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PROBLEMS AND SOLUTIONS

Proposals and solutions must be legible and should appear on separate sheets, each indicating the name of the sender. It would be preferred to submit your proposals and solutions as TeX files. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editors have supplied a solution. The editors encourage undergraduate and pre-college students to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the grade and school name. Solutions will be evaluated for publication by a committee of professors according to a combination of criteria. Questions concerning proposals and/or solutions can be sent by e-mail to: mathproblems-ks@hotmail.com.

Solutions to the problems in this issue should arrive before December 26, 2024

Problems

173. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Find all 2×2 matrices with real entries such that $\mathrm{e}^A = A - A^\top$, where $A^\top$ denotes the transpose of A and e^A denotes matrix exponentiation.

174. *Proposed by Vasile Cîrtoaje, Petroleum-Gas, University of Ploiesti, Romania.* Prove that 2 is the least positive value of the constant k such that

$$\left(\frac{ka_1 + a_2 + \cdots + a_7}{k+6}\right)^2 \geq \frac{a_1a_2 + a_2a_3 + \cdots + a_8a_1}{8}$$

whenever $a_1 \geq a_2 \geq \cdots \geq a_8 \geq 0$.

175. *Proposed by Michel Bataille, Rouen, France.* Let m and n be coprime positive integers such that $mn + 1$ is a square-free divisor of $m^4 + n^4 + 2$. Prove that $m = n = 1$.

176. *Proposed by Stănescu Florin, Şerban Cioculescu School, Găeşti, Romania.* Let $(a_n)_{n\geq 1}$ be a sequence of real numbers defined recursively by

$$a_1 \in (0,1], \qquad a_{n+1} = \frac{1-e^{-n^2 a_n}}{(n+1)^2}.$$

Compute the limit

$$\lim_{n\to\infty} \frac{n^4}{\ln n}\left((n^2-1)(1-e^{a_n}) - n(e^{-na_n}-1)\right).$$

177. *Proposed by Dion Aliu, University of York, United Kingdom.* Let $S(n)$ be the biggest square divisor of n. Let $\{a_n\}_{n=1}^{\infty}$ be defined as follows:

$$a_1 = a, a_2 = b$$

and

$$a_{n+1} = S(a_n) + S(a_{n-1}).$$

Prove that for every choice of $a, b \in \mathbb{N}$, the sequence is eventually constant.

178. *Proposed by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy.* Compute the integral

$$\int_0^{\infty} \frac{\ln(1-x+x^2)}{1+x^2}\mathrm{d}x.$$

179. *Proposed by Joe Santmyer, Las Cruces, New Mexico, USA.* If $a \geq 0$ show that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{1+ae^{-\pi\tan(\theta)}} \;=\; \pi\left(\frac{1-a+\ln(a)}{(1-a)\ln(a)}\right)$$

No problem is ever permanently closed. We will be very pleased to consider publishing new solutions or comments on previous problems.

Solutions

166. *Proposed by Mohsen Soltanifar, University of Toronto, Canada.*
Let X and Y be two continuous real valued random variables with strictly monotone cumulative distribution functions F_X, F_Y, respectively.

(i) Given $a \in [0,1]$, find functions $g, h : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{t\to+\infty} F_{X,Y}(g(t), h(t)) = a.$$

(ii) How many functions g, h satisfy the condition in part (i)?

Solution by the proposer.
(i) Define $g(t) = t, h(t) = e^{-t} + F_Y^{-1}(a), \quad (-\infty < t < \infty)$. Then, we have

$$\begin{aligned}
&|F_{X,Y}(t, e^{-t} + F_Y^{-1}(a)) - a| \quad = \\
&|F_{X,Y}(t, e^{-t} + F_Y^{-1}(a)) - F_{X,Y}(+\infty, e^{-t} + F_Y^{-1}(a)) + F_{X,Y}(+\infty, e^{-t} + F_Y^{-1}(a)) - a| \quad \leq \\
&|F_{X,Y}(t, e^{-t} + F_Y^{-1}(a)) - F_{X,Y}(+\infty, e^{-t} + F_Y^{-1}(a))| + |F_{X,Y}(+\infty, e^{-t} + F_Y^{-1}(a)) - a| \quad = \\
&|P(t < X, Y \leq e^{-t} + F_Y^{-1}(a))| + |F_Y(e^{-t} + F_Y^{-1}(a)) - a| \quad \leq \\
&1 - F_X(t) + |F_Y(e^{-t} + F_Y^{-1}(a)) - a| \\
&(-\infty < t < \infty). \qquad (1)
\end{aligned}$$

Now, considering $F_X(+\infty) = 1$ and $F_Y(F_Y^{-1}(a)) = a$, an application of the Squeeze Theorem on (1) yields the plausible result.
(ii) Indeed, there are uncountably many such functions. To see this, define continuous functions $u, v : \mathbb{R} \to \mathbb{R}$ such that $\lim_{t\to+\infty} u(t) = +\infty$ and $\lim_{t\to+\infty} v(t) = 0$. Now set:

$$\begin{aligned}
g(t) &= u(t) \\
h(t) &= v(t) + F_Y^{-1}(a) \qquad (-\infty < t < \infty). \qquad (2)
\end{aligned}$$

Accordingly, a repeated argument similar to part (i) yields the desired result. ▲

Reference.
[1] Evans , M.J & Rosenthal, J.S(2003). Probability and Statistics: The Science of Uncertainty (undergraduate-level textbook). W.H. Freeman and Co.

167. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Calculate

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots\right)\left(\frac{1}{n^2} - \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} - \cdots\right).$$

Solution by Michel Bataille, Rouen, France. Let $A_n = \sum_{k=0}^{\infty} \frac{(-1)^k}{n+k}$, $B_n = \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)^2}$ and let $S = \sum_{n=1}^{\infty} A_n B_n$ be the sum to be evaluated.

From $A_n = \int_0^1 \frac{u^{n-1}}{1+u}\, du$ and $B_n > 0$ (since $B_n > \frac{1}{n^2} - \frac{1}{(n+1)^2}$) we obtain

$$S = \sum_{n=1}^{\infty} \int_0^1 \frac{u^{n-1}B_n}{1+u}\, du = \int_0^1 \left(\sum_{n=1}^{\infty} B_n u^{n-1}\right) \frac{du}{1+u}.$$

Observing that $B_{n+1} + B_n = \frac{1}{n^2}$, we deduce that

$$(1+u)\sum_{n=1}^{\infty} B_n u^{n-1} = B_1 + \sum_{n=1}^{\infty} \frac{u^n}{n^2} = B_1 + \mathrm{Li}_2(u)$$

where $\mathrm{Li}_2(u) = \sum_{n=1}^{\infty} \frac{u^n}{n^2} = -\int_0^u \frac{\ln(1-t)}{t}\, dt$ is the dilogarithm function. Therefore

$$S = \int_0^1 \frac{B_1}{(1+u)^2}\, du - I$$

where $I = \int_0^1 \left(\int_0^u \frac{\ln(1-t)}{t}\, dt\right) \frac{du}{(1+u)^2}$.

Since $B_1 = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$ and $\int_0^1 \frac{1}{[1+u)^2}\, du = \left[\frac{-1}{1+u}\right]_0^1 = \frac{1}{2}$, we see that $S = \frac{\pi^2}{24} - I$.

Now, we have

$$\begin{aligned} I &= \int_0^1 \int_t^1 \left(\frac{du}{(1+u)^2}\right) \frac{\ln(1-t)}{t}\, dt = \int_0^1 \left(\frac{1}{1+t} - \frac{1}{2}\right) \frac{\ln(1-t)}{t}\, dt \\ &= \frac{1}{2}\int_0^1 \frac{\ln(1-t)}{t}\, dt - \int_0^1 \frac{\ln(1-t)}{1+t}\, dt = -\frac{\pi^2}{12} - \frac{1}{2}\int_0^1 \frac{\ln(u)}{1-\frac{u}{2}}\, du \\ &= -\frac{\pi^2}{12} - \frac{1}{2}\int_0^1 \left(\sum_{n=0}^{\infty} \frac{u^n \ln(u)}{2^n}\right) du = -\frac{\pi^2}{12} - \sum_{n=0}^{\infty} \frac{1}{2^{n+1}} \int_0^1 u^n \ln(u)\, du \\ &= -\frac{\pi^2}{12} + \sum_{n=0}^{\infty} \frac{1}{2^{n+1}(n+1)^2} = -\frac{\pi^2}{12} + \mathrm{Li}_2(1/2). \end{aligned}$$

It is known that $\mathrm{Li}_2(1/2) = \frac{\pi^2}{12} - \frac{(\ln(2))^2}{2}$ and it follows that

$$S = \frac{\pi^2}{24} + \frac{(\ln(2))^2}{2}.$$

▲

Also solved by Albert Stadler, Switzerland; Ángel Plaza Department of Mathematics, Universidad de Las Palmas de Gran Canaria, Spain; Moti Levy, Rehovot, Israel; Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria and the proposer.

168. *Proposed by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy.* Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\cos(2x)(1-\ln\cos x)}{x^2 + (1-\ln\cos x)^2} dx.$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria. The answer is $\frac{\pi}{4(1+\ln 2)^2}$.
Note that for $b>0$ and a real a we have

$$\frac{b}{a^2+b^2}=\int_0^\infty \cos(at)e^{-bt}\,dt.$$

Choosing $a=x$ and $b=1-\ln\cos x$ we conclude that

$$\frac{1-\ln\cos x}{x^2+(1-\ln\cos x)^2}=\int_0^\infty \cos(xt)e^{-(1-\ln\cos x)t}\,dt=\int_0^\infty \cos(xt)(\cos x)^t e^{-t}\,dt.$$

So we have

$$I\overset{\text{def}}{=}\int_0^{\pi/2}\frac{\cos(2x)(1-\ln\cos x)}{x^2+(1-\ln\cos x)^2}=\int_0^\infty e^{-t}F(t)\,dt,$$

with

$$F(t)=\int_0^{\pi/2}\cos(xt)(\cos x)^t\cos(2x)\,dx.$$

Using the parity of the integrand we see that

$$\begin{aligned}F(t)&=\frac{1}{2}\int_{-\pi/2}^{\pi/2}e^{ixt}(\cos x)^t\cos(2x)\,dx=\frac{1}{2^{t+1}}\int_{-\pi/2}^{\pi/2}\left(e^{2ix}+1\right)^t\cos(2x)\,dx\\&=\frac{1}{2^{2+t}}\int_{-\pi}^{\pi}(1+e^{ix})^t\cos x\,dx.\end{aligned}$$

Now, for $0<r<1$ we have

$$(1+re^{ix})^t=\sum_{n=0}^\infty\binom{t}{n}r^ne^{inx}.$$

Consequently, we have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(1+re^{ix})^t\cos x\,dx=\frac{1}{2}\binom{t}{1}r=\frac{rt}{2}.$$

Letting r tend to 1^- and using the dominated convergence theorem we conclude that

$$\int_{-\pi}^{\pi}(1+e^{ix})^t\cos x\,dx=\pi t.$$

It follows that

$$F(t)=\frac{\pi}{4}t2^{-t}.$$

Finally we obtain

$$I=\frac{\pi}{4}\int_0^\infty t(2e)^{-t}dt=\frac{\pi}{4}\int_0^\infty te^{-(1+\ln 2)t}dt=\frac{\pi}{4(1+\ln 2)^2}. \qquad \blacktriangle$$

Also solved by Albert Stadler, Switzerland; Moti Levy, Rehovot, Israel and the proposer.

169. *Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania.*
Let a,b,c,d,e be real numbers such that $a\ge b\ge c\ge d\ge e\ge 0$. Prove that

$$\left(\frac{ab+bc+cd+de+ea}{5}\right)^{1/2}\ge\left(\frac{abc+bcd+cde+dea+eab}{5}\right)^{1/3}.$$

Solution by Albert Stadler, Switzerland. We take the sixth power on both sides and see that we need to prove that

$$\left(\frac{ab+bc+cd+de+ea}{5}\right)^3 \geq \left(\frac{abc+bcd+cde+dea+eab}{5}\right)^2.$$

Let $f(a,b,c,d,e) = \left(\frac{ab+bc+cd+de+ea}{5}\right)^3 - \left(\frac{abc+bcd+cde+dea+eab}{5}\right)^2$. Further let $z = a-b, y = b-c, x = c-d, w = d-e, v = e$. By assumption, we have $v,w,x,y,z \geq 0$. Then $a = v+w+x+y+z, b = v+w+x+y, c = v+w+x, d = v+w, e = v$. A brute force approach (executed with Mathematica) reveals that

$$\begin{aligned} f(v+w+x+y+z, v+w+x+y, v+w+x, v+w, v) = &\frac{1}{25}\left(3v^4w^2 - v^4wz + 3v^4z^2\right) \\ &+ g(v,w,x,y,z) \end{aligned}$$

where $g(v,w,x,y,z)$ is a homogeneous polynomial of degree 6 all of whose coefficients are positive. By the AM-GM inequality, we have $3v^4w^2 + 3v^4z^2 \geq 6v^4wz$. Therefore

$$f(v+w+x+y+z, v+w+x+y, v+w+x, v+w, v) \geq 0,$$

and we are done. ▲

Also solved by the proposer. One incorrect solution was received.

170. *Proposed by Stănescu Florin, Şerban Cioculescu School, Găeşti, Romania.* Let $f : [0,1] \to \mathbb{R}$ be a three-times continuously differentiable function such that $f(0) = f'(0) = f(1)$ and $\left|f^{(3)}(t)\right| \leq 1$ for all $t \in [0,1]$. Prove

(i) $$|f(x)| \leq \frac{x(1-x)}{\sqrt{3}}\left(\int_0^x \frac{|f(t)|}{t(1-t)}dt\right)^{1/2} \quad \text{for all } x \in [0,1];$$

(ii) $$|f'(x)| \leq \frac{1-2x}{\sqrt{3}}\left(\int_0^x \frac{|f(t)|}{t(1-t)}dt\right)^{1/2} + \frac{x(1-x)}{6} \quad \text{for all } x \in [0,1/2];$$

(iii) $$9\int_0^1 \left(\frac{f(x)}{x}\right)^2 dx \leq \int_0^1 (1-x)^2 \frac{|f(x)|}{x}\, dx.$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria. Taylor's Theorem with integral remainder yields

$$f(x) = f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + \int_0^x \frac{(x-t)^2}{2} f^{(3)}(t)\, dt.$$

Hence

$$\begin{aligned} f(x) &= \frac{1}{2}f''(0)x^2 + \int_0^1 \frac{(x-t)^{+2}}{2} f^{(3)}(t)\, dt \\ 0 = f(1) &= \frac{1}{2}f''(0) + \int_0^1 \frac{(1-t)^2}{2} f^{(3)}(t)\, dt, \end{aligned}$$

where we used the notation y^+ for $\max(y,0)$. Replacing $f''(0)$ from the second formula in the first we get

$$f(x) = -\int_0^1 K(x,t) f^{(3)}(t)\, dt \tag{1}$$

where

$$K(x,t)=\frac{x^2(1-t)^2-(x-t)^{+2}}{2}. \tag{2}$$

Now let us consider $g(x)=\dfrac{f(x)}{x(1-x)}$. g is a continuously differentiable function on $(0,1)$ that can be extended by continuity on $[0,1]$ by setting $g(0)=f'(0)=0$ and $g(1)=-f'(1)$. According to (1) we have

$$g(x)=-\frac{1}{2}\int_0^1 L(x,t)f^{(3)}(t)\,dt \tag{3}$$

where

$$L(x,t)=\begin{cases}\dfrac{x(1-t)^2}{1-x} & \text{if } 0\le x\le t\le 1,\\ t(2-t)-\dfrac{t^2}{x} & \text{if } 0\le t<x\le 1.\end{cases}. \tag{4}$$

Applying the theorem of differentiation of an integral depending on a parameter, it follows that, for $x\in(0,1)$ we have

$$g'(x)=-\frac{1}{2}\int_0^1 M(x,t)f^{(3)}(t)\,dt \tag{5}$$

where

$$M(x,t)=\frac{\partial L}{\partial x}(x,t)=\begin{cases}\dfrac{(1-t)^2}{(1-x)^2} & \text{if } 0<x<t\le 1,\\ \dfrac{t^2}{x^2} & \text{if } 0<t<x<1.\end{cases} \tag{6}$$

In particular, taking $f(x)=x^2(x-1)/6$, we have $g'(x)=-1/6$ and applying (5) implies that

$$\int_0^1 M(x,t)dt=\frac{1}{3},\quad \text{for all } x\in(0,1).$$

Now, because $\left|f^{(3)}\right|\le 1$, and $M(x,t)\ge 0$ we conclude that

$$|g'(x)|\le\frac{1}{2}\int_0^1 M(x,t)\,dt=\frac{1}{6}. \tag{7}$$

We conclude, using (7) and the fact that $g(0)=0$, that

$$g(x)^2=2\int_0^x g(t)g'(t)dt\le 2\int_0^x |g(t)|\,|g'(t)|\,dt\le\frac{1}{3}\int_0^x |g(t)|\,dt. \tag{8}$$

Taking square roots yields (i).
Noting that $f(x)=(x-x^2)g(x)$ we see that $f'(x)=(1-2x)g(x)+x(1-x)g'(x)$. Applying (7) and (8) with the triangle inequality we see that

$$|f'(x)|\le\frac{|1-2x|}{\sqrt{3}}\left(\int_0^x |g(t)|\,dt\right)^{1/2}+\frac{x(1-x)}{6} \tag{9}$$

and (ii) follows.

On the other hand, using (8), we obtain

$$9\int_0^1 \left(\frac{f(x)}{x}\right)^2 = 9\int_0^1 (1-x)^2 g^2(x) \le \int_0^1 3(1-x)^2 \left(\int_0^x |g(t)|\ dt\right) dx$$
$$= \left[-(1-x)^3 \int_0^x |g(t)|\ dt\right]_0^1 + \int_0^1 (1-x)^3\, |g(x)|\, dx$$
$$= \int_0^1 (1-x)^2 \frac{|f(x)|}{x} dx,$$

which is (iii). Note that the inequalities in (i), (ii) and (iii) are sharp, because equality holds when $f(x) = x^2(1-x)/6$. ▲

Also solved by the proposer.

171. *Proposed by Michel Bataille, Rouen, France.* Find all real numbers ρ, α, ℓ with $\ell \neq 0$ such that

$$\lim_{n\to\infty} n^\alpha \rho^n \sum_{k=0}^{n} \binom{2n-2k}{n-k} \frac{(5n)^k}{k!} = \ell.$$

Solution by Moti Levy, Rehovot, Israel. Let

$$S := \sum_{k=0}^{n} \binom{2\,(n-k)}{n-k} \frac{(5n)^k}{k!}. \tag{3}$$

We will use the well-known identity

$$\binom{2n}{n} = (-4)^n \binom{-\frac{1}{2}}{n}. \tag{4}$$

We have

$$S = \sum_{k=0}^{n} (-4)^{n-k} \binom{-\frac{1}{2}}{n-k} \frac{(5n)^k}{k!} = (-4)^n \sum_{k=0}^{n} \binom{-\frac{1}{2}}{n-k} \frac{\left(-\frac{5n}{4}\right)^k}{k!}, \tag{5}$$

$$S = (-4)^n \left(\frac{\left(-\frac{5n}{4}\right)^n}{n!} + \binom{-\frac{1}{2}}{1} \frac{\left(-\frac{5n}{4}\right)^{n-1}}{(n-1)!} + \binom{-\frac{1}{2}}{2} \frac{\left(-\frac{5n}{4}\right)^{n-2}}{(n-2)!} + \cdots \right)$$
$$= \frac{(5n)^n}{n!} \left(1 + \binom{-\frac{1}{2}}{1} \left(\frac{-4}{5}\right) + \binom{-\frac{1}{2}}{2} \left(\frac{-4}{5n}\right)^2 (n-1)\, n + \cdots \right).$$

Now we use the fact

$$\frac{\prod_{k=0}^{m} (n-k)}{n^{m+1}} = 1 - \frac{m\,(m+1)}{2n} + O\left(\frac{1}{n^2}\right)$$

to write

$$\frac{n!}{(5n)^n} S = \sum_{k=0}^{n} \binom{-\frac{1}{2}}{k} \left(\frac{-4}{5}\right)^k \left(1 - \frac{k\,(k+1)}{2n} + O\left(\frac{1}{n^2}\right)\right).$$

The following generating functions are known:

$$\sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} z^k = \frac{1}{(1+z)^{\frac{1}{2}}},$$

$$\sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \frac{k\,(k+1)}{2} z^k = -\frac{z\,(z+4)}{8\,(1+z)^{\frac{5}{2}}}.$$

Hence we obtain

$$\sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \left(\frac{-4}{5}\right)^k = \sqrt{5}, \quad \sum_{k=0}^{\infty} \binom{-\frac{1}{2}}{k} \frac{k\,(k+1)}{2} \left(\frac{-4}{5}\right)^k = 8\sqrt{5}.$$

Therefore

$$S = \frac{(5n)^n}{n!} \left(\sqrt{5} + O\left(\frac{1}{n}\right)\right).$$

Using Stirling's approximation, we get

$$S \sim \sqrt{5}\frac{(5n)^n}{n!} \sim \sqrt{5}5^n n^n \frac{e^n}{\sqrt{2\pi n} n^n} = \sqrt{\frac{5}{2\pi}} \frac{(5e)^n}{\sqrt{n}}.$$

We conclude that $\rho = \frac{1}{5e}, \alpha = \frac{1}{2}$ and $l = \sqrt{\frac{5}{2\pi}}$. ▲

Also solved by Albert Stadler, Switzerland and the proposer.

172. *Proposed by Besfort Shala, University of Bristol, United Kingdom.* Let A, B and X be complex $n \times n$ matrices such that

$$X + AXB = AX + XB.$$

Prove that either A or B has 1 as an eigenvalue with multiplicity at least $\dfrac{\text{rank(X)}}{2}$.

Solution by Michel Bataille, Rouen, France. We denote by I and O, the unit and zero $n \times n$ matrices respectively and we assume that $X \neq O$.

First, we recall two results about ranks. Let M, N, P be complex $n \times n$ matrices. Then we have:
(1) $\text{rank}(MNP) + \text{rank}(N) \geq \text{rank}(MN) + \text{rank}(NP)$ (Frobenius' inequality)
(2) $\text{rank}(MP) \geq \text{rank}(M) + \text{rank}(P) - n$ (Sylvester's inequality, which follows from (1) by taking $N = I$).
Secondly, we remark that the given condition on A, X, B is equivalent to $(I - A)X(I - B) = O$. Therefore, applying (1) we obtain

$$0 = \text{rank}((I - A)X(I - B) \geq \text{rank}((I - A)X) + \text{rank}(X(I - B)) - \text{rank}(X).$$

But from (2), we have $\text{rank}((I-A)X) \geq \text{rank}(I-A) + \text{rank}(X) - n$ and $\text{rank}(X(I-B)) \geq \text{rank}(X) + \text{rank}(I - B) - n$; we deduce that

$$0 \geq \text{rank}(I - A) + \text{rank}(I - B) + \text{rank}(X) - 2n.$$

Now, the rank-nullity theorem yields

$$0 \geq n - \dim(\ker(I - A)) + n - \dim(\ker(I - B)) + \text{rank}(X) - 2n,$$

that is,

$$\dim(\ker(I - A)) + \dim(\ker(I - B) \geq \text{rank}(X).$$

It follows that at least one of the two integers $\dim(\ker(I - A)), \dim(\ker(I - B)$ is greater than or equal to $\frac{\text{rank}(X)}{2}$, say $\dim(\ker(I-A)) \geq \frac{\text{rank}(X)}{2}$. Then, $\dim(\ker(I-A)) \geq 1$, hence 1 is an eigenvalue of A. Its multiplicity (as a root of the characteristic polynomial), being greater than or equal to $\dim(\ker(I-A))$, is at least $\frac{\text{rank}(X)}{2}$. ▲

Also solved by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria; Moubinool Omarjee, Paris, France and the proposer. One incorrect solution was received.

Vasile Cîrtoaje's comment on problem 164, MathProblems, Vol 6, Issue 4. Moti Levy consumes the first page to prove the known inequality

$$2 \sum_{1 \leq i < j \leq n} a_i a_j \leq n(n-1)A^2.$$

Until (13), Moti proved the desired inequality for $A \leq \frac{1}{n-1}$. Then, he proved the desired inequality also for $A \leq \frac{1}{n-1}$ (see "we have from (13) and (15) that..."). Therefore, the proof for the main case $A \geq \frac{1}{n-1}$ is missing. The case $A \geq \frac{1}{n-1}$ is the main case, because for the equality constraint

$$\sum_{i=1}^{n} \frac{1}{a_i + 1} = n - 1,$$

by the AM-HM inequality

$$\sum_{i=1}^{n} (a_i + 1) \cdot \sum_{i=1}^{n} \frac{1}{a_i + 1} \geq n^2,$$

we get

$$A \geq \frac{1}{n-1}.$$

Solution of problem 164 by the proposers. We may assume WLOG that $0 \leq a_n \leq a_{n-1} \leq \cdots \leq a_1$.
Since

$$\sum_{i=1}^{n} \frac{1}{a_i + 1} \geq n - 1,$$

then

$$\sum_{i=1}^{n} a_i(a_i + 1) \leq 1.$$

This combined with the Cauchy-Bunyakovsky-Schwarz inequality gives:

$$\sum_{i=1}^{n} a_i(a_i + 1) \geq \left(\sum_{i=1}^{n} a_i(a_i + 1) \right)^2 \geq p^2.$$

Note that

$$\sum_{i=1}^{n} a_i(a_i + 1) \geq p^2$$

becomes an equality for either $a_1 = \cdots = a_n > 0$ or $a_1 = \cdots = a_k > 0$ and $a_{k+1} = \cdots = a_n = 0$ for $k = 2, 3, \ldots, n-1$.

Further,

$$\sum_{i=1}^{n} a_i(a_i+1) = 1$$

is equality for

$$\sum_{i=1}^{n} a_i(a_i+1) = p^2.$$

Thus, we have equality when $a_1 = \cdots = a_n = \frac{1}{n-1}$ or $a_1 = \cdots = a_k = \frac{1}{k-1}$ and $a_{k+1} = \cdots = a_n = 0$ for $k = 2, 3, \ldots, n-1$.
Obviously, $\sum_{i=1}^{n} a_i(a_i+1) \geq p^2$ is equivalent to $p \geq 2q$.
Let $p = \sum_{i=1}^{n} a_i$ and $q = \sum_{1\leq i<j\leq n} a_i a_j$, and assume that p and q have fixed values.
First, let $p < \frac{n}{n-1}$. Since $(p/n)^2 \geq \frac{2q}{n(n-1)}$, we have

$$\frac{n}{2q} \geq \frac{n^2}{(n-1)p^2}.$$

It suffices to prove that

$$(n-2)p+\frac{n^2}{(n-1)p^2} \geq \frac{2n^2-4n+1}{n-1} \quad\iff\quad (n-(n-1)p)(n+(n-1)p-(n-2)p^2) \geq 0,$$

which is true and strict for $0 < p < \frac{n}{n-1}$.
Now, let $p \geq \frac{n}{n-1}$. Since we fixed $\sum_{i=1}^{n} a_i$ and $\sum_{1\leq i<j\leq n} a_i a_j$, according to the Cirtoaje EV Theorem Corollary 1.4 for the function $\gamma \to \frac{1}{\gamma+1}$, the maximum value of $\sum_{i=1}^{n} \frac{1}{a_i+1}$ occurs for either $a_n = 0$ or $0 < a_n \leq a_{n-1} = \cdots = a_1$ $(*)$.
Case 1: We assume that there exist $0 < y \leq x$ such that $p = (n-1)x + y$ and $2q = (n-1)x((n-2)x + 2y)$. Since $p \geq \frac{n}{n-1}$, then $x \geq \frac{1}{n-1}$.
We get $\frac{n-1}{x+1} + \frac{1}{y+1}$. Hence

$$\frac{n-1}{x+1} + \frac{1}{y+1} \geq \sum_{i=1}^{n} \frac{1}{x_i+1} \geq n-1.$$

So, we have to prove that

$$(n-2)((n-1)x+y) + \frac{n}{(n-1)x((n-2)x+2y)} \geq \frac{2n^2-4n+1}{n-1}.$$

Since $\frac{n-1}{x+1} + \frac{1}{y+1} \geq n-1$, then $y \leq \frac{1-(n-2)x}{(n-1)x}$.
Thus $y \leq \min\left(x, \frac{1-(n-2)x}{(n-1)x}\right) = \frac{1-(n-2)x}{(n-1)x} =: m_x$.
Consider the function $y \to (n-2)((n-1)x+y) + \frac{n}{(n-1)x((n-2)x+2y)} : (0, m_x] \to \mathbb{R}$.
We prove that its first derivative is negative, i.e.

$$(n-2)(n-1)x((n-2)x+2y)^2 < 2n \quad \text{for} \quad y \in (0, m_x].$$

$$\frac{1}{n-1} \leq x < \frac{1}{n-2} \quad\implies\quad ((n-2)x+2y)^2 \leq \left((n-2)x + 2\cdot\frac{1-(n-2)x}{(n-1)x}\right)^2.$$

Note that the function $x \mapsto (n-2)x+2\frac{1-(n-2)x}{(n-1)x} : [\frac{1}{n-1}, \frac{1}{n-2}] \to \mathbb{R}$ is strictly convex, so its maximum is $\frac{n}{n-1}$.
Hence, $(n-2)(n-1)x((n-2)x+2y)^2 \leq n^2(n-2)x/(n-1)$ for $y \in (0, m_x]$.

But

$$n^2(n-2)x/(n-1)x < \frac{n^2}{n-1} < 2n \quad\Longrightarrow\quad (n-2)((n-1)x+y)+\frac{n}{(n-1)x((n-2)x+2y)} \geq$$

$$\geq (n-2)\cdot\frac{(n-1)^2x^2-(n-2)x+1}{(n-1)x}+\frac{n}{(n-1)(n-2)x^2-2(n-2)x+2}.$$

We now prove that if $x \in \left[\frac{1}{n-1}, \frac{1}{n-2}\right)$, then

$$(n-2)\cdot\frac{(n-1)^2x^2-(n-2)x+1}{(n-1)x}+\frac{n}{(n-1)(n-2)x^2-2(n-2)x+2} \geq \frac{2n^2-4n+1}{n-1}.$$

We denote $(n-1)x = z \in \left[1, \frac{n-1}{n-2}\right)$. So the latter reduces to:

$$(z-1)^2(n-1-(n-2)z)(2-z) \geq 0,$$

which is true , with equality iff $z = 1$, i.e., $a_1 = \cdots = a_n = \frac{1}{n-1}$.
Case 2: $a_n = 0$.
So

$$p = \sum_{i=1}^{n-1} a_i \quad \text{and} \quad 2q = 2\sum_{1\leq i<j\leq n-1} a_ia_j.$$

By $(*)$, we have

$$\sum_{i=1}^{n-1}\frac{1}{a_i+1} \geq n-2 \quad \text{so} \quad \sum_{i=1}^{n-1}\frac{a_i}{a_i+1} \leq 1.$$

Case 2a: $p \geq \frac{n-1}{n-2}$.
We need to show:

$$(n-2)\sum_{i=1}^{n-1} a_i + \frac{n}{2\sum_{1\leq i<j\leq n-1} a_ia_j} \geq \frac{2n^2-4n+1}{n-1}.$$

We proved $p \geq 2q \implies n \cdot 2q \geq np$.
Hence, it suffices to prove:

$$(n-2)p + n/p \geq \frac{2n^2-4n+1}{n-1} \iff$$

$$((n-1)p-n)((n-2)p-(n-1)) \geq 0,$$

which is true. Equality holds if and only if $p = \frac{n-1}{n-2}$.
This, combined with the above, gives equality if and only if $a_1 = \cdots = a_{n-1} = \frac{1}{n-2}$ and $a_n = 0$.
We are done.
Case 2b: $\frac{n}{n-1} \leq p < \frac{n-1}{n-2}$.
We prove that:

$$(n-2)p + n/(2q) > \frac{2n^2-4n+1}{n-1}.$$

Since:

$$(n-2)p^2 \geq 2(n-1)q,$$

then:

$$2q \leq (n-1)(n-2)p^2 \implies$$

$$(n-2)p + n/(2q) \geq (n-1)(n-2)p + \frac{n}{(n-1)(n-2)p^2}.$$

So it remains to prove:

$$(n-1)(n-2)p+\frac{n}{(n-1)(n-2)p^2}>\frac{2n^2-4n+1}{n-1}.$$

We denote $n-2p=z\in(0,1)$. So the latter reduces to:

$$(z-1)((n-1)^2z^2-n(n-2)z-n(n-2))>0,$$

which is true since the positive root of the quadratic is greater than 1.
The proof is complete. ▲

Reference.

`https://www.emis.de/journals/JIPAM/images/059_06_JIPAM/059_06_www.pdf#page=5`

MATHCONTEST SECTION

This section of the Journal offers readers an opportunity to solve interesting and elegant mathematical problems that have appeared in Math Contests around the world and that are most appropriate for undergraduate Math Olympiad training. Proposals are always welcome. The source of the proposals will appear when the solutions are published.

Proposals

120. Let $g:\mathbb{R}\to\mathbb{R}$ be continuous and such that $g(0)=0$ and $g(x)g(-x)>0$ for any $x>0$. Find all solutions $f:\mathbb{R}\to\mathbb{R}$ to the functional equation

$$g(f(x+y))=g(f(x))+g(f(y)),\ \text{for all } x,y\in\mathbb{R}.$$

121. Let $A=\{1,2,\ldots,n\}$. Prove that if $n>11$ then there is a bijective map $f:A\to A$ with the property that, for every $a\in A$, exactly one of $f(f(f(f(a))))=a$ and $f(f(f(f(f(a)))))=a$ holds.

122. A *magic square* of size n is an $n\times n$ array of real numbers such that all the rows, all the columns and the two main diagonals have the same sum. Determine the dimension, over $\mathbb{R}$, of the vector space of $n\times n$ magic squares.

123. For $u,v\in\mathbb{R}^4$, let $\langle u,v\rangle$ denote the usual dot product. Define a *vector field* to be a map $\omega:\mathbb{R}\to\mathbb{R}$ such that $\langle\omega(z),z\rangle=0$ for all $z\in\mathbb{R}^4$.
Find a maximal collection of vector fields $\{\omega_1,...,\omega_k\}$ such that the map Ω sending z to $\lambda_1\omega_1(z)+\cdots+\lambda_k\omega_k(z)$, with $\lambda_1,\ldots,\lambda_k\in\mathbb{R}$, is nonzero on $\mathbb{R}^4\backslash\{0\}$ unless $\lambda_1=\cdots=\lambda_k=0$.

124. For a continuously differentiable function $f:[0,1]\to\mathbb{R}$ with $f(1/2)=0$, show that

$$\left(\int_0^1 f(x)\mathrm{d}x\right)^2\le\frac{1}{4}\int_0^1\left(f'(x)\right)^2\mathrm{d}x.$$

Solutions

115. Find the minimum value of $2\sin x + 3\sin y - \cos(x-y)$.

(Training for the Kosovo Mathematical Olympiad)

Solution by Michel Bataille, France. Let $f(x,y) = 2\sin x + 3\sin y - \cos(x-y)$. We show that the minimum value of $f(x,y)$ is -6.
Since $\sin x \geq -1$ and $\sin y \geq -1$, we have $2\sin x + 3\sin y \geq -5$. Since $\cos(x-y) \leq 1$, we have $-\cos(x-y) \geq -1$. By addition, we deduce that $f(x,y) \geq -6$ for all $x, y \in \mathbb{R}$. Since $f\left(-\frac{\pi}{2}, -\frac{\pi}{2}\right) = -6$, we conclude that -6 is the minimum value of $f(x,y)$. ▲

Also solved by Albert Stadler, Switzerland.

116. Calculate

$$\lim_{n\to\infty} \frac{n!\left(1+\frac{1}{n}\right)^{n^2+n}}{n^{n+\frac{1}{2}}}.$$

(Jozsef Wildt IMC 2023)

Solution by Michel Bataille, France. Let $U_n = \frac{n!\left(1+\frac{1}{n}\right)^{n^2+n}}{n^{n+\frac{1}{2}}}$. Then we have

$$\ln(U_n) = \ln(n!) + (n^2+n)\ln\left(1+\frac{1}{n}\right) - \left(n+\frac{1}{2}\right)\ln(n).$$

We know that

$$\ln(n!) = n\ln(n) - n + \frac{\ln(n)}{2} + \ln(\sqrt{2\pi}) + o(1)$$

as $n \to \infty$. We deduce that

$$\ln(U_n) = -n + n^2\left(1+\frac{1}{n}\right)\ln\left(1+\frac{1}{n}\right) + \ln(\sqrt{2\pi}) + o(1).$$

Since $\ln\left(1+\frac{1}{n}\right) = \frac{1}{n} - \frac{1}{2n^2} + o(1/n^2)$, we readily obtain

$$\ln(U_n) = \frac{1}{2} + \ln(\sqrt{2\pi}) + o(1).$$

Thus, $\lim_{n\to\infty} \ln(U_n) = \frac{1}{2} + \ln(\sqrt{2\pi}) = \ln(\sqrt{2\pi e})$ so that

$$\lim_{n\to\infty} U_n = \sqrt{2\pi e}.$$ ▲

Also solved by Albert Stadler, Switzerland and Henry Ricardo, Westchester Area Math Circle.

117. Let f be a nonnegative and non increasing function on $[0,+\infty)$ and let g be a function defined on $[0,+\infty)$ such that $0 \leq g(x) \leq 2023$ with $\int_0^{+\infty} g(x)\,dx = 6069$. Prove that

$$\int_0^{+\infty} f(x)g(x)\,dx \leq 2023\int_0^3 f(x)\,dx.$$

(Jozsef Wildt IMC 2023)

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria. Let $M = 2023$ and $\lambda = 3$ so that $0 \leq g(x) \leq M$ and that $\int_0^{+\infty} g(x)\,dx = \lambda M$. For $0 \leq t \leq \lambda$ we have $(M - g(t))(f(t) - f(\lambda)) \geq 0$. Hence

$$\begin{aligned} M\int_0^\lambda f(t) &\geq \int_0^\lambda f(t)g(t)dt + f(\lambda)\left(\lambda M - \int_0^\lambda g(t)dt\right) \\ &= \int_0^\lambda f(t)g(t)dt + f(\lambda)\left(\int_0^\infty g(t)dt - \int_0^\lambda g(t)dt\right) \\ &= \int_0^\lambda f(t)g(t)dt + f(\lambda)\left(\int_\lambda^\infty g(t)dt\right) \\ &= \int_0^\lambda f(t)g(t)dt + \int_\lambda^\infty f(\lambda)g(t)dt \\ &\geq \int_0^\lambda f(t)g(t)dt + \int_\lambda^\infty f(t)g(t)dt \\ &= \int_0^\infty f(t)g(t)dt. \end{aligned}$$

▲

118. Let $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = af'(-x) + b, \quad \forall x \in \mathbb{R}.$$

(Jozsef Wildt IMC 2023)

Solution by Michel Bataille, Rouen, France. Suppose that f satisfies the conditions. Then, for all x we have $f(-x) = af'(x) + b$, hence

$$f'(x) = \frac{1}{a}\left(f(-x) - b\right). \tag{1}$$

Since $x \mapsto \frac{1}{a}\left(f(-x) - b\right)$ is differentiable, f' is differentiable as well and (1) gives

$$f''(x) = \frac{1}{a}\left(-f'(-x)\right) = \frac{1}{a}\left(\frac{1}{a}\left(b - f(x)\right)\right) = \frac{b}{a^2} - \frac{1}{a^2}f(x).$$

Therefore f is a solution of the differential equation $y'' + \frac{1}{a^2}\,y = \frac{b}{a^2}$. We deduce that for some real numbers A, B,

$$f(x) = b + A\cos(x/a) + B\sin(x/a).$$

Such a function satisfies $f'(x) = -\frac{A}{a}\sin(x/a) + \frac{B}{a}\cos(x/a)$, so we must have

$$b + A\cos(x/a) + B\sin(x/a) = A\sin(x/a) + B\cos(x/a) + b$$

for all x. Taking $x = 0$ shows that we must have $A = B$ so that $f(x) = b + A(\cos(x/a) + \sin(x/a))$.

Conversely, if A is any real number, the function $f : x \mapsto b + A(\cos(x/a) + \sin(x/a))$ is differentiable and satisfies $f(x) = af'(-x) + b$ as it is readily checked.

In conclusion the solutions are the functions $x \mapsto b + A(\cos(x/a) + \sin(x/a))$ where A is a real number. ▲

Also solved by Albert Stadler, Switzerland; Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria.

119. Let $n > 1$ and $A, B \in M_n(\mathbb{C})$ such that $f(z) = e^{zA}Be^{-zA}$ is bounded for all $z \in \mathbb{C}$. Compute

$$C = AB - BA \quad \text{and} \quad D = A^2B + BA^2.$$

(Jozsef Wildt IMC 2023)

Solution by Omri Simon, University of Bristol, United Kingdom. The map $f : \mathbb{C} \to M_n(\mathbb{C})$ is analytic, thus all of its components are entire functions $\mathbb{C} \to \mathbb{C}$. Since f is bounded, so are all of its components. By Liouville's theorem, we conclude that f is constant, thus all derivatives of f are 0. Note that $f'(z) = e^{zA}ABe^{-zA} - e^{zA}Be^{-zA}A$, therefore $C = f'(0) = AB - BA = 0$. It follows that A and B must commute, which means that $e^{zA} = \sum_{k=0}^{\infty} z^k A^k/k!$ and B commute. Therefore $f(z) = B$, which is indeed bounded for any choice of commuting A and B. This means that $D = 2A^2B$ cannot be determined in general. ▲

MATHNOTES SECTION

The best constant for some algebraic inequalities

Mihály Bencze and Marius Drăgan

Abstract. The purpose of this article is to determine the best constant for inequalities of the form

$$k \leq f(a, b, c), \qquad \text{or} \tag{1}$$

$$k \geq f(a, b, c), \tag{2}$$

where a, b, and c are positive numbers, and f is a rational, symmetric, and homogeneous function.

1. Introduction

In the following, we denote:

$$s = x + y + z, \quad p = xyz, \quad \alpha = (x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 9,$$

$$t = \frac{(x + y + z)^3}{xyz} \geq 27.$$

Lemma. *With the above notation, prove the equalities:*

1) $$\sum x^2 = \frac{s^3 - 2p\alpha}{s} \tag{3}$$

2) $$\sum x^3 = s^3 - 3\alpha p + 3p \tag{4}$$

3) $$\sum x^4 = \frac{(s^3 - 2\alpha p)^2}{s^2} - \frac{2\alpha^2 p^2}{s^2} + 4sp \tag{5}$$

4) $$\sum (x + y)^4 = s^4 - 12sp + \sum x^4 \tag{6}$$

5) $$\prod (y + z) = p\alpha - p \tag{7}$$

6) $$\sum xy(x + y) = p\alpha - 3p \tag{8}$$

7) $$\sum \frac{x}{y + z} = \frac{s^3 - 2\alpha p + 3p}{(\alpha - 1)p} \tag{9}$$

8) $$\sum \frac{x}{2x + y + z} = \frac{s^3 + 2\alpha p + 3p}{2s^3 + \alpha p + p} \tag{10}$$

9) $$\prod (x - y)^2 = (\alpha^2 + 18\alpha - 27)p^2 - \frac{4\alpha^3 p^3}{s^3} - 4ps^3 \tag{11}$$

10) $$\sum x(y - z)^2 = p(\alpha - 9) \tag{12}$$

11) $$\sum (x - y)^4 = \frac{2s^6 - 12\alpha p s^3 + 18\alpha^2 p^2}{s^2} \tag{13}$$

Proof.

1) $\sum x^2 = \left(\sum x\right)^2 - 2\sum xy = s^2 - 2\frac{p\alpha}{s} = \frac{s^3-2p\alpha}{s}$

2) $\sum x^3 = s\sum x^2 - \sum xy \sum x + 3xyz = s^3 - 3\alpha p + 3p$

3) $\sum x^4 = \left(\sum x^2\right)^2 - 2\left[\left(\sum xy\right)^2 - 2xyz\cdot\sum x\right] =$

$$= \frac{(s^3-2\alpha p)^2}{s^2} - \frac{2p^2\alpha^2}{s^2} + 4sp$$

4) $\sum(x+y)^4 = \sum(s-z)^4 = \sum(s^4 - 4s^3z + 6z^2s^2 - 4sz^3 + z^4) =$

$$= 3s^4 - 4s^4 + 6s^2\frac{s^3-2\alpha p}{s} - 4s(s^3 - 3\alpha p + 3p) + \sum z^4 = s^4 - 12ps + \sum x^4$$

5) $\prod(y+z) = \prod(s-x) = s^3 - s\cdot s^2 + \sum xys - p = p\alpha - p$

6) $\sum xy(x+y) = \prod(y+z) - 2p = p\alpha - 3p$

7) $\sum \frac{x}{y+z} = \frac{\sum x(s-y)(s-z)}{\prod(y+z)} = \frac{s\sum x^2 + 3xyz}{\prod(y+z)} = \frac{s^3-2\alpha p+3p}{(\alpha-1)p}$

8) $\sum \frac{x}{2x+y+z} = \frac{\sum x(s+y)(s+z)}{\prod(s+x)} =$

$$\frac{\sum x(s^2 + s(s-x) + yz)}{2s^3 + s\sum yz + xyz} = \frac{2s^3 - s\sum x^2 + 3xyz}{2s^3 + \alpha p + p} = \frac{s^3 + 2\alpha p + 3p}{2s^3 + \alpha p + p}$$

9) We denote $A = x^2y + y^2z + z^2x$, $B = xy^2 + yz^2 + zx^2$.
We have $A + B = p\alpha - 3p$.

$$A\cdot B = \sum_{\text{cyclic}} x^2y + \sum_{\text{cyclic}} xy^2 = \sum x^3y^3 + 3x^2y^2z^2 + xyz\sum x^3 =$$

$$\frac{p^3\alpha^3}{s^3} - 6\alpha p^2 + 9p^2 + ps^3.$$

We have

$$\left[\prod(x-y)\right]^2 = (A-B)^2 = (A+B)^2 - 4AB = (\alpha^2 + 18\alpha - 27)p^2 - \frac{4p^3\alpha^3}{s^3} - 4ps^3$$

10) $\sum_{\text{cyclic}} x(y-z)^2 = \sum x\left(\sum x^2 - x^2 - 2yz\right) = \sum x \sum x^2 - \sum x^3 - 6xyz =$

$$= p(\alpha - 9)$$

11) $\sum(x-y)^4 = 2\left(\sum x\right)^4 - 12\sum xy\left(\sum x\right)^2 + 18\left(\sum xy\right)^2 =$

$$= \frac{2s^6 - 12\alpha ps^3 + 18\alpha^2p^2}{s^2}$$

From Schur's inequality, we have

$$s^3 + 9p \geq 4\sum xy \quad \text{or} \quad t = \frac{s^3}{p} \geq 4\alpha - 9 = t_3.$$

Also from Schur's inequality

$$\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)^3 + \frac{9}{xyz} \geq 4\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right),$$

$$\text{or} \quad \frac{\left(\sum xy\right)^3}{p^3} + \frac{9}{p} \geq 4\frac{\alpha}{s}\frac{s}{p} \quad \text{or} \quad t \leq \frac{\alpha^3}{4\alpha - 9} = t_4.$$

So $t \in [t_3, t_4]$.

In the following, we will find the best bounds for t where we fix $\alpha \geq 9$.
It is known that if x, y, z are the positive roots of the equation $x^3 - sx^2 + Rx - p = 0$, where $s = \sum x,\ R = \sum xy,\ p = xyz$, then

$$18sRp - 4s^3p + s^2R^2 - 4R^3 - 27p^2 \geq 0 \quad \text{or}$$

$$18p^2\alpha - 4s^3p + p^2\alpha^2 - \frac{4p^3\alpha^3}{s^3} - 27p^2 \geq 0 \quad \text{or}$$

$$4t^2 - (\alpha^2 + 18\alpha - 27)t + 4\alpha^3 \leq 0 \quad \text{or} \quad t \in [t_1, t_2],$$

where $\ t_{1,2} = \frac{\alpha^2+18\alpha-27\pm\sqrt{(\alpha-1)(\alpha-9)^3}}{8}$.
We have $\ t_3 \leq t_1 \leq t \leq t_2 \leq t_4$.
Thus, using the above substitutions, inequality (1) may be written as

$$k \leq u(t),\ \forall\, t \in [t_1, t_2]. \tag{14}$$

If $\ t_\alpha$ is the minimum point, then $k \leq u(t_\alpha) \leq u(t), \forall\, t \in [t_1, t_2]$. $\ t_\alpha = t_1$ if u is increasing on $[t_1, t_2]$, $t_\alpha = t_2$ if $\ u$ is decreasing, and t_α is between t_1 and t_2 if $t_\alpha \in (t_1, t_2)$.
So the best constant is $k_0 = \inf\limits_{\alpha \geq 9} u(t_\alpha)$. The inequality (2) may be written as

$$k \geq u(t),\ \forall\, t \in [t_1, t_2]. \tag{15}$$

In the same way, if t_α is the maximum point, $u(t) \leq u(t_\alpha) \leq k$, $\forall\, t \in [t_1, t_2]$ where $t_\alpha = t_2$ if u is increasing, $t_\alpha = t_1$ if f is decreasing on $[t_1, t_2]$, and t_α is between t_1 and t_2 if $t_\alpha \in (t_1, t_2)$.
Thus, the best constant is $t_0 = \sup\limits_{\alpha \geq 9} u(t_\alpha)$.
Using this method, we solve the following problems.

2. Applications

Problem 1. Find the best constant k such that the inequality

$$(a-b)^4 + (b-c)^4 + (c-a)^4 \leq k(a^4 + b^4 + c^4 - abc(a+b+c))$$

holds for all positive numbers a, b, c.

Solution. Using (3) and (13), the inequality in the problem is equivalent to

$$\frac{2s^6 - 12\alpha ps^3 + 18\alpha^2p^2}{s^2} \leq k\left(\frac{(s^3 - 2p\alpha)^2}{s^2} - \frac{2p^2\alpha^2}{s^2} + 3sp\right),$$

or equivalently,

$$f(t) = \frac{2t^2 - 12\alpha t + 18\alpha^2}{(t - 2\alpha)^2 - 2\alpha^2 + 3t} \leq k, \quad \forall\, t \in [t_1, t_2],$$

(an inequality of the type (15)) with

$$f'(t) = \frac{2(t - 3\alpha)\left[(2\alpha + 3)t - (8\alpha^2 - 9\alpha)\right]}{\left[2\alpha^2 - 4\alpha t + t^2 + 3t\right]^2}.$$

Since $3\alpha \leq 4\alpha - 9 \leq t_1 \leq t$, and

$$\frac{8\alpha^2 - 9\alpha}{2\alpha + 3} \leq 4\alpha - 9 \leq t_1 \leq t,$$

for each $\alpha \geq 9$, it follows that f is increasing on $[t_1, t_2]$. Therefore,

$$f(t) \leq f(t_2) \leq k, \quad \forall\, t \in [t_1, t_2],$$

which implies that

$$k \geq \frac{2t_2^2 - 12\alpha t_2 + 18\alpha^2}{(t_2 - 2\alpha)^2 - 2\alpha^2 + 3t_2}, \quad \forall\, \alpha \geq 9,$$

or equivalently,

$$k \geq \frac{(\alpha - 9)\left[\alpha + 3 + \sqrt{(\alpha-1)(\alpha-9)}\right]^2}{\alpha^3 - 3\alpha^2 + 31\alpha - 45 + (\alpha^2 + 2\alpha - 15)\sqrt{(\alpha-1)(\alpha-9)}} = g(\alpha).$$

Thus, the best constant is

$$k_0 = \sup_{\alpha \geq 9} g(\alpha) = 2.$$

Therefore,

$$\sum (a-b)^4 \leq 2\left(\sum a^4 - abc \sum a\right).$$

Since $t \leq \frac{\alpha^3}{4\alpha - 9} = t_3$, we obtain

$$k \geq \frac{2t_3^2 - 12\alpha t_3 + 18\alpha^2}{(t_3 - 2\alpha)^2 - 2\alpha^2 + 3t_3} = \frac{2(\alpha-9)(\alpha-3)^2}{\alpha^3 - 7\alpha^2 + 17\alpha - 18} = h(\alpha).$$

Thus,

$$k \geq k_1 = \sup_{\alpha > 9} h(\alpha) = 2.$$

Problem 2. Find the best constant $\beta \in [3, 6]$ such that the inequality

$$\beta \frac{a^2 + b^2 + c^2}{ab + bc + ca} + (6 - \beta)\frac{ab + bc + ca}{a^2 + b^2 + c^2} \leq \frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}$$

holds for all $a, b, c > 0$.

Solution. The inequality from the statement can be written as

$$\beta \frac{s^3 - 2p\alpha}{p\alpha} + \frac{p\alpha(6-\beta)}{s^3 - 2p\alpha} \leq \alpha - 3 \quad \text{or}$$

$$\frac{\beta(t-2)}{\alpha} + \frac{(6-\beta)\alpha}{t - 2\alpha} - \alpha + 3 \leq 0, \quad \forall\, t \in [t_1, t_2]. \tag{16}$$

We consider the function

$$f : [t_1, t_2] \to \mathbb{R}, \quad f(t) = \frac{\beta(t-2)}{\alpha} + \frac{(6-\beta)\alpha}{t - 2\alpha} - \alpha - 3$$

with

$$f'(t) = \frac{\beta}{\alpha} - \frac{(6-\beta)\alpha}{(t-2\alpha)^2}.$$

$$\begin{aligned} f'(t) &= \beta\left(\frac{1}{\alpha} + \frac{\alpha}{(t-2\alpha)^2}\right) - \frac{6\alpha}{(t-2\alpha)^2} \geq \frac{3}{\alpha} + \frac{3\alpha}{(t-2\alpha)^2} - \frac{6\alpha}{(t-2\alpha)^2} \\ &= \frac{3}{\alpha} - \frac{3\alpha}{(t-2\alpha)^2} = 3\left[\frac{(t-2\alpha)^2 - \alpha^2}{\alpha(t-2\alpha)^2}\right] = \frac{3(t-3\alpha)(t-\alpha)}{\alpha(t-2\alpha)^2} \geq 0, \end{aligned}$$

since $t > 4\alpha - 9$, or $(t-\alpha)(t-\alpha) \geq (3\alpha - 9)(\alpha - 9) \geq 0$, for all $\alpha \geq 9$.
Thus, f is an increasing function.
It follows that $f(t) \leq f(t_2) \leq 0$, for all $t \in [t_1, t_2]$, or

$$\beta\left(\frac{t_2 - 2\alpha}{\alpha} - \frac{\alpha}{t_2 - 2\alpha}\right) \leq \alpha - 3 - \frac{6\alpha}{t_2 - 2\alpha},$$

since $t_2 - 2\alpha \geq \alpha$, we obtain $(t_2 - 2\alpha)^2 \geq \alpha^2$, or

$$\frac{t_2 - 2\alpha}{\alpha} - \frac{\alpha}{t_2 - 2\alpha} \geq 0.$$

But $t_2 \geq 4\alpha - 9$, so

$$\alpha - 3 - \frac{6\alpha}{t_2 - 2\alpha} \geq \alpha - 3 - \frac{6\alpha}{2\alpha - 9} = \frac{(2\alpha - 3)(\alpha - 9)}{2\alpha - 9} > 0.$$

Thus,

$$\beta \leq \left(\alpha - 3 - \frac{6\alpha}{t_2 - 2\alpha}\right) \Bigg/ \left(\frac{t_2 - 2\alpha}{\alpha} - \frac{\alpha}{t_2 - 2\alpha}\right) = g(\alpha) =$$

$$= \frac{8\alpha\left(\alpha^3 - \alpha^2 - 81\alpha + 81 + (\alpha - 3)\sqrt{(\alpha-1)(\alpha-9)^3}\right)}{\left(\alpha^2 - 6\alpha - 27 + \sqrt{(\alpha-1)(\alpha-9)^3}\right)\left(\alpha^2 + 10\alpha - 27 + \sqrt{(\alpha-1)(\alpha-9)^3}\right)}.$$

Thus, the best constant is $\beta = \min_{\alpha \geq 9} g(\alpha) = 4$. Therefore,

$$4\frac{a^2 + b^2 + c^2}{ab + bc + ca} + 2\frac{ab + bc + ca}{a^2 + b^2 + c^2} \geq \frac{a+b}{c} + \frac{b+c}{a} + \frac{c+a}{b},$$

which represents problem 4497, authored by Hhat Tung from Crux V 45/410.

Problem 3. Find the best constant k such that the inequality

$$\left[(x-y)^2 + (y-z)^2 + (z-x)^2\right] \geq k(x-y)^2(y-z)^2(z-x)^2$$

holds true for all positive numbers x, y, z.

Solution. The inequality from the statement can be written as

$$k \leq \frac{\left[\sum (x-y)^2\right]^3}{\prod (x-y)^2} = \frac{8\left[\sum x^2 - \sum xy\right]^3}{\alpha^2 + 18\alpha - 27 - \frac{4\alpha^3}{t} - 4t} =$$

$$= \frac{8\left[\frac{s^3 - 3p\alpha}{s}\right]^3 t}{p^2\left(-4t^2 + (\alpha^2 + 18\alpha - 27)t - 4\alpha^3\right)} = \frac{8(t - 3\alpha)^3}{-4t^2 + (\alpha^2 + 18\alpha - 27)t - 4\alpha^3}.$$

We consider the function $f : [t_1, t_2] \to \mathbb{R}$

$$f(t) = \frac{8(t - 3\alpha)^3}{-4t^2 + (\alpha^2 + 18\alpha - 27)t - 4\alpha^3},$$

$$f'(t) = \frac{8(2t - 9\alpha + 27)(\alpha^2 - 3\alpha - 2t)(3\alpha - t)^2}{(4\alpha^3 - \alpha^2 t - 18\alpha t + 4t^2 + 2t)^2}.$$

If $\alpha \geq 9$, we have

$$t_1 \leq \frac{9\alpha - 27}{2} \leq t_2 \leq \frac{\alpha^2 - 3\alpha}{2}.$$

Thus, if $t \in \left[t_1, \frac{9\alpha - 27}{2}\right)$, we have $f'(t) \leq 0$, and if $t \in \left(\frac{9\alpha - 27}{2}, t_2\right]$, we have $f'(t) \geq 0$.
It follows that $t_\alpha = \frac{9\alpha - 27}{2}$ is a minimum point.

Thus, $f(t) \geq f\left(\frac{9\alpha-27}{2}\right) = 54$. Therefore, $k = 54$ is the best constant

Problem 4. Find the best constant k such that the inequality

$$(a^2+b^2+c^2)(a^2+b^2+c^2-ab-bc-ca) \geq$$
$$\geq k(a+b+c)(a^2b+ab^2+b^2c+bc^2+c^2a+ac^2-6abc)$$

holds true for all positive numbers a, b, c.

Solution. Without loss of generality, we consider $\alpha > 9$. The inequality in the statement can be rewritten in an equivalent form as

$$\begin{aligned} k &\leq \frac{\sum a^2\left(\sum a^2 - \sum ab\right)}{\sum a\left(\sum ab(b+c) - 6abc\right)} \\ &= \frac{\left(\dfrac{s^3-2p\alpha}{s}\right)\left(\dfrac{s^3-2p\alpha}{s} - \dfrac{2p\alpha}{s}\right)}{s(p\alpha - 3p - 6p)} \\ &= \frac{(t-2\alpha)(t-3\alpha)}{t(\alpha-9)} = \frac{1}{\alpha-9}\left(1-\frac{2\alpha}{t}\right)(t-3\alpha). \end{aligned}$$

We define the function $f : [t_1, t_2] \to \mathbb{R}$ as

$$f(t) = \frac{1}{\alpha-9}\left(1-\frac{2\alpha}{t}\right)(t-3\alpha),$$

which is an increasing function, since it is the product of positive increasing functions.
Therefore, $f(t_1) \leq f(t)$, for all $t \in [t_1, t_2]$; thus, $k \leq f(t_1) \leq f(t)$, for all $t \in [t_1, t_2]$.
The best constant is

$$k_0 = \inf_{\alpha>9} f(t_1) = \inf_{\alpha>9} \frac{(t_1-2\alpha)(t_1-3\alpha)}{t_1(\alpha-9)} =$$
$$= \inf_{\alpha>9} \frac{\left(\alpha^2+2\alpha-27-\sqrt{(\alpha-1)(\alpha-9)^3}\right)\left(\alpha^2-6\alpha-27-\sqrt{(\alpha-1)(\alpha-9)^3}\right)}{8(\alpha-9)(\alpha^2+18\alpha-27-\sqrt{(\alpha-9)^3(\alpha-1)})}$$
$$= \sqrt{6}-2 \quad \text{at} \quad a = 3+3\sqrt{6}.$$

Thus, we have

$$(a^2+b^2+c^2)(a^2+b^2+c^2-ab-bc-ca) \geq$$
$$\geq \left(\sqrt{6}-2\right)(a^2b+b^2c+b^2c+bc^2+c^2a+ca^2-6abc).$$

Problem 5. Find the best constant β such that

$$\beta\frac{a^3+b^3+c^3}{abc} + \frac{(8-24\beta)abc}{(a+b)(b+c)(c+a)} \geq 1, \tag{17}$$

holds true for all $a, b, c > 0$.

Solution. The inequality from the statement can be written as

$$\beta(t-3\alpha+3) + \frac{(8-24\beta)}{\alpha-1} \geq 1 \quad \text{for } t \in [t_1, t_2]. \tag{18}$$

But we also have

$$\beta(t-3\alpha+3)+\frac{8-24\beta}{\alpha-1} \geq \beta(t_1-3\alpha+3)+\frac{8-24\beta}{\alpha-1}.$$

From (18), it follows that

$$\beta(t_1-3\alpha+3)+\frac{8-24\beta}{\alpha-1} \geq 1 \quad \text{or} \quad \beta\left(t_1-3\alpha+3-\frac{24}{\alpha-1}\right) \geq \frac{\alpha-9}{\alpha-1}.$$

This gives

$$\beta \geq \frac{\alpha-9}{\alpha-1}\cdot\frac{1}{t_1-3\alpha+3-\frac{24}{\alpha-1}} =$$

$$= \frac{8(\alpha-9)}{-189+\alpha^3-7\alpha^2+3\alpha-(\alpha-1)\sqrt{(\alpha-1)(\alpha-9)^3}}.$$

Thus, the best constant is

$$\beta_0 = \sup_{\alpha>9}\frac{8(\alpha-9)}{\alpha^3-7\alpha^2+3\alpha-189-(\alpha-1)\sqrt{(\alpha-1)(\alpha-9)^3}} \overset{\text{WA}}{=} \frac{17-5\sqrt{5}}{82}.$$

So, the best inequality of the type given in the statement is

$$\left(17-5\sqrt{5}\right)\frac{a^3+b^3+c^3}{abc}+\frac{248+120\sqrt{5}}{(a+b)(b+c)(c+a)} \geq 82.$$

In a similar manner, we search for the reverse inequality of type (17), where the best constant is

$$\beta_1 = \inf_{\alpha>9}\frac{8(\alpha-9)}{\alpha^3-7\alpha^2+3\alpha-189-(\alpha-1)\sqrt{(\alpha-9)^3(\alpha-1)}} = 0.$$

We then obtain

$$\frac{8abc}{\prod(a+b)} \leq 1,$$

which represents the Cesaro inequality.

Problem 6. Find the best constant $\beta \leq \frac{1}{8}$ such that the inequality

$$(3-24\beta)\frac{abc}{a^3+b^3+c^3}+\beta\frac{(a+b)(b+c)(c+a)}{abc} \geq 1$$

holds true for all $a, b, c > 0$.

Solution. The inequality from the statement is equivalent to

$$\frac{3-24\beta}{t-3\alpha+3}+\beta(\alpha-1) \geq 1, \quad \forall\, t \in [t_1, t_2]. \tag{19}$$

But since $t \leq t_2 = \frac{\alpha^2+18\alpha-27+\sqrt{(\alpha-1)(\alpha-9)^3}}{8}$, we obtain

$$\frac{3-24\beta}{t-3\alpha+3}+\beta(\alpha-1) \geq \frac{3-24\beta}{t_2-3\alpha+3}+\beta(\alpha-1).$$

Thus, from (19), we have

$$\frac{3-24\beta}{t_2-3\alpha+3}+\beta(\alpha-1) \geq 1,$$

or

$$\beta\left(\alpha-1-\frac{24}{t_2-3\alpha+3}\right)\geq 1-\frac{3}{t_2-3\alpha+3}\quad \text{or}$$

$$\beta\geq\frac{t_2-3\alpha}{(\alpha-1)t_2-3\alpha^2+6\alpha-27}\quad \text{or}$$

$$\beta\geq\frac{\alpha^2-6\alpha-27+\sqrt{(\alpha-9)^3(\alpha-1)}}{\alpha^3-7\alpha^2+3\alpha-189+(\alpha-1)\sqrt{(\alpha-1)(\alpha-9)^3}}=f(\alpha),\quad \alpha\geq 9.$$

Thus, the best constant is

$$\beta_0=\sup_{\alpha>9}f(x)\simeq 0.100588\quad \text{at}\quad \alpha\simeq 9.06109.$$

Therefore, for all $\beta\geq\beta_0$, the inequality from the statement is true.

Moreover, since $t\leq t_4=\frac{\alpha^3}{4\alpha-9}$, it follows that

$$\beta\geq\frac{t_3-3\alpha}{(\alpha-1)t_3-3\alpha^2+6\alpha-27}=u(\alpha),\quad \forall\,\alpha\geq 9.$$

Thus, $\beta_1=\sup\limits_{\alpha\geq 9}u(\alpha)=\frac{2}{19}$. We then obtain the inequality

$$\frac{9\,abc}{a^3+b^3+c^3}+\frac{2(a+b)(b+c)(c+a)}{abc}\geq 19.$$

Problem 7. Find the best constant k such that the inequality

$$k\left[x(y-z)^2+y(z-x)^2+z(x-y)^2\right]^3\geq xyz\prod(x-y)^2$$

holds true for all positive numbers x,y,z.

Solution. The inequality from the statement can be written as

$$k\geq\frac{\alpha^2+18\alpha-27-\frac{4\alpha^3}{t}-4t}{(\alpha-9)^3}.$$

We consider the function $f:[t_1,t_2]\to\mathbb{R}$ defined as

$$f(t)=\frac{\alpha^2+18\alpha-27-\frac{4\alpha^3}{t}-4t}{(\alpha-9)^3},$$

with

$$f'(t)=\frac{1}{(\alpha-9)^3}\cdot\frac{4(\alpha^3-t^2)}{(\alpha-9)^3}.$$

Since $t_1\leq\sqrt{\alpha^3}\leq t_2$, it follows that $\sqrt{\alpha^3}$ is a maximum point for f.
Thus, $f(t)\leq f\left(\sqrt{\alpha^3}\right)$. But $k\geq f(t)$ for all $t\in[t_1,t_2]$, so $k\geq f\left(\sqrt{\alpha^3}\right)\geq f(t)$.
It follows that

$$k\geq\frac{\alpha^2+18\alpha-27-\frac{4\alpha^3}{t}-4t}{(\alpha-9)^3}=\frac{\left(\sqrt{\alpha}-3\right)^3\left(\sqrt{\alpha}+1\right)}{\left(\sqrt{\alpha}-3\right)^3\left(\sqrt{\alpha}+3\right)^3}=\frac{\sqrt{\alpha}+1}{\left(\sqrt{\alpha}+3\right)^3}.$$

Therefore, the best constant is

$$k_0=\sup_{\alpha\geq 9}\frac{\sqrt{\alpha}+1}{\left(\sqrt{\alpha}+3\right)^3}=\frac{1}{54}.$$

Problem 8. Find the best real constant $u \geq \frac{2}{27}$ such that the inequality

$$u \sum \frac{a}{b+c} + \frac{4-6u}{3} \sum \frac{a}{2a+b+c} \geq 1$$

holds true for all $a, b, c > 0$.

Solution. Since

$$\sum \frac{a}{b+c} = \frac{t-2\alpha+3}{\alpha-1}, \quad \sum \frac{a}{2a+b+c} = \frac{t+2\alpha+3}{2t+\alpha+1},$$

the inequality from the statement is equivalent to

$$f(t) = \frac{u(t-2\alpha+3)}{\alpha-1} + \frac{(4-6u)}{3} \frac{(t+2\alpha+3)}{2t+\alpha+1} \geq 1,$$

with

$$\begin{aligned} f'(t) &= \frac{u}{\alpha-1} + \frac{(6\alpha+10)(3u-2)}{3(2t+\alpha+1)^2} \\ &= \frac{3u(2t+\alpha+1)^2 + (\alpha-1)(6\alpha+10)(3u-2)}{(3\alpha-3)(2t+\alpha+1)^2}. \end{aligned}$$

We have

$$\begin{gathered} 3u(2t+\alpha+1)^2 + (\alpha-1)(6\alpha+10)(3u-2) \geq \\ \geq 3u(9\alpha-17)^2 + (\alpha-1)(6\alpha+10)(3u-2) = \\ = u\left[3(9\alpha-17)^2 + (3\alpha-3)(6\alpha+10)\right] - (2\alpha-2)(6\alpha+10) \geq \\ \geq \frac{2}{9}\left[(9\alpha-17)^2 + (\alpha-1)(6\alpha+10)\right] - (2\alpha-2)(6\alpha+10) \geq 0. \end{gathered}$$

The last inequality is equivalent to

$$(\alpha-9)\left(\alpha-\frac{41}{33}\right) \geq 0,$$

which is true. Therefore, f is an increasing function on $[t_3, t_4]$. Hence, $f(t) \geq f(t_3)$ for all $t \in [t_3, t_4]$, with $t_3 = 4\alpha - 9$. Thus,

$$u\left[\frac{t_3-2\alpha+3}{\alpha-1} - \frac{2(t_3+2\alpha+3)}{2t_3+\alpha+1}\right] \geq 1 - \frac{4}{3} \frac{(t_3+2\alpha+3)}{2t_3+\alpha+1},$$

or

$$u \geq \frac{(\alpha-1)(2t_3-5\alpha-9)}{3\left[2t_3^2+(9-5\alpha)t_3-6\alpha^2-\alpha+9\right]} \quad \text{or}$$

$$u \geq \frac{\alpha-1}{6\alpha-10}, \quad \text{so} \quad u \geq \sup_{\alpha \geq 9} \frac{\alpha-1}{6\alpha-10} = \frac{2}{11} = u_1.$$

Therefore,

$$6 \sum \frac{a}{b+c} + 32 \sum \frac{a}{2a+b+c} \geq 33.$$

In the following, we find the best constant $u \geq \frac{2}{27}$ with the property given in the statement. We have

$$3u(2t+\alpha+1)^2+(\alpha-1)(6\alpha+10)(3u-2) \geq$$
$$\geq 3u(2t_1+\alpha+1)^2+(\alpha-1)(6\alpha+10)(3u-2) \geq$$
$$\geq 3\left[3(2t_1+\alpha+1)^2+(3\alpha-3)(6\alpha+10)\right]-(2\alpha-2)(6\alpha+10) \geq$$
$$\geq \frac{2}{27}\left(3\left(\frac{1}{4}\left(\alpha^2+18\alpha-27-\sqrt{(\alpha-1)(\alpha-9)^3}\right)+\alpha+1\right)^2+\right.$$
$$\left.+3(\alpha-1)(6\alpha+10)\right)-(2\alpha-2)(6\alpha+10) \geq 0 \Leftrightarrow$$
$$\Leftrightarrow (\alpha-1)(\alpha-9)\left(\alpha^2+18\alpha+141-(\alpha+23)\sqrt{(\alpha-1)(\alpha-9)}\right) \geq 0, \quad \forall\, \alpha \geq 9.$$

Thus, we have

$$u \geq \frac{(\alpha-1)(2t_1-5\alpha-9)}{3\left[2t_1^2+(9-5\alpha)t_1-6\alpha^2-\alpha+9\right]} =$$
$$g(\alpha)=\frac{8}{3\left(\alpha+11-\sqrt{(\alpha-1)(\alpha-9)}\right)}, \quad \forall\, \alpha \geq 9.$$

Therefore, the best constant is $u_2 = \sup\limits_{\alpha \geq 9} g(\alpha)$, or $u_2 = 0.166663$. We have

$$F(u)=\frac{u(t-2\alpha+3)}{\alpha-1}+\frac{(4-6u)}{3}\frac{t+2\alpha+3}{2t+\alpha+1}=$$
$$=\left(\frac{t-2\alpha+3}{\alpha-1}-\frac{2(t+2\alpha+3)}{2t+\alpha+1}\right)u+\frac{4(t+2\alpha+3)}{3(2t+\alpha+1)}, \quad \forall\, u \geq \frac{2}{27}.$$

We have

$$(t-2\alpha+3)(2t+\alpha+1)-2(\alpha-1)(t+2\alpha+3)=$$
$$=2t^2+(9-5\alpha)t-6\alpha^2-\alpha+9 \geq 2t_3^2+(9-5\alpha)t_3-6\alpha^2-\alpha+9=$$
$$=2(\alpha-9)(3\alpha-5) \geq 0.$$

It follows that $F : \left[\frac{2}{27}, +\infty\right) \to \mathbb{R}$ is increasing. Thus, $F(u_1) \geq F(u_2) = f(t) \geq 1$, which proves that u_2 is better than u_1. Moreover, u_2 is the best constant since t_1 and t_2 are the best bounds for t.

Problem 9. Find the best constant $\gamma \leq \frac{8}{15}$ such that the inequality

$$(16-27\gamma)(a^4+b^4+c^4)+\gamma(a+b+c)^4 \geq (a+b)^4+(b+c)^4+(c+a)^4$$

holds true for all positive numbers a, b, c.

Solution. We have

$$\sum(a+b)^4=\sum(s-c)^4=\sum\left(s^4-4s^3c+6s^2c^2-4sc^3+c^4\right)=$$
$$=3s^4-4s^4+6s^2\frac{s^3-2p\alpha}{s}-4s(s^3-3\alpha p+3p)+\sum c^4=$$
$$=s^4-12sp+\sum c^4 \quad \text{and}$$

$$\sum c^4 = \left(\sum c^2\right)^2 - 2\sum a^2b^2 = \left(\sum c^2\right)^2 - 2\left(\sum ab\right)^2 + 4abc\sum a =$$
$$= \frac{(cs^3 - 2p\alpha)^2}{s^2} - \frac{2p^2\alpha^2}{s^2} + 4sp.$$

Thus, the inequality from the statement is equivalent to

$$(16 - 27\gamma)\sum a^4 + \gamma s^4 \geq s^4 - 12sp + \sum a^4 \quad \text{or}$$
$$(15 - 27\gamma)\sum a^4 + (\gamma - 1)s^4 + 12sp \geq 0 \quad \text{or}$$
$$(15 - 27\gamma)\left[\frac{(s^3 - 2p\alpha)^2}{s^2} - \frac{2p^2\alpha^2}{s^2} + 4sp\right] + 12sp + (\gamma - 1)s^4 \geq 0,$$

or

$$(15 - 27\gamma)\left[\frac{(t - 2\alpha)^2}{t} - \frac{2\alpha^2}{t} + 4\right] + 12 + (\gamma - 1)t \geq 0,$$
$$(15 - 27\gamma)\frac{(t - 2\alpha)^2}{t} - \frac{2(15 - 27\gamma)\alpha^2}{t} + 60 - 108\gamma + 12 + (\gamma - 1)t \geq 0,$$

or

$$f(t) = (15 - 27\gamma)\frac{(t - 2\alpha)^2}{t} - \frac{2(15 - 27\gamma)\alpha^2}{t} + 72 - 108\gamma + (\gamma - 1)t \geq 0, \quad (20)$$

with

$$f'(t) = \frac{(15 - 27\gamma)(t^2 - 4\alpha^2)}{t^2} + \frac{2(15 - 27\gamma)\alpha^2}{t^2} + \gamma - 1 =$$
$$= \frac{15 - 27\gamma}{t^2}\left(t^2 - 2\alpha^2\right) + \gamma - 1.$$

But since

$$\frac{t^2 - 2\alpha^2}{t^2} \geq \frac{(4\alpha - 9)^2 - 2\alpha^2}{(4\alpha - 9)^2} \geq \frac{7}{9},$$

the last inequality holds for $\alpha \geq 9$. Therefore,

$$f'(t) \geq \frac{7}{9}(15 - 27\gamma) + \gamma - 1 = \frac{4}{3}(8 - 15\gamma) \geq 0.$$

Thus, f is an increasing function, meaning $f(t) \geq f(4\alpha - 9) \geq 0$, or

$$\frac{-2(\alpha - 9)(19\alpha\gamma - 7\alpha - 63\gamma + 27)}{4\alpha - 9} \geq 0,$$

which implies

$$19\alpha\gamma - 7\alpha - 63\gamma + 27 \leq 0,$$

or

$$\gamma(19\alpha - 63) \leq 7\alpha - 27, \quad \forall\, \alpha \geq 9,$$

or

$$\gamma \leq \inf_{\alpha \geq 9} \frac{7\alpha - 27}{19\alpha - 63} = \frac{1}{3}.$$

We obtain $\gamma \leq \frac{1}{3}$. If $\gamma_0 = \frac{1}{8}$, we obtain the inequality

$$7\sum a^4 + \frac{1}{3}\sum(a + b + c)^4 \geq \sum(a + b)^4.$$

Also, from (20), we have $f(t) \geq f(t_1) \geq 0$, or

$$-27\alpha^2 u + 15\alpha^2 + 54\alpha t_1 u - 30\alpha t_1 - 13t_1^2 u + 7t_1^2 - 54t_1 u + 36t_1 \geq 0,$$

or

$$u(27\alpha^2 - 54\alpha t_1 + 13t_1^2 + 54t_1) \le 15\alpha^2 - 30\alpha t_1 + 7t_1^2 + 36t_1,$$

and since $27\alpha^2 - 54\alpha t_1 + 13t_1^2 + 54t_1 \ge 0$, for all $\alpha \ge 9$, it follows that

$$u \le \frac{7\alpha^4 - 92\alpha^3 + 354\alpha^2 - 972\alpha + 1215 - (7\alpha^2 + 6\alpha - 45)\sqrt{(\alpha-1)(\alpha-9)^3}}{13\alpha^4 - 164\alpha^3 + 702\alpha^2 - 2916\alpha + 3645 - (13\alpha^2 + 18\alpha - 135)\sqrt{(\alpha-1)(\alpha-9)^3}} =$$

$$= \frac{(7\alpha - 15)(\alpha^2 - 2\alpha + 9) - (7\alpha^2 + 6\alpha - 45)\sqrt{(\alpha-1)(\alpha-9)}}{13\alpha^3 - 47\alpha^2 + 279\alpha - 405 - (13\alpha^2 + 18\alpha - 135)\sqrt{(\alpha-1)(\alpha-9)}} = g(\alpha),$$

taking into account that the denominator is positive for $\alpha \ge 9$.
Thus, the best constant for $\gamma \le \frac{8}{15}$ is $\gamma_1 = \inf\limits_{\alpha \ge 9} g(\alpha) \overset{\text{WA}}{=} 0.368508$, at $\alpha \simeq 2308.24$, which is better than $\gamma_2 = \frac{1}{3}$, since $\gamma_1 \ge \gamma_2$ and γ_1 is the best constant.

Problem 10. Find the best constant k such that

$$\frac{(4-3k)(x+y)(y+z)(z+x)}{32xyz} + k\left(\frac{x}{2x+y+z} + \frac{y}{x+2y+z} + \frac{z}{x+y+2z}\right) \ge 1$$

holds true for all positive numbers x, y, z.

Solution. From (5) and (8), the inequality from the statement can be written as

$$\frac{4-3k}{32}(\alpha - 1) + k\frac{(t + 2\alpha + 3)}{2t + \alpha + 1} \ge 1 \quad \text{or}$$

$$\frac{\alpha - 1}{9} - 1 + k\left(\frac{t + 2\alpha + 3}{2t + \alpha + 1} - \frac{3\alpha - 3}{32}\right) \ge 0 \quad \text{or}$$

$$\frac{\alpha - 9}{8} + k\frac{(38 - 6\alpha)t + 64\alpha - 3\alpha^2 + 99}{32(2t + \alpha + 1)} \ge 0 \quad \text{or}$$

$$\frac{k\left[(6\alpha - 38)t + 3\alpha^2 - 64\alpha - 99\right]}{2t + \alpha + 1} \le 4(\alpha - 9).$$

But we also have

$$(6\alpha-38)t+3\alpha^2-64\alpha-99 \ge (6\alpha-38)(4\alpha-9)+3\alpha^2-64\alpha-99-27(\alpha-1)(\alpha-9) = 0, \quad \forall\, \alpha \ge 9.$$

Thus, we obtain

$$k \le f(t) = \frac{(4\alpha - 36)(2t + \alpha + 1)}{(6\alpha - 38)t + 3\alpha^2 - 64\alpha - 99}, \quad \text{with}$$

$$f'(t) = \frac{-128(\alpha - 9)(3\alpha + 5)}{(3\alpha^2 + \alpha(6t - 64) - 38t - 99)^2} \le 0.$$

So, f is decreasing on $[t_1, t_2]$, and thus $k \le f(t_2) \le f(t)$ for all $t \in [t_1, t_2]$. Therefore, the best constant is

$$k_0 = \inf_{\alpha \ge 9} f(t_2) = \inf_{\alpha \ge 9} \frac{4\left[(\alpha - 1)(9\alpha + 31) - 8\sqrt{(\alpha-1)(\alpha-9)}\right]}{27\alpha^2 + 114\alpha - 77} =$$

$$= \frac{4}{37}\left(7 + 2\sqrt{3}\right) \simeq 1.13125 \quad \text{at} \quad \alpha = 5 + \frac{8}{\sqrt{3}}.$$

Thus, if $k \le \frac{4}{37}\left(7 + 2\sqrt{3}\right)$, the inequality from the statement is true.

Problem 11. Find the best constant λ such that the inequality

$$\frac{x^3 + y^3 + z^3}{3xyz} + \left[\frac{8xyz}{(x+y)(y+z)(z+x)}\right]^\lambda \ge 2$$

holds true for every positive real numbers x, y, z.
Solution. The inequality from the statement is equivalent to

$$\frac{s^3 - 3p\alpha + 3p}{3p} + \left[\frac{8p}{p\alpha - p}\right]^\lambda \geq 2 \quad \Leftrightarrow \quad \frac{t}{3} - \alpha + 1 + \left(\frac{8}{\alpha - 1}\right)^\lambda \geq 2.$$

Since $f : [t_1, t_2] \to \mathbb{R}$ is increasing, we have $f(t) \geq f(t_1) \geq 2$ for all $t \in [t_1, t_2]$, or

$$\frac{t_1}{3} - \alpha + 1 + \left(\frac{8}{\alpha - 1}\right)^\lambda \geq 2 \quad \Leftrightarrow$$

$$\lambda \leq \frac{\ln\left(\frac{51+6\alpha-\alpha^2+\sqrt{(\alpha-1)(\alpha-9)^3}}{24}\right)}{\ln\left(\frac{8}{\alpha-1}\right)} = g(\alpha), \quad \alpha \in [9, +\infty).$$

Thus, the best constant λ_0 is

$$\lambda_0 = \inf_{\alpha \in [9, +\infty)} g(\alpha) \overset{\text{WA}}{\simeq} 3.84663 \quad \text{at} \quad \alpha \simeq 9.11215.$$

Remark. This problem improves Problem 8 (Bonus Problem) from *Crux Mathematicorum*, June 2021.

JUNIOR PROBLEMS

Solutions to the problems in this issue should arrive before December 26, 2024

Proposals

81. *Proposed by Mihaly Bencze, Braşov, Romania and Neculai Stanciu, Buzău, Romania.* Let x, y and z be positive real numbers. Show that

$$\sum_{\text{cyc}} \frac{2}{x+y} \leq \sum_{\text{cyc}} \frac{8x}{3y^2+2yz+3z^2} \leq \frac{1}{xyz} \sum_{\text{cyc}} x^2.$$

82. *Proposed by George-Florin Şerban, National Pedagogical College "D.P.Perpessiciu", Braila, Romania.* Find all prime numbers p and q such that $p^3 + q^3 - 7pq = 2023$.

83. *Proposed by Fiton Hoxha, Technical University of Munich, Germany.*
A permutation of the numbers $1, 2, \ldots, n$ is given. In each step, if the first element of the permutation is $k \neq 1$, we form a new permutation by moving k to the k-th position. We say that the process terminates if eventually the first element becomes 1.

(a) Show that no matter what initial permutation we start with, the process will always terminate.
(b) Find the maximal number of steps that the process could take before terminating.

Example: For $n = 6$, the process could look like this:

$$361245 \to 613245 \to 132456.$$

84. *Proposed by Dren Neziri, University of Primorska, Slovenia.*
Let $ABCD$ be a trapezoid with $AB \parallel CD$ and let O be the intersection of its diagonals. Let X and Y be points in the circumcircles of triangles OAB and OCD, respectively, with the following properties: (i) we have $OX = OY$, (ii) the points X and O lie on opposite sides with respect to AB, and (iii) the points Y and O lie on opposite sides with respect to CD. Prove that $YC \geq XA$ if and only if $XB \geq YD$.

85. *Proposed by George-Florin Şerban, National Pedagogical College "D.P.Perpessiciu", Braila, Romania.* Prove that the equation $x^{2021} + y^{2022} = z^{2023}$ has infinitely many solutions in positive integers.

No problem is ever permanently closed. We will be very pleased to consider publishing new solutions or comments on previous problems.

Solutions

76. *Proposed by Mihaly Bencze and Neculai Stanciu, Romania.*
Let $ABCDEFGHIJKLM$ be a regular 13-gon. Prove that

$$\frac{AC-AB}{AD-AC}=\frac{AF}{AE}.$$

Solution by Michel Bataille, Rouen, France. Without loss of generality, we assume that the successive vertices $A, B, \ldots, L, M$ of the 13-gon are the points of the complex plane with affixes $\exp(2k\pi i/13)$ $(k=0,1,2,\ldots,12)$. Then, we have

$$AF=|1-\exp(10\pi i/13)|=|\exp(5\pi i/13)(\exp(-5\pi i/13)-\exp(5\pi i/13))|=|-2i\sin(5\pi/13)|,$$

that is, $AF=2\sin(5\pi/13)$.
In the same way, we obtain $AE=2\sin(4\pi/13)$, $AC=\sin(2\pi/13)$, $AB=\sin(\pi/13)$, $AD=\sin(3\pi/13)$. It follows that the equality to be proved is equivalent to

$$2\sin(4\pi/13)\left(\sin(2\pi/13)-\sin(\pi/13)\right)=2\sin(5\pi/13)\left(\sin(3\pi/13)-\sin(2\pi/13)\right). \tag{1}$$

Since $2\sin a\sin b=\cos(a-b)-\cos(a+b)$, the left-hand side L of (1) rewrites as

$$L=\cos(2\pi/13)-\cos(6\pi/13)-\cos(3\pi/13)+\cos(5\pi/13)$$

while the right-hand side R rewrites as

$$R=\cos(2\pi/13)-\cos(8\pi/13)-\cos(3\pi/13)+\cos(7\pi/13).$$

Since $\cos(8\pi/13)=-\cos(\pi-8\pi/13)=-\cos(5\pi/13)$ and similarly, $\cos(7\pi/13)=-\cos(6\pi/13)$, we see that $L=R$ and we are done. ▲

Also solved by the proposers.

77. *Proposed by Titu Zvonaru, Comănești, Romania.*
Let a, b, c be positive real numbers. Prove that

$$\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)\left(\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\right)\geq 3(a^2+b^2+c^2).$$

Solution by the proposer. All sums in the solution are cyclic. By substracting $(\sum a)(\sum(a^2+b^2)/(a+b))$ from both sides, the given inequality is equivalent to

$$\left(\sum\frac{a^2+b^2}{a+b}\right)\left(\sum\frac{a^2}{b}-\sum a\right)\geq\left(\sum a\right)\left(\frac{3\sum a^2}{\sum a}-\sum\frac{a^2+b^2}{a+b}\right). \tag{1}$$

We have

$$\sum\frac{a^2}{b}-\sum a=\sum\left(\frac{a^2}{b}-2a+b\right)=\sum\frac{(a-b)^2}{b}, \tag{2}$$

$$\begin{aligned}\frac{3\sum a^2}{\sum a}-\sum\frac{a^2+b^2}{a+b}&=\sum\left(\frac{\sum a^2}{\sum a}-\frac{b^2+c^2}{b+c}\right)=\sum\frac{ab(a-b)+ac(a-c)}{(b+c)(a+b+c)}\\&=\sum\frac{ab(a-b)}{(b+c)(a+b+c)}+\sum\frac{ac(a-c)}{(b+c)(a+b+c)}\\&=\sum\frac{ab(a-b)}{(b+c)(a+b+c)}+\sum\frac{ab(b-a)}{(c+a)(a+b+c)}\\&=\sum\frac{ab(a-b)(c+a-b-c)}{(a+c)(b+c)(a+b+c)}\\&=\sum\frac{ab(a-b)^2}{(a+c)(b+c)(a+b+c)}\qquad(3)\end{aligned}$$

By (2) and (3) and dividing through by $(a-b)^2$ whenever this is non-zero, it follows that to prove (1) it suffices to show that

$$(a+c)(b+c)\left(\sum\frac{a^2+b^2}{a+b}\right)\geq ab^2.$$

The last inequality is true because the expression on the left-hand side is certainly bigger than $(a+c)(b^2+c^2)>ab^2$. The proof also shows that equality holds if and only if $a=b=c$. ▲

78. *Proposed by Dren Neziri, University of Primorska, Slovenia.* Let BC be a fixed chord of a fixed circle ω. Let A be a variable point in the major arc $\widehat{BC}$. Define

$$F(A)=\begin{cases}\text{point on the side } AC \text{ such that } BA+AM=MC \text{ if } AB\leq AC;\\\text{point on the side } AB \text{ such that } BM=MA+AC \text{ if } AB>AC.\end{cases}$$

Find the perimeter, in terms of BC and $\angle BAC$, of the locus of $F(A)$ as A moves along major arc $\widehat{BC}$.

Solution by the proposer. Let N be the midpoint of major arc $\widehat{BC}$ and P the midpoint of BC. We will check only the case when $AB>AC$. The other one is similar.

Let's denote $F(A)$ with M and let X be a point in ray BA, not in segment AB such that $AX=AC$. Then $BM=MA+AC=MA+AX=MX$, so M is the midpoint of BX. Since $AC=AX$, it follows that $\angle ACX=\angle AXC$ and $\angle ACX+\angle AXC=\angle BAC$, so $\angle BXC=\angle AXC=\frac{\angle BAC}{2}$. As A moves along the major arc $\widehat{BC}$ the angle $\angle BAC$ remains constant, so angle $\angle BXC$ also remains constant. Hence as A moves along the arc $\widehat{CN}$, point X moves along an arc of some circle. Since $\angle BNC=2\angle BXC$ and $BN=NC$, it follows that N is the center of the that circle. By applying homothety with factor $1/2$ around B, since M is the midpoint of BX we have that M moves along the circle (call it Ω) that has the midpoint of BN (call it O) as the center and $\frac{BN}{2}$ as the length of radius.

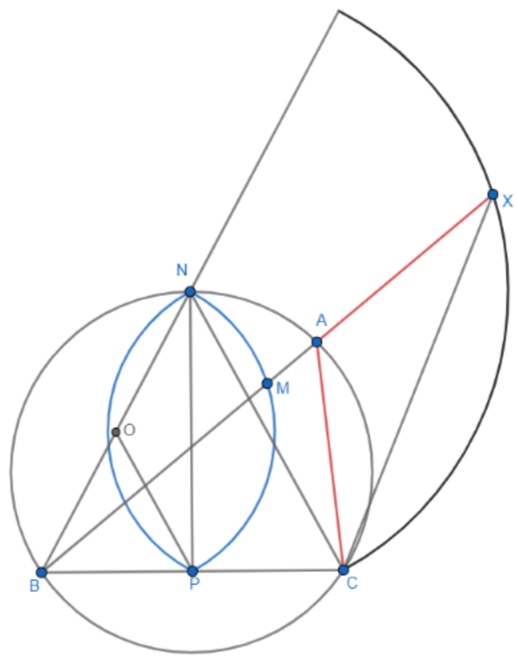
Since we are checking the movement of M as A moves from C to N, by continuity we have that M moves along the arc $\widehat{PN}$ of Ω. Similarly we get the result for the case when A moves from N to B. Now we compute the perimeter of this locus.

The diameter of Ω is equal to

$$BN = \frac{BP}{\sin(\frac{\angle BAC}{2})} = \frac{BC}{2\sin(\frac{\angle BAC}{2})}$$

and $\angle NOP = 2\angle NBC = 180° - \angle BAC$. Computing it for the two cases together we get

$$\frac{BC \cdot \pi \cdot (180° - \angle BAC)}{\sin(\frac{\angle BAC}{2}) \cdot 180°}.$$ ▲



79. *Proposed by Besfort Shala, University of Bristol, United Kingdom.*
Let F_n be the Fibonacci sequence given by $F_1 = 1, F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 1$. For a given positive integer d, let $f(d)$ be the least positive integer such that $d \mid F_{f(d)}$. Prove that for coprime positive integers m and n, we have that $f(mn) = f(m)f(n)$ if and only if $f(m)$ and $f(n)$ are coprime.

Solution by Orges Ajvazi, Kosovo. Let us first prove that if $f(mn) = f(m)f(n)$, then $\gcd(f(m), f(n)) = 1$. Note that $mn \mid F_{f(mn)} = F_{f(m)f(n)}$. Suppose that $\gcd(f(m), f(n)) = d$ so that $\text{lcm}(f(m), f(n)) = f(m)f(n)/d$.
It is known that $m \mid F_{f(m)}$ but also that $m \mid F_{f(m)\cdot x}$ for any positive integer x since the Fibonacci sequence is a divisibility sequence. It follows from this that $m \mid F_{f(m)f(n)/d}$ and $n \mid F_{f(m)f(n)/d}$. Since m and n are coprime, it follows that $mn \mid F_{f(m)f(n)/d}$, but by the minimality definition of $f(mn)$, it follows that $d = 1$.
Next let us prove that if $\gcd(f(m), f(n)) = 1$, then $f(mn) = f(m)f(n)$. Let $f(mn) = r$. Since $mn \mid F_r$, we have $m \mid F_r$ and $n \mid F_r$. Since $m \mid F_r$ and $m \mid F_{f(m)}$ and since the Fibonacci sequence is a strong divisibility sequence, it follows that $m \mid \gcd(F_r, F_{f(m)}) = F_{\gcd(r, f(m)} = F_{f(m)}$ (the last equality holds because m does

not divide a Fibonacci term before $F_{f(m)}$). From this we obtain that $f(m) \mid r$. Similarly we get $f(n) \mid r$, from where it follows that $r \geq f(m)f(n)$ (since $f(m)$ and $f(n)$ are coprime). But mn clearly divides $F_{f(m)f(n)}$ (again because the Fibonacci sequence is a divisibility sequence), so we have $r = f(m)f(n)$, as desired. ▲

Also solved by the proposer.

80. *Proposed by Valmir Krasniqi, Institute of Science, Technology, Engineering and Mathematics, Republic of Kosova.* Let $x, y, z > 0$ such that $xyz = 1$. Prove that

$$\sqrt{\frac{x^3+3}{x^2+y+2z}} + \sqrt{\frac{y^3+3}{y^2+z+2x}} + \sqrt{\frac{z^3+3}{z^2+x+2y}} \geq 3.$$

Comment. We have not received any solution for this problem and the author's solution to the problem contained several errors. The problem remains open, and we will accept solutions in the current issue's deadline.