

$$(2, 3, \dots, n-1, 1, n).$$

Its first $n-1$ entries are now π_{n-1} , so a further T_{n-1} moves give the identity permutation. Hence

$$T_n = 2T_{n-1} + 1 = 2^{n-1} - 1.$$

Together with part (a), this proves that the maximal number of moves is $2^{n-1} - 1$.

Also solved by the proposer.

84. *Proposed by Dren Neziri, University of Primorska, Slovenia.* Let $ABCD$ be a trapezoid with $AB \parallel CD$ and let O be the intersection of its diagonals. Let X and Y be points on the circumcircles of triangles OAB and OCD , respectively, with the following properties: (i) we have $OX = OY$, (ii) the points X and O lie on opposite sides with respect to AB , and (iii) the points Y and O lie on opposite sides with respect to CD . Prove that $YC \geq XA$ if and only if $XB \geq YD$.

Solution by the editors. Put

$$k = \frac{CD}{AB} = \frac{OC}{OA} = \frac{OD}{OB} > 0,$$

where the equalities follow from $AB \parallel CD$ and the similarity of triangles OAB and OCD . Since X and O lie on opposite sides of AB , the cyclic order of the first quadrilateral is O, A, X, B . Likewise, the cyclic order of the second is O, C, Y, D . Ptolemy's theorem therefore gives

$$AB \cdot OX = OA \cdot XB + OB \cdot XA,$$

and

$$CD \cdot OY = OC \cdot YD + OD \cdot YC.$$

Dividing the second equality by k and using $OX = OY$, we obtain

$$OA \cdot XB + OB \cdot XA = OA \cdot YD + OB \cdot YC.$$

Consequently,

$$OA(XB - YD) = OB(YC - XA).$$

Since OA and OB are positive, the two differences have the same sign, proving

$$YC \geq XA \iff XB \geq YD.$$

Also solved by the proposer.

85. *Proposed by George-Florin Șerban, National Pedagogical College "D.P.Perpessiciu", Braila, Romania.* Prove that the equation $x^{2021} + y^{2022} = z^{2023}$ has infinitely many solutions in positive integers.

Solution by Michel Bataille, Rouen, France. More generally, we show that, for any positive integer k , the equation $x^{2k-1} + y^{2k} = z^{2k+1}$ has infinitely many solutions in positive integers.

Let n be any positive integer and let

$$a_n = 2k^2 + 2kn(2k+1), \quad b_n = k(2k-1) + n(4k^2-1), \quad c_n = 2k^2 - 2k + 1 + 2kn(2k-1).$$

Then, the infinitely many triples $(2^{a_n}, 2^{b_n}, 2^{c_n})$ ($n \in \mathbb{N}$) are solutions for (x, y, z) . Indeed, we have

$$\begin{aligned} (2^{a_n})^{2k-1} + (2^{b_n})^{2k} &= 2^{2k^2(2k-1)} \cdot 2^{2kn(4k^2-1)} + 2^{2k^2(2k-1)} \cdot 2^{2kn(4k^2-1)} \\ &= 2 \cdot 2^{2k^2(2k-1)} \cdot 2^{2kn(4k^2-1)} \\ &= 2^{2k^2(2k-1)+1} \cdot 2^{2kn(4k^2-1)} = (2^{c_n})^{2k+1}. \end{aligned}$$

Also solved by the proposer.