Mathproblems
ISSN: 2217-446X, url: http://www.mathproblems-ks.org
Volume 6, Issue 4 (2024), Pages 635-663

PROBLEMS AND SOLUTIONS

Proposals and solutions must be legible and should appear on separate sheets, each indicating the name of the sender. It would be preferred to submit your proposals and solutions as `TeX` files. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editors have supplied a solution. The editors encourage undergraduate and pre-college students to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the grade and school name. Solutions will be evaluated for publication by a committee of professors according to a combination of criteria. Questions concerning proposals and/or solutions can be sent by e-mail to: `mathproblems-ks@hotmail.com`.

Solutions to the problems in this issue should arrive before May 30, 2024

Problems

166. *Proposed by Mohsen Soltanifar, University of Toronto, Canada.*
Let X and Y be two continuous real valued random variables with strictly monotone cumulative distribution functions F_X, F_Y, respectively.

(i) Given $a \in [0,1]$, find functions $g, h : \mathbb{R} \to \mathbb{R}$ such that

$$\lim_{t \to +\infty} F_{X,Y}(g(t), h(t)) = a.$$

(ii) How many functions g, h satisfy the condition in part (i)?

167. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Calculate

$$\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+2} - \cdots \right) \left(\frac{1}{n^2} - \frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} - \cdots \right).$$

168. *Proposed by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italy.* Evaluate

$$\int_0^{\frac{\pi}{2}} \frac{\cos(2x)(1-\ln\cos x)}{x^2+(1-\ln\cos x)^2}dx.$$

169. *Proposed by Vasile Cîrtoaje, Petroleum-Gas University of Ploiesti, Romania.* Let a, b, c, d, e be real numbers such that $a \geq b \geq c \geq d \geq e \geq 0$. Prove that

$$\left(\frac{ab+bc+cd+de+ea}{5}\right)^{1/2} \geq \left(\frac{abc+bcd+cde+dea+eab}{5}\right)^{1/3}.$$

170. *Proposed by Stănescu Florin, Şerban Cioculescu School, Găeşti, Romania.* Let $f : [0,1] \to \mathbb{R}$ be a three-times continuously differentiable function such that $f(0) = f'(0) = f(1)$ and $\left|f^{(3)}(t)\right| \leq 1$ for all $t \in [0,1]$. Prove

(i) $\quad |f(x)| \leq \dfrac{x(1-x)}{\sqrt{3}} \left(\displaystyle\int_0^x \frac{|f(t)|}{t(1-t)}dt\right)^{1/2}$ for all $x \in [0,1]$;

(ii) $\quad |f'(x)| \leq \dfrac{1-2x}{\sqrt{3}} \left(\displaystyle\int_0^x \frac{|f(t)|}{t(1-t)}dt\right)^{1/2} + \dfrac{x(1-x)}{6}$ for all $x \in [0,1/2]$;

(iii) $\quad 9\displaystyle\int_0^1 \left(\frac{f(x)}{x}\right)^2 dx \leq \int_0^1 (1-x)^2 \frac{|f(x)|}{x}\, dx.$

171. *Proposed by Michel Bataille, Rouen, France.* Find all real numbers ρ, α, ℓ with $\ell \neq 0$ such that

$$\lim_{n\to\infty} n^\alpha \rho^n \sum_{k=0}^{n} \binom{2n-2k}{n-k} \frac{(5n)^k}{k!} = \ell.$$

172. *Proposed by Besfort Shala, University of Bristol, United Kingdom.* Let A, B and X be complex $n \times n$ matrices such that

$$X + AXB = AX + XB.$$

Prove that either A or B has 1 as an eigenvalue with multiplicity at least $\dfrac{\text{rank(X)}}{2}$.

No problem is ever permanently closed. We will be very pleased to consider publishing new solutions or comments on previous problems.

Solutions

159. *Proposed by Mohsen Soltanifar, University of Toronto, Canada.*
Let $n \geq 1$ and consider discrete uniform probability distribution on the set $S_n = \{1, 2, \ldots, n\}$ and for given $0 < p < 1$, $N(p, n)$ be the number of events of S_n with probability p. Define a function $f : (0, 1) \to \mathbb{R} \cup \{-\infty\}$ via:

$$f(p) = \limsup_{n\to\infty} \Big(\frac{\log_2(N(p, n))}{n}\Big).$$

Compute the maximum value of the function f. (Corrected version)

Solution by Moti Levy, Rehovot, Israel. We have

$$N(p, n) = \begin{cases} \binom{n}{pn} = \frac{n!}{(pn)!((1-p)n)!} & \text{if } pn \text{ is a nonnegative integer,} \\ 0 & \text{otherwise.} \end{cases}$$

It follows that the limit $\lim_{n\to\infty}\left(\frac{\log_2(N(p,n))}{n}\right)$ does not exist if p is a rational number. Moreover, we have $\lim_{n\to\infty}\left(\frac{\log_2(N(p,n))}{n}\right) = -\infty$ if p is an irrational number. Therefore, we rephrase the problem as follows:

$$f(p) = \limsup_{n\to\infty}\left(\frac{\log_2(N(p, n))}{n}\right).$$

The asymptotic formula for $\ln(n!)$ is well known to be

$$\ln(n!) = n(\ln(n) - 1) + O(\ln(n)).$$

Therefore

$$\begin{aligned} \ln\left(\binom{n}{pn}\right) &= \ln(n!) - \ln((pn)!) - \ln(((1-p)n)!) \\ &= n(\ln(n) - 1) - pn(\ln(pn) - 1) - (1-p)n(\ln((1-p)n) - 1) + O(\ln(n)) \\ &= -((1-p)\ln(1-p) + p\ln(p)) + O(\ln(n)), \end{aligned}$$

$$f(p) = \limsup_{n\to\infty}\left(\frac{\log_2(N(p, n))}{n}\right) = -\frac{1}{\ln(2)}((1-p)\ln(1-p) + p\ln(p)),$$

$$\frac{df}{dp} = \frac{1}{\ln 2}(\ln(1-p) - \ln p),$$

$$\frac{d^2 f}{dp^2} = \frac{d\left(\frac{1}{\ln 2}(\ln(1-p) - \ln p)\right)}{dp} = -\frac{1}{p(\ln 2)(1-p)} \leq 0 \quad \text{for} \quad 0 < p < 1.$$

The maximum occurs at $p = \frac{1}{2}$ and the maximum value is 1.

Also solved by the proposer.

160. *Proposed by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy.* Evaluate

$$\int_0^\infty \frac{e^{-x}(1-e^{-2x})(1-e^{-4x})(1-e^{-6x})}{x(1+e^{-14x})}\mathrm{d}x.$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria. Let the considered integral be denoted by I. Clearly we have

$$I = \int_0^\infty \frac{(e^x - e^{-x})(e^{2x}-e^{-2x})(e^{3x}-e^{-3x})}{x(e^{7x}+e^{-7x})}\,dx.$$

But we also have

$$(e^x - e^{-x})(e^{2x}-e^{-2x})(e^{3x}-e^{-3x}) = e^{6x}-e^{-6x}-(e^{4x}-e^{-4x})-(e^{2x}-e^{-2x}).$$

Hence

$$\begin{aligned} I &= \int_0^\infty \frac{e^{6x}-e^{-6x}}{x(e^{7x}+e^{-7x})}\,dx - \int_0^\infty \frac{e^{4x}-e^{-4x}}{x(e^{7x}+e^{-7x})}\,dx - \int_0^\infty \frac{e^{2x}-e^{-2x}}{x(e^{7x}+e^{-7x})}\,dx \\ &= F\left(\frac{6}{7}\right) - F\left(\frac{4}{7}\right) - F\left(\frac{2}{7}\right), \qquad (*) \end{aligned}$$

where

$$F(\alpha) = \int_0^\infty \frac{e^{\alpha t}-e^{-\alpha t}}{t(e^t+e^{-t})}\,dt.$$

Now, for $0 \le \alpha < 1$ we have

$$\begin{aligned} F'(\alpha) &= \int_0^\infty \frac{e^{\alpha t}+e^{-\alpha t}}{e^t+e^{-t}}\,dt = \int_{-\infty}^\infty \frac{e^{\alpha t}}{e^t+e^{-t}}\,dt \\ &= \frac{1}{2}\int_{-\infty}^\infty \frac{e^{(\alpha+1)u/2}}{e^u+1}\,du. \end{aligned}$$

The change of variables $v = 1/(1+e^u)$ shows that

$$F'(\alpha) = \int_0^1 (1-v)^{(\alpha-1)/2} v^{-(\alpha+1)/2}\,dv = \frac{1}{2}B\left(\frac{1+\alpha}{2}, \frac{1-\alpha}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma\left(\frac{1-\alpha}{2}\right)$$

and using the reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$ we conclude that

$$F'(\alpha) = \frac{\pi}{2\sin\left(\frac{\pi}{2}(1-\alpha)\right)} = \frac{\pi}{2\cos(\pi\alpha/2)}.$$

Thus, since $F(0) = 0$, we get by integration

$$F(\alpha) = -\ln\tan\left(\frac{\pi(1-\alpha)}{4}\right),$$

Using this in $(*)$ we obtain

$$I = \ln\left(\cot\left(\frac{\pi}{28}\right)\tan\left(\frac{3\pi}{28}\right)\tan\left(\frac{5\pi}{28}\right)\right),$$

which is the required result. ▲

Also solved by Albert Stadler, Switzerland; Moti Levy, Rehovot, Israel; Michel Bataille, Rouen, France and the proposer.

161. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Find all integer 2×2 matrices satisfying the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} 2^a & 2^b \\ 2^c & 2^d \end{pmatrix}.$$

Solution 1 by Thammadol Tansrivorarat, University of Bristol, United Kingdom. Since

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} 2^a & 2^b \\ 2^c & 2^d \end{pmatrix},$$

it is clear that $a \geq 0, b \geq 0, c \geq 0, d \geq 0$.
Note that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix} = \begin{pmatrix} 2^a & 2^b \\ 2^c & 2^d \end{pmatrix} \qquad (1).$$

Since $b(a+d)$ and $c(a+d)$ are powers of 2, so are b and c. Then bc is a power of 2. Let $bc = 2^x$ for some non-negative integer x. By (1), we must have $a^2 + 2^x = 2^a$, or $a^2 = 2^a - 2^x$.
Claim: It is impossible to have $x < a - 1$.
Proof. Assume that $x < a - 1$. Then, $a^2 = 2^x(2^{a-x} - 1)$, where $2^{a-x} - 1 \equiv -1 \pmod 4$. This is impossible as $\gcd(2^x, 2^{a-x} - 1) = 1$ implies that $2^{a-x} - 1$ is a square of some integer. But this cannot be true since a square must be congruent to 0 or 1 modulo 4.
From the above claim, we have $x = a - 1$ or $x = a$.
Case 1: $x = a$. In this case, $bc = 2^a$ or $a = 0$. Then $bc = 1$ or equivalently, $b = 1$ and $c = 1$. Hence,

$$\begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix} = \begin{pmatrix} 1 & d \\ d & d^2 + 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 2^d \end{pmatrix}.$$

Hence, we have $d = 2$. But this is false since $5 = d^2 + 1 \neq 2^d = 4$.
Case 2: $x = a - 1$. In this case, we have $bc = 2^{a-1}$ and $a^2 = 2^{a-1}$. As $a^2 = 2^{a-1}$, a can only be a power of 2, say $a = 2^u$. So, $2^{2u} = 2^{2^u - 1} \implies 2u = 2^u - 1 \implies u = 0$. So, we have $a = 1$. As $bc = 2^a - 1$, $bc = 1$ which means $b = 1$ and $c = 1$. Then,

$$\begin{pmatrix} 1 & 1 \\ 1 & d \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & d \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2^d \end{pmatrix} \implies d^2 + 1 = 2^d.$$

If $d > 1$, then $d^2 = 2^d - 1 \equiv 3 \pmod 4$, which leads to a contradiction.
If $d = 0$, then

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix},$$

a contradiction.
If $d = 1$, then

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix},$$

satisfying the problem's equation.
After checking all cases, the only solution is $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$. ▲

Solution 2 by the Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA. The only solution is $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.
From the given equation we have $a^2 + bc = 2^a$, $(a+d)b = 2^b$, $(a+d)c = 2^c$, and $bc + d^2 = 2^d$. Since the left sides of these equations are integers, each of a, b, c, and d must be nonnegative. From the second and third equations, both b and c must be nonzero, and hence powers of 2. From the determinants of the matrices in the given equation, $2^{a+d} - 2^{b+c} = (ad - bc)^2 \geq 0$, so that $a + d \geq b + c$. From the first and fourth equations, we see that $2^a - a^2 = bc = 2^d - d^2$. Consider the function $f(x) = 2^x - x^2$; then $f(a) = f(d) = bc$. Since $f'(x) = 2^x \cdot \ln 2 - 2x > \frac{1}{2} \cdot (2^x - 4x) \geq 0$ for $x \geq 4$, then $f(x)$ is increasing for $x \geq 4$. Thus, if $a \neq d$, then either $\{a, d\} = \{2, 4\}$ and $bc = 0$, or $\{a, d\} = \{0, 1\}$ and $bc = 1$. The former case is impossible since b and c are nonzero. In the latter case, $(a, b, c, d) = (0, 1, 1, 1)$ or $(1, 1, 1, 0)$, so that either

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2^a & 2^b \\ 2^c & 2^d \end{pmatrix}$$

or

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^2 = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \neq \begin{pmatrix} 2 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 2^a & 2^b \\ 2^c & 2^d \end{pmatrix},$$

both of which contradict the given matrix equation. Therefore, we must have $a = d$. From the second and third equations, we know that $\frac{2^b}{b} = 2a = \frac{2^c}{c}$. Consider $g(x) = \frac{2^x}{x}$; then $g(b) = g(c) = 2a$. Since $g'(x) = \frac{2^x}{x^2}(x \ln 2 - 1) > 0$ for $x > \frac{1}{\ln 2}$, then $g(x)$ is increasing for $x \geq 2$. That means that either $b = c$ or $\{b, c\} = \{1, 2\}$ and $a = 1$. In the latter case, that would mean $a^2 + bc = 1 + 2 = 3 \neq 2 = 2^a$, contradicting the first equation. Therefore, we must have $b = c$ and $a = d$.
Since $a + d \geq b + c$, then $a \geq b \geq 1$. Since $2ab = 2^b$, then a and b must both be powers of 2. Setting $a = 2^i$ and $b = 2^j$, where i and j are nonnegative integers with $i \geq j$, we see from the first equation that

$$a^2 + b^2 = 2^{2i} + 2^{2j} = 2^{2j}\left(2^{2(i-j)} + 1\right) = 2^a.$$

If $i > j$, then 2^a has an odd divisor larger than 1, which is impossible, so $a = b$. Thus, $2a^2 = 2^a$, so that $2^{2i+1} = 2^a$ and $a = 2i + 1$ is odd. Since the only odd power of two is 1, then we must have $a = b = c = d = 1$. So,

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}^2 = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix} = \begin{pmatrix} 2^1 & 2^1 \\ 2^1 & 2^1 \end{pmatrix},$$

and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is the only solution to the given matrix equation. ▲

Also solved by Albert Stadler, Switzerland; Michel Bataille, Rouen, France and the proposer.

162. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" Secondary School and "Dr. C. Angelescu High School", Buzău, Romania.* Let a be a positive real number. Compute

$$\lim_{x\to\infty}\left((x+a)\sin\frac{1}{x+a}\sqrt[x+1]{\Gamma(x+2)}-x\sin\frac{1}{x}\sqrt[x]{\Gamma(x+1)}\right).$$

Solution by Moti Levy, Rehovot, Israel. Stirling's asymptotic formula for $\ln(\Gamma(x+1))$ is

$$\ln(\Gamma(x+1))=x\ln(x)-x+\frac{1}{2}\ln(2\pi x)+\frac{1}{12x}+O\left(\frac{1}{x^3}\right). \tag{1}$$

After division of both sides of (1) by x and exponentiating, we get an asymptotic formula of $(\Gamma(x+1))^{\frac{1}{x}}$, namely

$$(\Gamma(x+1))^{\frac{1}{x}}=\frac{x}{e}\left(1+\frac{\ln(2\pi x)}{2x}+O\left(\frac{\ln^2(x)}{x^2}\right)\right). \tag{2}$$

An asymptotic formula for $\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}$ is

$$\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}=1-\frac{1}{6x^2}+O\left(\frac{1}{x^4}\right). \tag{3}$$

By (2) and (3) we have,

$$\begin{aligned}\frac{\sin\left(\frac{1}{x+a}\right)}{\frac{1}{x+a}}(\Gamma(x+2))^{\frac{1}{x+1}}&=\left(1-\frac{1}{6(x+a)^2}+O\left(\frac{1}{x^4}\right)\right)\left(\frac{x+1}{e}+\frac{x+1}{e}\frac{\ln(2\pi(x+1))}{2(x+1)}+O\left(\frac{\ln^2(x)}{x}\right)\right) \qquad (4)\\&=\left(1-\frac{1}{6(x+a)^2}\right)\left(\frac{x+1}{e}+\frac{x+1}{e}\frac{\ln(2\pi(x+1))}{2(x+1)}\right)+O\left(\frac{\ln^2(x)}{x}\right).\end{aligned}$$

Similarly, we have

$$\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}(\Gamma(x+1))^{\frac{1}{x}}=\left(1-\frac{1}{6x^2}\right)\left(\frac{x}{e}+\frac{x}{e}\frac{\ln(2\pi x)}{2x}\right)+O\left(\frac{\ln^2(x)}{x}\right). \tag{5}$$

By subtracting (5) from (4) we get,

$$\begin{aligned}&\left(\frac{\sin\left(\frac{1}{x+a}\right)}{\frac{1}{x+a}}(\Gamma(x+2))^{\frac{1}{x+1}}-\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}(\Gamma(x+1))^{\frac{1}{x+1}}\right)\\&=\left(1-\frac{1}{6(x+a)^2}\right)\left(\frac{x+1}{e}+\frac{x+1}{e}\frac{\ln(2\pi(x+1))}{2(x+1)}\right)-\left(1-\frac{1}{6x^2}\right)\left(\frac{x}{e}+\frac{x}{e}\frac{\ln(2\pi x)}{2x}\right)+O\left(\frac{\ln^2(x)}{x}\right).\end{aligned} \tag{6}$$

Taking the limit of (6) gives

$$\lim_{x\to\infty}\left(\frac{\sin\left(\frac{1}{x+a}\right)}{\frac{1}{x+a}}\left(\Gamma\left(x+2\right)\right)^{\frac{1}{x+1}}-\frac{\sin\left(\frac{1}{x}\right)}{\frac{1}{x}}\left(\Gamma\left(x+1\right)\right)^{\frac{1}{x+1}}\right)$$
$$=\lim_{x\to\infty}\left(1-\frac{1}{6\left(x+a\right)^2}\right)\left(\frac{x+1}{e}+\frac{x+1}{e}\frac{\ln\left(2\pi\left(x+1\right)\right)}{2\left(x+1\right)}\right)-\left(1-\frac{1}{6x^2}\right)\left(\frac{x}{e}+\frac{x}{e}\frac{\ln\left(2\pi x\right)}{2x}\right)$$
$$=\lim_{x\to\infty}\left(\frac{x+1}{e}-\frac{x}{e}+\frac{1}{e}\frac{\ln\left(2\pi\left(x+1\right)\right)}{2}-\frac{1}{e}\frac{\ln\left(2\pi x\right)}{2}\right)$$
$$=\lim_{x\to\infty}\left(\frac{1}{e}+\frac{1}{2e}\ln\left(1+\frac{1}{x}\right)\right)=\frac{1}{e}.$$

▲

Also solved by the proposer.

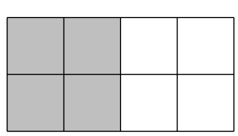
163. *Proposed by Valmir Krasniqi, Institute of Science, Technology, Engineering and Mathematics, Republic of Kosova.* In each square of an 8×8 grid, there is a lamp. We say that two lamps on the grid are neighbors if they share a vertex or an edge. Initially, some lamps are lit. Every minute, each lamp that has at least three lit neighboring lamps will also light up. What is the smallest number of lamps that need to be lit initially to ensure that after some time all the lamps will be lit?

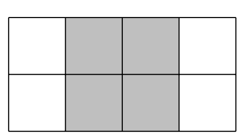
Solution by Fiton Hoxha, Technical University of Munich, Germany. The answer is 21. We shall show this by first examining some smaller grids. If a configuration of lit lamps does not transition to a fully lit grid then we shall say that it is *incomplete*, otherwise we will say that it is *complete*.

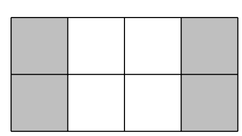
Claim 1. *An incomplete configuration of a 4×4 (or larger) grid has at most 4 lit lamps in any 2×4 subgrid.*

Proof. If we had 5 lit lamps, then one half of the grid would have at least 3 lit lamps, which completes to 4 after one minute. Because we also have lit lamps on the other half, this would lead to a complete configuration. □

After careful examination, we see that in incomplete configurations the only possible 2×4 subgrids with 4 lit lamps up to symmetry are these:


1.1

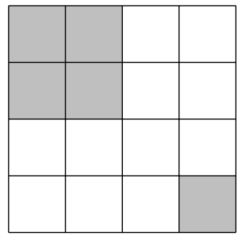

1.2


1.3

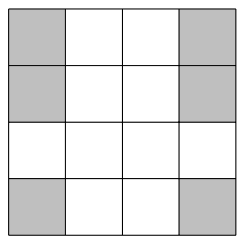
Claim 2. *An incomplete configuration in a 4×4 grid has at most 6 lit lamps.*

Proof. Assume that there is an incomplete configuration with 7 lit lamps. Consider the upper and lower 2×4 grids inside the 4×4 grid. One of these, say the upper one, must have at least 4 lit lamps. By Claim 1 it must have exactly 4 lit lamps. So it's one of 1.1, 1.2 or 1.3.

If it is 1.1, then we may have at most one lit lamp on the lower half:

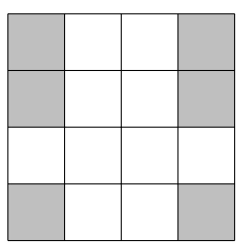


If it is 1.2, then we can not have any lit lamps on the lower half. And finally, if it is 1.3 then we may have at most two lit lamps on the lower half:

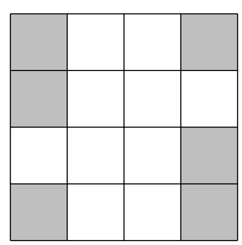


In any case, the number of lit lamps is at most 6. □

After also considering the cases in which both 2×4 halves have three lit lamps, we conclude that the only incomplete configurations with 6 lit lamps on a 4×4 grid up to reflection and rotation are these:

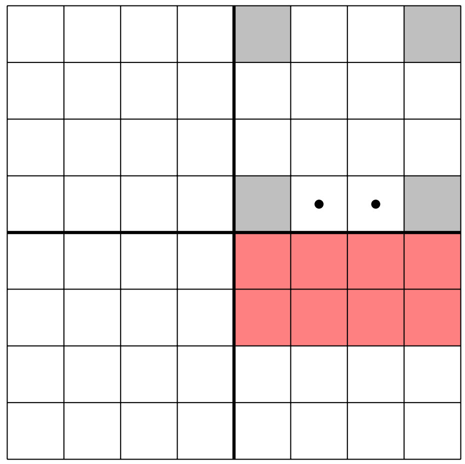


2.1



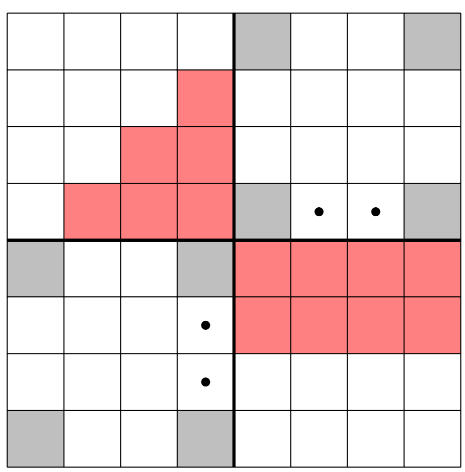
2.2

Now we move on to the 8×8 case. Assume that there exists an incomplete 8×8 grid with 21 lit lamps. Consider the four 4×4 grids inside this 8×8 grid. One of them must have at least 6 lit lamps. Assume w.l.o.g that it's the upper right one. By Claim 2, it has exactly 6 lit lamps and it is one of 2.1 or 2.2 (or their reflections or rotations). Notice that in all of these cases, on any two neighbouring sides of the 4×4 square we have three lit lamps on one of them and two lit lamps on the other. Assume w.l.o.g that we have three lit lamps on the lower side of the upper right 4×4 grid. Noticing also that in all of these configurations the four corner lamps are lit, we represent the current state with the following diagram:

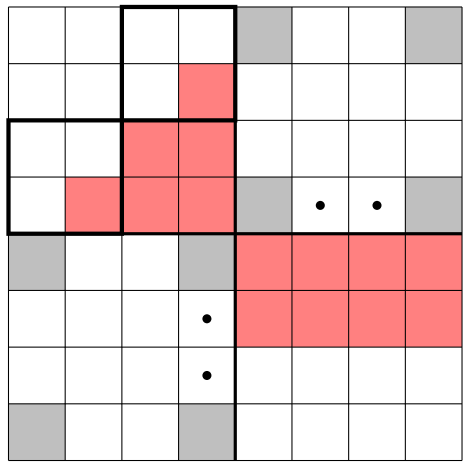


The two dots imply that exactly one of those squares has a lit lamp. Because of the three lamps on the lower side, we can not have any lit lamps inside the red shaded area. By Claim 1, the lower right 4×4 grid can have at most 4 lit lamps.

Since we have at most 10 lit lamps on the right side of the 8×8 grid, then we must have at least 11 lit lamps on the left side, so one of the 4×4 grids has 6 lit lamps. If that is the upper left 4×4 grid, then it must have three lit lamps on the lower side, since having three lit lamps on the right side would lead to a complete configuration. But if it has three lit lamps on the lower side then the lower left 4×4 grid has at most 4 lit lamps as we saw before, leading to a maximal number of 20 lit lamps. So the only option left is that the lower left 4×4 grid has 6 lit lamps, with three of those on the right side for similar reasons. We have the following diagram:

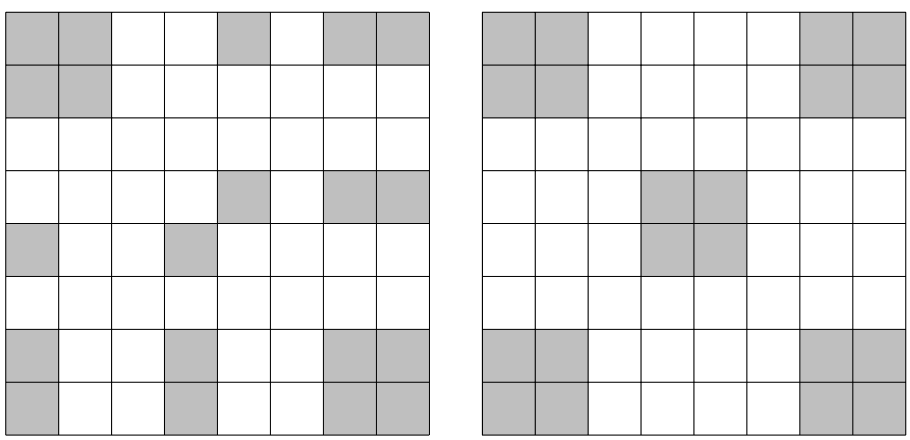


Clearly we can not have any lit lamps in the red shaded area in the upper left 4×4 grid. We must have 5 lit lamps in the remaining squares. Consider the two marked 2×2 grids:



We see that these 2×2 grids can have at most 1 lit lamp. This leaves 3 lit lamps for the corner 2×2 grid, but this is clearly not possible.

We conclude that it is not possible to have an incomplete configuration with 21 lit lamps on the 8×8 grid, i.e., all configurations with 21 lit lamps are complete. Indeed 21 is the smallest number for which this holds since we have the following incomplete configurations with 20 lit lamps, among many others:



▲

Also solved by the proposer. One incorrect solution was received.

164. *Proposed by Vasile Cirtoaje, Petroleum-Gas University of Ploieşti, Romania and Leonard Giugiuc, Romania.* Let $n \geq 3$ be an integer and let $a_1, a_2, \ldots, a_n$ be non-negative real numbers such that $\sum_{i=1}^{n} \frac{1}{a_i+1} \geq n-1$ and $\sum_{1 \leq i<j \leq n} a_i a_j > 0$. Prove

$$(n-2)\sum_{i=1}^{n} a_i + \frac{n}{2\sum_{1\leq i<j\leq n} a_i a_j} \geq \frac{2n^2-4n+1}{n-1}.$$

When is equality attained?

Solution by Moti Levy, Rehovot, Israel.
Recall the **Arithmetic Mean Theorem**, which may be found in the book "Mathematical Inequalities, Volume 5" by Vasile Cirtoaje: Let $F: \mathbb{A} \to \mathbb{R}$ be a symmetric continuous function on $\mathbb{A} \subseteq \mathbb{R}^n$ satisfying

$$F(a_1, a_2, \ldots, a_n) \geq F\Big(\frac{a_1+a_n}{2}, a_2, \ldots, \frac{a_1+a_n}{2}\Big)$$

for all $a_1, a_2, \ldots, a_n \in \mathbb{A}$ such that $a_1 \geq a_2 \geq \ldots \geq a_n$ or $a_1 \leq a_2 \leq \ldots \leq a_n$. Then, for all $(a_1, a_2, \ldots, a_n) \in \mathbb{A}$, the following inequality holds:

$$F(a_1, a_2, \ldots, a_n) \geq F(A, A, \ldots, A)$$

where

$$A = \frac{a_1+a_2+\ldots+a_n}{n}.$$

Let $\mathbb{A} := \{(a_1, a_2, ..., a_n) : a_1, a_2, \ldots, a_n \geq 0\}$. Without loss of generality, assume that $a_1 \geq a_2 \geq \ldots \geq a_n$.
Let

$$F(a_1, a_2, \ldots, a_n) := (n-2)\sum_{i=1}^{n} a_i + \frac{n}{2\sum_{i=1\leq i<j\leq n}^{n} a_i a_j}. \tag{7}$$

We have

$$F\left(\frac{a_1+a_n}{2}, a_2, \ldots, \frac{a_1+a_n}{2}\right) = (n-2)\sum_{i=1}^{n} a_i + \frac{n}{2\left(\sum_{\substack{(i,j),1\leq i<j\leq n \\ (i,j)\neq(1,n)}}^{n} a_i a_j + \left(\frac{a_1+a_n}{2}\right)^2\right)}. \tag{8}$$

By the AM-GM inequality we have

$$\sum_{\substack{(i,j),1\leq i<j\leq n \\ (i,j)\neq(1,n)}}^{n} a_i a_j + \left(\frac{a_1+a_n}{2}\right)^2 \geq \sum_{i=1\leq i<j\leq n}^{n} a_i a_j. \tag{9}$$

It follows from (8) and (9) that

$$F(a_1, a_2, \ldots, a_n) \geq F\left(\frac{a_1+a_n}{2}, a_2, \ldots, \frac{a_1+a_n}{2}\right). \tag{10}$$

Then by the Arithmetic Mean Theorem we obtain

$$F(a_1, a_2, \ldots, a_n) = (n-2)\sum_{i=1}^{n} a_i + \frac{n}{2\sum_{i=1\leq i<j\leq n}^{n} a_i a_j} \geq F(A, A, \ldots, A) = (n-2)\,nA + \frac{1}{(n-1)\,A^2}.$$

Let $g\left(A\right) := F(A, A, \ldots, A)$. The function $g\left(A\right)$ is strictly decreasing on the interval $\left(0, \frac{1}{n-1}\right]$ since

$$\frac{dg}{dA} = -\frac{2}{A^3\left(n-1\right)} + n\left(n-2\right) < 0 \text{ for } n \geq 3. \tag{11}$$

Now

$$g\left(\frac{1}{n-1}\right) = \frac{2n^2-4n+1}{n-1}, \tag{12}$$

hence it follows from (11) and (12) that

$$g\left(A\right) \geq \frac{2n^2-4n+1}{n-1} \quad \text{for} \quad 0 < A \leq \frac{1}{n-1}. \tag{13}$$

The constraint $\sum_{i=1}^{n} \frac{1}{a_i+1} \geq n-1$ implies that the harmonic mean of $(a_i+1)_{i=1}^{n}$ is less than $\frac{n}{n-1}$, that is, we have

$$\frac{n}{\sum_{i=1}^{n} \frac{1}{a_i+1}} \leq \frac{n}{n-1}.$$

By the super-additivity of the harmonic mean function, we have

$$\frac{1}{\sum_{i=1}^{n} \frac{1}{a_i+b_i}} \geq \frac{1}{\sum_{i=1}^{n} \frac{1}{a_i}} + \frac{1}{\sum_{i=1}^{n} \frac{1}{b_i}}. \tag{14}$$

Hence

$$\frac{n}{n-1} \geq \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i+1}} \geq \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} + 1,$$

that is,

$$\frac{1}{n-1} \geq \frac{n}{\sum_{i=1}^{n} \frac{1}{a_i}} := H. \tag{15}$$

Since harmonic mean is less than or equal the arithmetic mean, we have from (13) and (15) that

$$g\left(A\right) \geq \frac{2n^2-4n+1}{n-1} \quad \text{for} \quad 0 < H \leq A \leq \frac{1}{n-1}.$$

We conclude that

$$(n-2)\sum_{i=1}^{n} a_i + \frac{n}{2\sum_{i=1 \leq i<j \leq n}^{n} a_i a_j} \geq \frac{2n^2-4n+1}{n-1}, \quad \text{for } H \leq \frac{1}{n-1}.$$

Equality is attained when $a_1 = a_2 \cdots = a_n = \frac{1}{n-1}$. ▲

Also solved by the proposer.

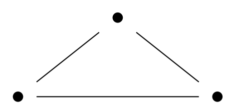
165. *Proposed by Besfort Shala, University of Bristol, Bristol, United Kingdom.* Let n be a positive integer. Suppose that there exists a simple graph G with n vertices and precisely $\lceil n^2/3 \rceil$ edges that can be drawn in the plane with less than $12(n/12)^4$ unordered pairs of edges that intersect at a point other than the vertices of the graph. Find all possible values of n.

Solution by the proposer. For a simple graph G with v vertices and e edges, let $c(G)$ be the minimum number of edge crossings needed to draw G on the plane. Removing $c(G)$ edges from G and any isolated vertices that may remain, we obtain a planar graph H with $e - C(G)$ edges and $v' \leq v$ vertices. By Euler's theorem for planar graphs, we have $v' - (e - c(G)) + f' = 2$, where f' is the number of faces of H (including the unbounded face). Combining this with the standard facts that in a planar graph each edge separates at most two faces and any face is bounded by at least three edges, we have $3f' \leq 2(e - c(G))$. Therefore

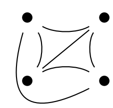
$$6 - 3v' + (e - c(G)) = 3f' \leq 2(e - c(G))$$

which gives $c(G) \geq e - 3v' + 6 \geq e - 3v + 6$ since $v' \leq v$. Now, taking $v = n, e = \lceil n^2/3 \rceil$ we get $12(n/12)^4 > c(G) \geq \lceil n^2/3 \rceil - 3n + 6$ which can only hold for $3 \leq n \leq 6$ and $n \geq 18$.
Let us first show that $3 \leq n \leq 6$ are possible. Note that in this range we have $12(n/12)^4 < 1$, so all of our graphs must be planar.

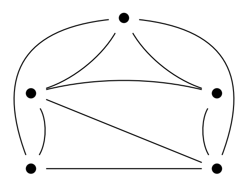
$n = 3 \quad e = \lceil 3^2/3 \rceil = 3$



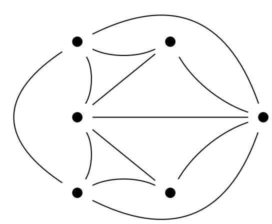
$n = 4 \quad e = \lceil 4^2/3 \rceil = 6$



$n = 5 \quad e = \lceil 5^2/3 \rceil = 9$



$n = 6 \quad e = \lceil 6^2/3 \rceil = 12$



Next we rule out $n \geq 18$ using an idea of Székely to bootstrap the inequality $c(G) \geq e - 3v + 6$ via a probabilistic method. More precisely, let K be a random induced subgraph of G where each vertex is included independently with probability p (a parameter to be optimized later). Letting V, E and C be the random number of vertices, edges and minimum number of crossings of K, respectively, we claim that $\mathbb{E}V = pv, \mathbb{E}E = p^2 E$ and $\mathbb{E}C = p^4 c(G)$. The first two expectations are obvious, whereas the last one follows from the fact that each edge crossing in G (that may not be trivially resolved) involves 4 vertices that we would have to include in K. Now for each such random induced subgraph K we have $C \geq E - 3V + 6 \geq E - 3V$, thus $p^4 c(G) \geq p^2 e - 3pv$ after taking expectations. Optimizing the choice of p in terms of v and e gives $p = 4v/e$ (note that $4v \leq e$ certainly holds for our graph G and $n \geq 18$). This gives $c(G) \geq e^3/(4^3 v^2) \geq n^4/12^3 = 12(n/12)^4$, as desired. ▲

MATHCONTEST SECTION

This section of the Journal offers readers an opportunity to solve interesting and elegant mathematical problems that have appeared in Math Contests around the world and that are most appropriate for undergraduate Math Olympiad training. Proposals are always welcome. The source of the proposals will appear when the solutions are published.

Proposals

115. Find the minimum value of $2\sin x + 3\sin y - \cos(x-y)$.

116. Calculate

$$\lim_{n\to\infty} \frac{n!\left(1+\frac{1}{n}\right)^{n^2+n}}{n^{n+\frac{1}{2}}}.$$

117. Let f be a nonnegative and non increasing function on $[0,+\infty)$ and let g be a function defined on $[0,+\infty)$ such that $0 \le g(x) \le 2023$ with $\int_0^{+\infty} g(x)\,dx = 6069$. Prove that

$$\int_0^{+\infty} f(x)g(x)\,dx \le 2023\int_0^3 f(x)\,dx.$$

118. Let $a \in \mathbb{R}^*$ and $b \in \mathbb{R}$. Find all differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x) = af'(-x) + b, \quad \forall x \in \mathbb{R}.$$

119. Let $n > 1$ and $A, B \in M_n(\mathbb{C})$ such that $f(z) = e^{zA}Be^{-zA}$ is bounded for all $z \in \mathbb{C}$. Compute

$$C = AB - BA \quad \text{and} \quad D = A^2B + BA^2.$$

Solutions

110. Let n be a positive integer and let A be a real $n\times n$ matrix such that $A^{2023}=I$, where I is the $n\times n$ identity matrix. Suppose there exists a real number λ such that $(A-\lambda I)^2=0$. Prove that $A=I$.

(Kosovo IMC TST 2023)

Solution by Albert Stadler, Switzerland. By the binomial theorem, we have

$$I=A^{2023}=(A-I+I)^{2023}=\sum_{j=0}^{2023}\binom{2023}{j}(A-I)^jI^{2023-j}=^{2023}I+2023(A-I)^{2022},$$

where we have taken into account that $(A-\lambda I)^j=0$ for all $j\geq 2$. We solve the above equation for A and find that

$$A=cI$$

for some real number c. The equation $(A-\lambda I)^2=(cI-\lambda I)^2=0$ then implies that $c=\lambda$. We conclude that $I=A^{2023}=(cI)^{2023}=c^{2023}I$ and $c^{2023}=1$, so that $\lambda=c=1$ and $A=I$. ▲

Also solved by Michel Bataille, Rouen, France.

111. Show that the sequence $x_n=\sin^2(n)$ is divergent.

(Kosovo IMC TST 2021)

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria. Suppose that $\lim\limits_{n\to\infty}x_n=\ell$. Then $\lim\limits_{n\to\infty}\cos^2 n=1-\ell$. But $\cos(2n)=2\cos^2 n-1$, so taking the limit as n tends to $+\infty$, we get $\lim\limits_{n\to\infty}\cos(2n)=1-2\ell\triangleq L$. Taking the limit as n tends to $+\infty$ in the equality

$$\cos(2n+2)+\cos(2n-2)=2\cos(2n)\cos 2,$$

we see that $(1-\cos 2)L=0$, hence $L=0$, and $\ell=1/2$. It follows that

$$\frac{1}{2}=\lim_{n\to\infty}\sin^2(2n)=1-\lim_{n\to\infty}\cos^2(2n)=1-0=1$$

which is absurd. Thus the sequence $(x_n)_n$ is divergent. ▲

Also solved by Michel Bataille, Rouen, France; Eagle Problem Solvers, Georgia Southern University, Statesboro, GA and Savannah, GA.

112. Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous function such that for any zero x_0 of f, there exists $n\in\mathbb{N}$ and a continuous function $g:\mathbb{R}\to\mathbb{R}$ such that $g(x_0)\neq 0$ and

$$f(x)=(x-x_0)^ng(x)$$

for all $x\in\mathbb{R}$.

(a) Prove that f has a finite number of zeros in the interval $[-M,M]$ for any $M>0$.
(b) Give an example of such a function with infinitely many zeros.
(c) Is it possible for f to have uncountably many zeros?

(Kosovo IMC TST 2022)

Solution by Besfort Shala, University of Bristol, United Kingdom. Suppose f has infinitely many zeros in $[-M, M]$. By compactness, we get a convergent subsequence of distinct zeros of f, which converges to a zero x_0 of f by continuity of f. Now $f(x) = (x - x_0)^n g(x)$ for some n and continuous g with $g(x_0) \neq 0$. But g is non-zero in a neighborhood of x_0 by continuity and $(x - x_0)^n \neq 0$ for $x \neq x_0$, contradicting that there is a sequence of distinct zeros of f converging to x_0.
An example of such a function with infinitely many zeros is $f(x) = \sin x$. Indeed, for any zero $k\pi$ of f, one has $f(x) = (x - k\pi)g(x)$ where $g(x) = f(x)/(x - k\pi)$ for $x \neq k\pi$ and $\lim_{x\to k\pi} f(x)/(x - k\pi) = (-1)^k \neq 0$ (so we set $g(k\pi) = (-1)^k$, making g continuous and non-zero at $k\pi$).
Such f cannot have uncountably many zeros, since for any $M \in \mathbb{N}$ it has a finite number of zeros, so it has at most countably many zeros in $\mathbb{R} = \bigcup_{M\in\mathbb{N}}[-M, M]$ (as a countable union of finite sets is countable). ▲

113. Let k be a positive integer and let $\varphi_k : \mathbb{N} \to \mathbb{R}$ be the multiplicative function defined by $\varphi_k(p^\alpha) = p^{(\alpha-1)k}(p^k - 1)$ on prime powers. Find all real β such that the limit

$$\lim_{n\to\infty} \frac{\frac{\varphi_k(1)}{1} + \frac{\varphi_k(2)}{2} + \cdots + \frac{\varphi_k(n)}{n}}{n^\beta}$$

exists and is finite. What is the value of the limit in each case?

(Kosovo IMC TST 2022)

Solution by Albert Stadler, Switzerland. Clearly, $1 \leq \varphi_k(p^\alpha) < p^{\alpha k}$. By multiplicativity, $1 \leq \varphi_k(n) < n^k$ for all natural numbers n.
So the Dirichlet series

$$\zeta_k(s) = \sum_{n=1}^{\infty} \frac{\varphi_k(n)}{n^s}$$

converges absolutely for $\mathrm{Re}(s) > k + 1$.
We next express $\zeta_k(s)$ in terms of the Riemann zeta function $\zeta(s)$ and find

$$\zeta_k(s) = \prod_p \left(1 + \frac{\varphi_k(p)}{p^s} + \frac{\varphi_k(p^2)}{p^{2s}} + \dots\right) =$$

$$= \prod_p \left(1 + \frac{p^k(p^k - 1)}{p^{s+k}} + \frac{p^{2k}(p^k - 1)}{p^{2s+k}} + \dots\right) =$$

$$= \prod_p \left(1 + \frac{(p^k - 1)}{p^k} \frac{p^{k-s}}{1 - p^{k-s}}\right) = \prod_p \left(\frac{1 - p^{-s}}{1 - p^{k-s}}\right) = \frac{\zeta(s-k)}{\zeta(s)}.$$

We deduce from

$$\sum_{n=1}^{\infty} \frac{\varphi_k(n)}{n^s} = \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \sum_{n=1}^{\infty} \frac{n^k}{n^s}$$

that

$$\varphi_k(n) = n^k \sum_{d|n} \frac{\mu(d)}{d^k}.$$

Hence

$$\sum_{n=1}^{N} \frac{\varphi_k(n)}{n} = \sum_{n=1}^{N} n^{k-1} \sum_{d|n} \frac{\mu(d)}{d^k} =$$

$$= \sum_{d\le N} \frac{\mu(d)}{d^k} \sum_{j\le \frac{N}{d}} (jd)^{k-1} = \sum_{d\le N} \frac{\mu(d)}{d} \sum_{j\le \frac{N}{d}} j^{k-1} =$$

$$= \sum_{d\le N} \frac{\mu(d)}{d} \left(\frac{1}{k} \frac{N^k}{d^k} + O\left(\frac{N^{k-1}}{d^{k-1}} \right) \right) =$$

$$= \frac{1}{k} N^k \sum_{d\le N} \frac{\mu(d)}{d^{k+1}} + O\left(N^{k-1} \sum_{d\le N} \frac{1}{d^k} \right) =$$

$$= \frac{1}{k(k+1)} N^k + O(1) + O\left(\max\left(N^{k-1}, \log N\right)\right),$$

taking into account that

$$\sum_{d>N} \frac{1}{d^{k+1}} = O\left(\frac{1}{N^k}\right),$$

$$\sum_{d\le N} \frac{1}{d} = O(\log N),$$

$$\sum_{d\le N} \frac{1}{d^k} = O(1), \quad k > 1.$$

We conclude that the limit exists for all $\beta \ge k$ and equals

$$\lim_{n\to\infty} \frac{\frac{\varphi_k(1)}{1} + \frac{\varphi_k(2)}{2} + \cdots + \frac{\varphi_k(n)}{n}}{n^\beta} = \begin{cases} \frac{1}{k(k+1)}, & \beta = k \\ 0, & \beta > k \end{cases}.$$

▲

114. (a) Find a constant $c > 0$ satisfying the following property: there exists $A \ge 1$ such that for all integers $n \ge A$ and all configurations of n points inside a unit square, there are two points at distance at most $\dfrac{c}{\sqrt{n}}$.

(b) Find a constant $d > 0$ satisfying the following property: there exists $B \ge 1$ such that for all integers $n \ge B$, there exists a configuration of n points inside a unit square such that all mutual distances between pairs of points are at least $\dfrac{d}{\sqrt{n}}$.

(Kosovo IMC TST 2023)

Solution by Besfort Shala, University of Bristol, United Kingdom. For $n \ge 2$, and for any configuration of n points inside the unit square, let δ be the minimum distance between two points of the configuration. The n discs of radius $\delta/2$ which are centred at the points of the configuration do not overlap, and they are all included in a square of side $1 + \delta$. Comparing the areas, we deduce that

$$n\pi(\delta/2)^2 \le (1+\delta)^2.$$

Since the maximal possible distance between points in a unit square is $\sqrt{2}$, we have

$$n\pi(\delta/2)^2 \le (1+\sqrt{2})^2 = 3 + 2\sqrt{2} \le 6,$$

which implies

$$\delta \le \sqrt{\frac{24}{n\pi}}.$$

This solves the first part of the problem, for $c = \sqrt{24/\pi}$. The constant can be improved as follows: from the bound on δ proven above, we deduce

$$n\pi(\delta/2)^2 \leq \left(1 + \sqrt{\frac{24}{n\pi}}\right)^2$$

and then

$$\delta \leq 2\left(1 + \sqrt{\frac{24}{n\pi}}\right)\frac{1}{\sqrt{n\pi}}$$

which is smaller than $c/\sqrt{n}$ for n sufficiently large, as soon as $c > 2/\sqrt{\pi}$.

On the other hand, for all $n \geq 2$, we can consider a square $k \times k$ grid for $k = \lceil\sqrt{n}\rceil$, with $\lceil\sqrt{n}\rceil^2 - n$ points removed. If the grid is chosen as the set of points $(a/(k+1), b/(k+1))$ $(a, b \in \{1, \ldots, k\})$ of the unit square $[0,1]^2$, the all mutual distances between the points are larger than or equal to

$$\frac{1}{k+1} = \frac{1}{\lceil\sqrt{n}\rceil + 1} \geq \frac{1}{\sqrt{n} + 2}$$

which is larger than $d/\sqrt{n}$ for n sufficiently large, as soon as $d > 1$. ▲

MATHNOTES SECTION

Recurrence for Euler polynomials originating from Multiple Zeta Values theory

Moti Levy

December 3, 2023

Abstract. A recurrence formula for Euler polynomials of odd degree is proved. The recurrence was conjectured by Marian Gencev in his article concerning multilpe zeta values.

1. Introduction

Multiple zeta values (MZVs) are generalization of the single zeta values $\zeta(s)$. The multiple zeta values are defined by

$$\zeta(s_1,\dots,s_k):=\sum_{n_1>\cdots>n_k\geq 1}\frac{1}{n_1^{s_1}\cdots n_k^{s_k}}$$

for positive integers $s_1,\dots,s_k$ with $s_1>1$. This notation is extended to *alternating* multiple zeta values by putting bar over those exponents with an associated sign in the numerator. For example

$$\zeta\left(\overline{3},5,\overline{1}\right)=\sum_{n_1>n_2>n_3\geq 1}\frac{(-1)^{n_1+n_3}}{n_1^3 n_2^5 n_3}.$$

Linear combinations of MZVs may sometimes be expressed by the ordinary value zeta function and standard functions.
Surprisingly, in the following two examples, the coefficients are related to Euler polynomials.
Example from [1]:
The sum (16) can be expressed as a rational polynomial in the ordinary zeta values $\zeta(i)$, $i\geq 2$.

$$\sum_{j=2}^{k}E_{2\left\lfloor\frac{j-1}{2}\right\rfloor+1}(0)\,\zeta(\overline{j},\underbrace{1\dots,1}_{k-j}),\tag{16}$$

where $E_n(x)$ are the Euler polynomials.
The Euler polynomials are defined by the exponential generating function

$$\frac{2e^{xt}}{e^t+1}=\sum_{n=0}^{\infty}E_n(x)\frac{t^n}{n!}.$$

Example from [2]:

Let us adopt the notation $s_{n,p}$ and $\sigma_{n,p}$ according to Nielsen and define

$$s_{n,p}:=\zeta(n+1,\underbrace{1\ldots,1}_{p-1})\qquad\qquad \sigma_{n,p}=(-1)^p\,\zeta(\overline{n+1},\underbrace{1\ldots,1}_{p-1}). \tag{17}$$

In contrast to $\sigma_{n,p}$, the closed form of $s_{n,p}$ is known for all admissible integers n,p, see [3].
Kölbig [3] corrected Nielsen and gave us this equation:

$$s_{n,p}=\sum_{q=1}^{n}\binom{n+p-q-1}{p-1}\sigma_{q,n+p-q}+\sum_{q=1}^{p}\binom{n+p-q-1}{n-1}\sigma_{q,n+p-q}. \tag{18}$$

Equation (18) can be regarded linear system of equations with $\sigma_{q,n+p-q}$ as the unknowns.
As an example, for $n+p=5$, one finds two equations,

$$\begin{aligned}s_{1,4}&=\sum_{q=1}^{1}\binom{1+4-q-1}{4-1}\sigma_{q,1+4-q}+\sum_{q=1}^{4}\binom{1+4-q-1}{1-1}\sigma_{q,1+4-q}\\&=2\sigma_{1,4}+\sigma_{2,3}+\sigma_{3,2}+\sigma_{4,1}.\end{aligned} \tag{19}$$

$$\begin{aligned}s_{2,3}&=\sum_{q=1}^{2}\binom{2+3-q-1}{3-1}\sigma_{q,2+3-q}+\sum_{q=1}^{3}\binom{2+3-q-1}{2-1}\sigma_{q,2+3-q}\\&=6\sigma_{1,4}+3\sigma_{2,3}+\sigma_{3,2},\end{aligned} \tag{20}$$

or in matrix notation,

$$\begin{bmatrix}2&1&1&1\\6&3&1&0\end{bmatrix}\begin{bmatrix}\sigma_{1,4}\\\sigma_{2,3}\\\sigma_{3,2}\\\sigma_{4,1}\end{bmatrix}=\begin{bmatrix}s_{1,4}\\s_{2,3}\end{bmatrix}. \tag{21}$$

The reduced row echelon form of $\begin{bmatrix}2&1&1&1\\6&3&1&0\end{bmatrix}$ is $\begin{bmatrix}1&\frac{1}{2}&0&-\frac{1}{4}\\0&0&1&\frac{3}{2}\end{bmatrix}$. Gencev observed that the last column constitutes the coefficients of Euler polynomial $E_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{4}$, with opposite signs.
Gencev formulated a recurrence formula for the reduced row echelon form and stated the following conjecture:
Conjecture: Let the sequence $(A_{s,m})_{s=1}^{m}$ be defined by the following recursion,

$$A_{s,m}+\sum_{i=1}^{s-1}A_{i,m}\binom{m+s-2i}{m-s}\frac{m-i}{m+s-2i}-\frac{1}{2}\binom{m+s-1}{m-s}\frac{2m-1}{m+s-1}=0,\quad s=1,\ldots,m,\quad m\geq 1 \tag{22}$$

then $(-A_{s,m})_{s=1}^{m}$ are the coefficients of the Euler polynomial of degree $2m-1$, i.e.,
$E_{2m-1(x)}=x^{2m-1}-A_{1,m}x^{2m-2}-A_{2.m}x^{2m-4}-A_{3,m}x^{2m-6}-\cdots A_{j,m}x^{2m-2j}\cdots-A_{m,m}.$ ■

2. Proof of Gencev's conjecture

It is known that Euler's polynomial can be expressed by the Bernoulli's numbers B_i (See short letter by Cheon [4]),

$$E_n\left(x\right) = \frac{2}{n+1}\sum_{i=1}^{n+1}\left(1-2^i\right)\binom{n+1}{i}B_i x^{n+1-i} \tag{23}$$

For example, let $n = 3$,

$$\begin{aligned} E_3\left(x\right) &= \frac{1}{2}\left(\left(1-2\right)\binom{4}{1}B_1x^3 + \left(1-2^2\right)\binom{4}{2}B_2x^2 + \left(1-2^4\right)\binom{4}{4}B_4\right) \\ &\frac{1}{2}\left(\left(1-2\right)\binom{4}{1}\left(-\frac{1}{2}\right)x^3 + \left(1-2^2\right)\binom{4}{2}\frac{1}{6}x^2 + \left(1-2^4\right)\binom{4}{4}\left(-\frac{1}{30}\right)\right) \\ &= x^3 - \frac{3}{2}x^2 + \frac{1}{4}. \end{aligned}$$

Set $n = 2m-1$ in (23) to get (24),

$$E_{2m-1}\left(x\right) = \frac{1}{m}\sum_{i=1}^{2m}\left(1-2^i\right)\binom{2m}{i}B_i x^{2m-i} \tag{24}$$

Our aim is to show that the coefficients of the Euler polynomial of degree $2m-1$ satisfy the recurrence defined in (22), that is, if

$$A_{s,m} = -\left[x^{2m-2s}\right]E_{2m-1}\left(x\right) \tag{25}$$

then the recurrence (22) is satisfied for $s = 1,\ldots,m, \quad m \geq 1$. We argue that once we prove this, then any sequence of numbers which satisfy the recurrence $s = 1,\ldots,m, \quad m \geq 1$ must be equal to the Euler polynomial coefficients, since (22) may be viewed as linear system of equations with non singular matrix, i.e., with a single solution.

Before delving into the proof details, we give an outline of the proof:
1) Express the Euler polynomial coefficients by Bernoulli's mumblers.
2) Plugging this expression into the LHS of the recurrence (22).
3) Simplification of the LHS and showing that it is equal to the terms with even indexes of the convolution of two sequences.
4) Finding the generating functions of these two sequences.
5) The generating function of the LHS is the product of the generating functions of these two sequences.
6) Showing that the even coefficients of generating function of the LHS are equal to zero, which implies that the LHS is indeed zero.

Step 1)
It follows from (24) that

$$A_{s,m} = \frac{1}{m}\binom{2m}{2s}\left(2^{2s}-1\right)B_{2s} \tag{26}$$

Step 2)
Now we plug-in (26) into the LHS of the recurrence (22) and get an expression

involving the Bernoulli's numbers

$$\frac{1}{m}\binom{2m}{2s}\left(2^{2s}-1\right)B_{2s}+\frac{1}{m}\sum_{i=1}^{s-1}\binom{2m}{2i}\binom{m+s-2i}{m-s}\frac{m-i}{m+s-2i}\left(2^{2i}-1\right)B_{2i}\frac{1}{2}\binom{m+s-1}{m-s}\frac{2m-1}{m+s-1} \tag{27}$$

Step 3)
Simplification of (27) gives the recurrence

$$\frac{1}{m}\sum_{i=1}^{s}\binom{2m}{2i}\binom{m+s-2i}{2s-2i}\frac{m-i}{m+s-2i}\left(2^{2i}-1\right)B_{2i}-\frac{1}{2}\binom{m+s-1}{m-s}\frac{2m-1}{m+s-1}$$

and further simplification gives

$$\frac{1}{m}\sum_{k=1}^{2s}\binom{2m}{k}\binom{m+s-k}{m-s}\frac{2m-k}{m+s-k}\left(2^{k}-1\right)B_{k}$$

which is equivalent to

$$\sum_{k=1}^{2s}\frac{(m+s-k-1)!}{(2m-k-1)!\,(2s-k)!}\left(2^{k}-1\right)\frac{B_k}{k!} \tag{28}$$

Let $m^{'} := m-s$. Then (28) can be rewritten as

$$\sum_{k=0}^{2s}\frac{\left(m^{'}-1+(2s-k)\right)!}{(2m^{'}-1+(2s-k))!\,(2s-k)!}\left(2^{k}-1\right)\frac{B_k}{k!} \tag{29}$$

Let

$$a_n := \frac{\left(m^{'}-1+n\right)!}{(2m^{'}-1+n)!\,(n)!},$$

$$b_n := \left(2^n-1\right)\frac{B_n}{n!}.$$

We observe that (29) is convolution of the sequences $(a_n)_{n\geq 0}, (b_n)_{n\geq 0}$,

$$c_n := \sum_{k=0}^{n} a_{n-k}b_k.$$

Step 4)
Let $A(z)$, $B(z)$ be the generating function of the sequences $(a_n)_{n\geq 0}, (b_n)_{n\geq 0}$ respectively.

$$A(z) = \sum_{n=0}^{\infty}\frac{\left(m^{'}-1+n\right)!}{(2m^{'}-1+n)!n!}z^n, \tag{30}$$

$$B(z) = \sum_{n=0}^{\infty}\left(2^n-1\right)\frac{B_n}{n!}. \tag{31}$$

The following lemma explains how to express a binomial sum as a hypergeometric function [5]:

Lemma : Let $(\alpha_k)_{k\geq 0}$ satisfies the following conditions:

$$\alpha_0 = 1,$$
$$\frac{\alpha_{k+1}}{\alpha_k} = \frac{1}{k+1}\frac{(k+a)}{(k+c)}z.$$

Then

$$\sum_{k=0}^{\infty} \alpha_k = {}_1F_1\left(a,b;c;z\right),$$

where ${}_1F_1(a,b;c;z)$ is the confluent hypergeometric function.■

Applying the lemma on the binomial sum (30) gives the generating function of $(a_n)_{n\geq 0}$,

$$A\left(z\right) = \frac{\left(m^{'}-1\right)!}{(2m^{'}-1)!}{}_1F_1\left(m^{'},2m^{'};z\right). \tag{32}$$

It is known that the exponential generating function of the Bernoulli's numbers is

$$\sum_{n=0}^{\infty} \frac{B_n}{n!}z^n = \frac{z}{e^z-1}.$$

It follows that the generating function of $(b_n)_{n\geq 0}$ is

$$B\left(z\right) = \sum_{n=0}^{\infty} b_n z^n = \frac{2z}{e^{2z}-1} - \frac{z}{e^z-1}. \tag{33}$$

Step 5)

The generating function of $C\left(z\right)$ is the product of $A(z)$ and $B\left(z\right)$:

$$C\left(z\right) := A\left(z\right)B\left(z\right) = \frac{\left(m^{'}-1\right)!}{(2m^{'}-1)!}{}_1F_1\left(m^{'},2m^{'},z\right)\left(\frac{2z}{e^{2z}-1} - \frac{z}{e^z-1}\right).$$

Step 6)

The **even** coefficients of $C\left(z\right)$ are expressed by the sum in (29).

The confluent hypergeometric ${}_1F_1\left(m^{'},2m^{'},z\right)$ can be expressed by the confluent hypergeometric limit function ${}_0F_1\left(m^{'}+\frac{1}{2},2m^{'},\frac{z^2}{16}\right)$, (see Kummer's transformation in [6])

$${}_1F_1\left(m^{'},2m^{'};z\right) = e^{\frac{z}{2}}{}_0F_1\left(m^{'}+\frac{1}{2},2m^{'},\frac{z^2}{16}\right).$$

$$\begin{aligned} C\left(z\right) &= \frac{\left(m^{'}-1\right)!}{(2m^{'}-1)!}e^{\frac{z}{2}}{}_0F_1\left(m^{'}+\frac{1}{2},2m^{'},\frac{z^2}{16}\right)\left(\frac{2z}{e^{2z}-1} - \frac{z}{e^z-1}\right) \\ &= -\frac{\left(m^{'}-1\right)!}{(2m^{'}-1)!}{}_0F_1\left(m^{'}+\frac{1}{2},2m^{'},\frac{z^2}{16}\right)\frac{z}{e^{\frac{z}{2}}+e^{-\frac{z}{2}}} \end{aligned}$$

We observe immediately that $C\left(z\right)$ is odd function, which implies that the even coefficients of $C\left(z\right)$ are indeed zero. Thus the conjecture is proved to be correct.

References

[1] M. E. Hoffman, M. Kuba, M. Levy, G. Louchard, An asymptotic series for an integral, Ramanujan J. 53 (2020) 1-25.

[2] M. Gencev, "On parametric K-fold series and its connection to Nielsen-Kolbig relations, J. of Math. Anal. Appl. 522 (2023) 126978.

[3] K. S. Kolbig, Nielsen generalized polylogarithms, SIAM J. Math. Anal. 17 (1986) 1232-1258.

[4] Gi-Sang Cheon, A Note on the Bernoulli and Euler Polynomials, Applied Mathematics Letters, 16 (2003) 365-368.

[5] Graham, Knuth, Patashnik, Concrete Mathematics, 2nd edition, Addison-Wesley

[6] Confluent hypergeometric function, Wikipedia entry.

JUNIOR PROBLEMS

Solutions to the problems in this issue should arrive before May 30, 2024

Proposals

76. *Proposed by Mihaly Bencze and Neculai Stanciu, Romania.*
Let $ABCDEFGHIJKLM$ be a regular 13-gon. Prove that

$$\frac{AC-AB}{AD-AC}=\frac{AF}{AE}.$$

77. *Proposed by Titu Zvonaru, Comănești, Romania.*
Let a, b, c be positive real numbers. Prove that

$$\left(\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\right)\left(\frac{a^2+b^2}{a+b}+\frac{b^2+c^2}{b+c}+\frac{c^2+a^2}{c+a}\right)\geq 3(a^2+b^2+c^2).$$

78. *Proposed by Dren Neziri, University of Primorska, Slovenia.* Let BC be a fixed chord of a fixed circle ω. Let A be a variable point in the major arc $\widehat{BC}$. Define

$$F(A)=\begin{cases}\text{point on the side } AC \text{ such that } BA+AM=MC \text{ if } AB\leq AC;\\ \text{point on the side } AB \text{ such that } BM=MA+AC \text{ if } AB>AC.\end{cases}$$

Find the perimeter, in terms of BC and $\angle BAC$, of the locus of $F(A)$ as A moves along major arc $\widehat{BC}$.

79. *Proposed by Besfort Shala, University of Bristol, United Kingdom.*
Let F_n be the Fibonacci sequence given by $F_1=1, F_2=1$ and $F_{n+2}=F_{n+1}+F_n$ for all $n\geq 1$. For a given positive integer d, let $f(d)$ be the least positive integer such that $d \mid F_{f(d)}$. Prove that for coprime positive integers m and n, we have that $f(mn)=f(m)f(n)$ if and only if $f(m)$ and $f(n)$ are coprime.

80. *Proposed by Valmir Krasniqi, Institute of Science, Technology, Engineering and Mathematics, Republic of Kosova.* Let $x, y, z>0$ such that $xyz=1$. Prove that

$$\sqrt{\frac{x^3+3}{x^2+y+2z}}+\sqrt{\frac{y^3+3}{y^2+z+2x}}+\sqrt{\frac{z^3+3}{z^2+x+2y}}\geq 3.$$

No problem is ever permanently closed. We will be very pleased to consider publishing new solutions or comments on previous problems.

Solutions

71. *Proposed by Mihaly Bencze, Braşov, Romania, and Neculai Stanciu, Buzău, Romania.* Let $a_1, a_2, \ldots, a_n$ be positive real numbers and define $a_{n+1} = a_1$. Prove

$$\sum_{k=1}^{n} \frac{a_k^2}{a_k + a_{k+1}} \geq \frac{1}{2} \sum_{k=1}^{n} a_k \geq \sum_{k=1}^{n} \frac{a_k a_{k+1}^2}{a_k^2 + a_{k+1}^2}.$$

Solution by Henry Ricardo, Westchester Area Math Circle. Applying the Cauchy-Schwarz (C-S) and AGM inequalities, we have

$$\begin{aligned}
\sum_{k=1}^{n} \frac{a_k^2}{a_k + a_{k+1}} &\overset{\text{C-S}}{\geq} \frac{\left(\sum_{k=1}^{n} a_k\right)^2}{\sum_{k=1}^{n} (a_k + a_{k+1})} \\
&= \frac{\left(\sum_{k=1}^{n} a_k\right)^2}{2\sum_{k=1}^{n} a_k} \\
&= \frac{1}{2} \sum_{k=1}^{n} a_k \\
&= \frac{1}{2} \sum_{k=1}^{n} a_{k+1} \\
&= \sum_{k=1}^{n} \frac{a_k a_{k+1}^2}{2 a_k a_{k+1}} \\
&\overset{\text{AGM}}{\geq} \sum_{k=1}^{n} \frac{a_k a_{k+1}^2}{a_k^2 + a_{k+1}^2}.
\end{aligned}$$

▲

Also solved by Titu Zvonaru, Comănești, Romania; Edi Berisha, Republic of Kosova; NYSS Problem Solving Group, Republic of Kosova; Michel Bataille, Rouen, France and the proposer.

72. *Proposed by George-Florin Şerban, National Pedagocial College "D. P. Perpessicius", Braila, Romania.* A 4-digit positive integer $\overline{abcd}$ is called *special* if it satisfies

$$\overline{abcd} = (a + b + c + d)(a^2 + b^2 + c^2 + d^2)^2.$$

One easily verifies that 2023 is a special number. Find all special numbers.

Solution by the proposer. Let $s = a + b + c + d$, $t = a^2 + b^2 + c^2 + d^2$, and check cases of the former.
For $s \leq 4$, it is clear that $st^2 \leq 4(8)^2 = 1024$, which clearly gets surpassed by $\overline{abcd}$.
If $s = 5$, $\overline{abcd} = 5000 = 5 \cdot 625 = 3125$, $\overline{abcd} = 4100 = 5 \cdot 289 = 1445$, in other cases the max $\left(st^2\right) = 5 \cdot 169 = 845$.

If $s=6$, $\overline{abcd}=6000=6\cdot 1296=7776$, $\overline{abcd}=5100=6\cdot 676=4056$, $\overline{abcd}=4200=6\cdot 400=2400$, $\overline{abcd}=4110=6\cdot 676=1944$, $\overline{abcd}=3300=6\cdot 324=1944$, $\overline{abcd}=3120=6\cdot 196=1176$, $\overline{abcd}=2220=6\cdot 144=864$.
If $s=7$, $\overline{abcd}=7000=7\cdot 2401=16807$, $\overline{abcd}=6100=7\cdot 1369=9583$, $\overline{abcd}=5200=7\cdot 841=5887$, $\overline{abcd}=5110=7\cdot 729=5103$, $\overline{abcd}=4300=7\cdot 625=4375$, $\overline{abcd}=4210=7\cdot 441=3087$, $\overline{abcd}=4111=7\cdot 361=2527$, $\overline{abcd}=3310=7\cdot 361=2527$, $\overline{abcd}=2023=7\cdot 289$, (A), $\overline{abcd}=3112=7\cdot 225=1575$, $\overline{abcd}=2221=7\cdot 169=1183$.
If $s=8$, if a digit is greater or equal to 6 then $st^2\geq 8\cdot 1296=10368$ then the greatest digit is 5, $\overline{abcd}=5300=8\cdot 1156=9248$, $\overline{abcd}=5210=8\cdot 900=7200$, $\overline{abcd}=5111=8\cdot 784=6272$, $\overline{abcd}=4400=8\cdot 1024=8192$, $\overline{abcd}=4310=8\cdot 676=5408$, $\overline{abcd}=4220=8\cdot 576=4608$, $\overline{abcd}=4211=8\cdot 484=3872$, $\overline{abcd}=3320=8\cdot 484=3872$, $\overline{abcd}=3221=8\cdot 324=2592$, $\overline{abcd}=3311=8\cdot 400=3200$, $\overline{abcd}=2222=8\cdot 256=2048$.
If $s=9$, then the greatest digit is 6 then $st^2\geq 9\cdot 1296=11664$ then the greatest digit is 5, $\overline{abcd}=5400=9\cdot 1681=15129$, $\overline{abcd}=5310=9\cdot 1225=11025$, $\overline{abcd}=5220=9\cdot 1089=9801$, $\overline{abcd}=5211=9\cdot 961=8649$, $\overline{abcd}=4410=9\cdot 1089=9801$, $\overline{abcd}=4311=9\cdot 729=6561$, $\overline{abcd}=4320=9\cdot 841=7569$, $\overline{abcd}=4221=9\cdot 625=5625$, $\overline{abcd}=3330=9\cdot 729=6561$, $\overline{abcd}=3312=9\cdot 529=4761$, $\overline{abcd}=3222=9\cdot 441=3969$.
It is easy to check that if one of a,b,c,d is 7 $st^2\geq 9999$, which is impossible. Instead we get that $\overline{abcd}\leq 6100$. But, using Hölder's inequality, we get

$$16\times 6100\geq 16(\overline{abcd})=(1+1+1+1)^2(s)(t)^2\geq s^5\Leftrightarrow s<10.$$

Since s is an integer, we have $s\leq 9$, which we already considered. In conclusion, the only special numbers are $\overline{abcd}=2023$ and 2400. ▲

73. *Proposed by Leonard Giugiuc, Romania.* Let x,y and z be non-negative real numbers, no two of which are simultaneously zero, such that $x+y+z=3$. Find the least upper bound of the expression

$$\frac{(x+y)(y+z)(z+x)}{xy+yz+zx}\cdot\left(\frac{1}{x+3}+\frac{1}{y+3}+\frac{1}{z+3}\right).$$

Solution by Michel Bataille, Rouen, France. We denote by $\mathcal{C}$ the constraints $x,y,z\geq 0,(x+y)(y+z)(z+x)\neq 0,x+y+z=3$ and set

$$g(x,y,z)=\frac{(x+y)(y+z)(z+x)}{xy+yz+zx},\quad h(x,y,z)=\frac{1}{x+3}+\frac{1}{y+3}+\frac{1}{z+3}$$

and $f(x,y,z)=g(x,y,z)h(x,y,z)$.
We claim that the least upper bound of $f(x,y,z)$ for (x,y,z) satisfying $\mathcal{C}$ is $\dfrac{5}{2}$.

Let M denote the desired l.u.b. and let $t\in(0,3/2)$. Since $(t,t,3-2t)$ satisfies $\mathcal{C}$, we have $f(t,t,3-2t)\leq M$, that is,

$$\left(3-\frac{t(3-2t)}{6-3t}\right)\cdot\frac{15-3t}{(3+t)(6-2t)}\leq M. \tag{1}$$

(Note that $g(x,y,z)=\frac{(x+y+z)(xy+yz+zx)-xyz}{xy+yz+zx}=(x+y+z)-\frac{xyz}{xy+yz+zx}$.)
Letting t approach 0 in (1), we obtain $\frac{5}{2}\leq M$.

Thus, the claim follows if we show that $f(x,y,z) \leq \frac{5}{2}$ whenever (x,y,z) satisfies $\mathcal{C}$. Let (x,y,z) be such a triple. Then we have $g(x,y,z) = 3 - \frac{xyz}{xy+yz+zx} \leq 3$ and since $h(x,y,z) > 0$ it is sufficient to prove that $h(x,y,z) \leq \frac{5}{6}$. Setting $m = xy+yz+zx, p = xyz$, a simple calculation gives

$$h(x,y,z) = \frac{m+45}{3m+p+54} = \frac{5}{6} \cdot \frac{6m+270}{15m+5p+270},$$

and $h(x,y,z) \leq \frac{5}{6}$ follows since $0 < 6m \leq 15m + 5p$. ▲

Also solved by Titu Zvonaru, Comănești, Romania and the proposer.

74. *Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam.* Given the triangle ABC. The internal angle bisectors from A, B, C meet sides BC, CA, AB at A_1, B_1, C_1 respectively. Prove

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} + \frac{\cos(\overrightarrow{BB_1}, \overrightarrow{CC_1})}{\cos\frac{A}{2}} + \frac{\cos(\overrightarrow{CC_1}, \overrightarrow{AA_1})}{\cos\frac{B}{2}} + \frac{\cos(\overrightarrow{AA_1}, \overrightarrow{BB_1})}{\cos\frac{C}{2}} = 0.$$

Solution by Titu Zvonaru, Comănești, Romania. We have

$$\cos(\overrightarrow{BB_1}, \overrightarrow{CC_1}) = -\cos\left(180^\circ - \frac{B+C}{2}\right) = -\cos\left(90^\circ - \frac{A}{2}\right) = -\sin\frac{A}{2}.$$

After cyclically using the above equality, the conclusion follows by definition of the tangent function. ▲

Also solved by the proposer.

75. *Proposed by Besfort Shala, University of Bristol, Bristol, United Kingdom.* Let n be a positive integer. Find the smallest value of k (in terms of n) such that the system of equations

$$x_1 + x_2 + \cdots + x_n = x_1^2 + x_2^2 + \cdots + x_n^2 = \cdots = x_1^k + x_2^k + \cdots + x_n^k$$

has a unique solution over the positive real numbers.

Solution by Edi Berisha, Republic of Kosova. It is clear that for any n and k, we have the solution $x_i = 1$ for all $1 \leq i \leq n$. Thus, we must find all k (which may or may not be dependent on n) such that there are no other solutions besides this. Note that the corresponding value of k for $n = 1$ is clearly $k = 2$. Now let $n > 1$. For $k = 2$, we always have another solution to the system of equations given by $x_1 = \frac{1}{2}(\sqrt{n}+1)$, and $x_i = \frac{1}{2}$ for $2 \leq i \leq n$.
Next, we claim that $k = 3$ (for all $n \geq 2$) is the smallest k that satisfies the condition of the problem. Indeed, we have

$$\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n x_i^3\right) = \left(\sum_{i=1}^n x_i^2\right)^2$$

On the other hand, by Cauchy-Schwarz we have

$$\left(\sum_{i=1}^n x_i\right)\left(\sum_{i=1}^n x_i^3\right) \geq \left(\sum_{i=1}^n x_i^2\right)^2$$

with equality if and only if $x_i = 1$ for all $1 \leq i \leq n$. ▲

Also solved by the proposer.