Mathproblems
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PROBLEMS AND SOLUTIONS

Proposals and solutions must be legible and should appear on separate sheets, each indicating the name of the sender. It would be preferred to submit your proposals and solutions as `TeX` files. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editors have supplied a solution. The editors encourage undergraduate and pre-college students to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the grade and school name. Solutions will be evaluated for publication by a committee of professors according to a combination of criteria. Questions concerning proposals and/or solutions can be sent by e-mail to: `mathproblems-ks@hotmail.com`.

Solutions to the problems in this issue should arrive before February 25, 2024

Problems

159. *Proposed by Mohsen Soltanifar, University of Toronto, Canada.*
Let $n \geq 1$ and consider discrete uniform probability distribution on the set $S_n = \{1, 2, \ldots, n\}$ and for given $0 < p < 1$, $N(p, n)$ be the number of events of S_n with probability p. Define a function $f : (0, 1) \to \mathbb{R}$ via:

$$f(p) = \lim_{n\to\infty} \Big(\frac{\log_2(N(p, n))}{n}\Big).$$

Compute the maximum value of the function f.

160. *Proposed by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy.* Evaluate

$$\int_0^\infty \frac{e^{-x}(1 - e^{-2x})(1 - e^{-4x})(1 - e^{-6x})}{x(1 + e^{-14x})} \mathrm{d}x.$$

161. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Find all integer 2×2 matrices satisfying the equation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} 2^a & 2^b \\ 2^c & 2^d \end{pmatrix}.$$

162. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" Secondary School and "Dr. C. Angelescu High School", Buzău, Romania.*
Let a be a positive real number. Compute

$$\lim_{x\to\infty} \left((x+a) \sin \frac{1}{x+a} \sqrt[x+1]{\Gamma(x+2)} - x \sin \frac{1}{x} \sqrt[x]{\Gamma(x+1)} \right).$$

163. *Proposed by Valmir Krasniqi, Institute of Science, Technology, Engineering and Mathematics, Republic of Kosova.* In each square of an 8×8 grid, there is a lamp. We say that two lamps on the grid are neighbors if they share a vertex or an edge. Initially, some lamps are lit. Every minute, each lamp that has at least three lit neighboring lamps will also light up. What is the smallest number of lamps that need to be lit initially to ensure that after some time all the lamps will be lit?

164. *Proposed by Vasile Cirtoaje, Petroleum-Gas University of Ploieşti, Romania and Leonard Giugiuc, Romania.* Let $n \geq 3$ be an integer and let $a_1, a_2, \ldots, a_n$ be non-negative real numbers such that $\sum_{i=1}^n \frac{1}{a_i+1} \geq n-1$ and $\sum_{1\leq i<j\leq n} a_i a_j > 0$. Prove

$$(n-2)\sum_{i=1}^n a_i + \frac{n}{2\sum_{1\leq i<j\leq n} a_i a_j} \geq \frac{2n^2-4n+1}{n-1}.$$

When is equality attained?

165. *Proposed by Besfort Shala, University of Bristol, Bristol, United Kingdom.*
Let n be a positive integer. Suppose that there exists a simple graph G with n vertices and precisely $\lceil n^2/3 \rceil$ edges that can be drawn in the plane with less than $12(n/12)^4$ unordered pairs of edges that intersect at a point other than the vertices of the graph. Find all possible values of n.

No problem is ever permanently closed. We will be very pleased to consider publishing new solutions or comments on previous problems.

Solutions

152. *Proposed by Marian Dinca, Bucharest, Romania and Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Rumania.*
Let a, b, c and d be the lenghts of the sides of a convex quadrilateral inscribed in a circle with radius R. Prove the inequality

$$\frac{a^2}{b+c+d-a}+\frac{b^2}{a+c+d-b}+\frac{c^2}{a+b+d-c}+\frac{d^2}{a+b+c-d}\geq 2\sqrt{2}R.$$

Solution by Michel Bataille, Rouen, France. Let O be the centre of the circumcircle of $ABCD$. Without loss of generality, we suppose that $ABCD$ is counter-clockwise oriented and denote by $2\alpha, 2\beta, 2\gamma, 2\delta$ the measures in $(0, 2\pi)$ of the directed angles $\angle(\overrightarrow{OA},\overrightarrow{OB}), \angle(\overrightarrow{OB},\overrightarrow{OC}), \angle(\overrightarrow{OC},\overrightarrow{OD}), \angle(\overrightarrow{OD},\overrightarrow{OA})$, respectively. Then, whether O is interior to $ABCD$ or not, $\alpha, \beta, \gamma, \delta \in (0, \pi),\ \alpha+\beta+\gamma+\delta = \pi$ and

$$a := AB = 2R\sin\alpha,\ b := BC = 2R\sin\beta,\ c := CD = 2R\sin\gamma,\ d := DA = 2R\sin\delta.$$

With the above notation, the required inequality rewrites as

$$\sum_{\text{cyclic}}\frac{\sin^2\alpha}{\sin\beta+\sin\gamma+\sin\delta-\sin\alpha}\geq\sqrt{2}. \tag{1}$$

Now, using $\alpha+\beta+\gamma+\delta = \pi$ and some classical trigonometric formulas, we obtain

$$\sin\beta+\sin\gamma+\sin\delta-\sin\alpha = 4\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\delta}{2}\sin\frac{\delta+\beta}{2}$$

so that (1) becomes

$$\sum_{\text{cyclic}}\frac{\sin^2\alpha}{\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\delta}{2}\sin\frac{\delta+\beta}{2}}\geq 4\sqrt{2}. \tag{2}$$

Since the function $x\mapsto\ln\sin x$ is concave on $(0, \pi)$, we have

$$\ln\sin\frac{\beta+\gamma}{2}+\ln\sin\frac{\gamma+\delta}{2}+\ln\sin\frac{\delta+\beta}{2}\leq 3\ln\sin\frac{\beta+\gamma+\delta}{3} = 3\ln\sin\left(\frac{\pi-\alpha}{3}\right)$$

and so

$$0<\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\delta}{2}\sin\frac{\delta+\beta}{2}\leq\sin^3\left(\frac{\pi-\alpha}{3}\right).$$

Thus, to prove (2), it is sufficient to prove

$$\sum_{\text{cyclic}}\frac{\sin^2\left(3\cdot\frac{\pi-\alpha}{3}\right)}{\sin^3\left(\frac{\pi-\alpha}{3}\right)}\geq 4\sqrt{2}. \tag{3}$$

To this aim, we consider the function f defined on $(0, \frac{\pi}{3})$ by

$$f(x) = \frac{\sin^2 3x}{\sin^3 x} = \frac{(3-4\sin^2 x)^2}{\sin x} = \frac{9}{\sin x}-24\sin x+16\sin^3 x = \psi(\sin x)$$

where $\psi(t) = \frac{9}{t} - 24t + 16t^3$. For $x \in (0, \frac{\pi}{3})$, we obtain $f'(x) = \psi'(\sin x)\cos x$ and

$$f''(x) = \psi''(\sin x)\cos^2 x - \psi'(\sin x)\sin x = \frac{9(2-\sin^2 x)}{\sin^3 x} + 24(5 - 6\sin^2 x)\sin x.$$

Since $x \in (0, \frac{\pi}{3})$, we have $5 - 6\sin^2 x > 0$, hence $f''(x) > 0$ and so f is convex on the interval $(0, \frac{\pi}{3})$.
Finally, Jensen's inequality yields

$$\sum_{\text{cyclic}} f\left(\frac{\pi-\alpha}{3}\right) \geq 4f\left(\frac{\pi}{3} - \frac{1}{3}\cdot\frac{\alpha+\beta+\gamma+\delta}{4}\right) = 4f\left(\frac{\pi}{4}\right).$$

Since $f\left(\frac{\pi}{4}\right) = \sqrt{2}$, (3) holds and we are done. ▲

Also solved by Albert Stadler, Switzerland; Moti Levy, Rehovot, Israel; Marius Dragan, Romania and the proposer.

153. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let $\alpha > 1$ and let $a, b \in \mathbb{R}$, $b \neq 0$. Calculate

$$\lim_{n\to\infty} \begin{pmatrix} 1 - \frac{a}{n^\alpha} & -\frac{b}{n} \\ \frac{b}{n} & 1 + \frac{a}{n^\alpha} \end{pmatrix}^n.$$

Solution by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy. The eigenvalues of the matrix $M \doteq \begin{pmatrix} 1 - \frac{\alpha}{n^\alpha} & \frac{-b}{n} \\ \frac{b}{n} & 1 + \frac{\alpha}{n^\alpha} \end{pmatrix}$ are

$$\lambda_+ = 1 + i\sqrt{\frac{b^2}{n^2} - \frac{a^2}{n^{2\alpha}}}, \qquad \lambda_- = 1 - i\sqrt{\frac{b^2}{n^2} - \frac{a^2}{n^{2\alpha}}}$$

and the eigenvectors are

$$\left(1, \frac{a}{bn^{\alpha-1}} - i\frac{n}{b}\sqrt{\frac{b^2}{n^2} - \frac{a^2}{n^{2\alpha}}}\right), \qquad \left(1, \frac{a}{bn^{\alpha-1}} + i\frac{n}{b}\sqrt{\frac{b^2}{n^2} - \frac{a^2}{n^{2\alpha}}}\right),$$

respectively. The matrices

$$P = \begin{pmatrix} 1 & 1 \\ \frac{a}{bn^{\alpha-1}} - i\frac{n}{b}\sqrt{\frac{b^2}{n^2} - \frac{a^2}{n^{2\alpha}}} & \frac{a}{bn^{\alpha-1}} + i\frac{n}{b}\sqrt{\frac{b^2}{n^2} - \frac{a^2}{n^{2\alpha}}} \end{pmatrix}$$

and

$$P^{-1} = \begin{pmatrix} \frac{1}{2} - \frac{a}{2in^\alpha\sqrt{\frac{b^2}{n^2} - \frac{a^2}{n^{2\alpha}}}} & \frac{-b}{2in\sqrt{\frac{b^2}{n^2} - \frac{a^2}{n^{2\alpha}}}} \\ \frac{1}{2} + \frac{a}{2in^\alpha\sqrt{\frac{b^2}{n^2} - \frac{a^2}{n^{2\alpha}}}} & \frac{b}{2in\sqrt{\frac{b^2}{n^2} - \frac{a^2}{n^{2\alpha}}}} \end{pmatrix}$$

are such that

$$M^n = P\begin{pmatrix} \lambda_+ & 0 \\ 0 & \lambda_- \end{pmatrix}^n P^{-1} = P\begin{pmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{pmatrix} P^{-1},$$

$$\lambda_+^n = \left(1 + \frac{b^2}{n^2} - \frac{a^2}{n^{2\alpha}}\right)^{\frac{n}{2}} e^{in\arctan\sqrt{\frac{b^2}{n^2} - \frac{a^2}{n^{2\alpha}}}} \to e^{i|b|},$$

$$\lambda_-^n = \left(1 + \frac{b^2}{n^2} - \frac{a^2}{n^{2\alpha}}\right)^{\frac{n}{2}} e^{-in\arctan\sqrt{\frac{b^2}{n^2} - \frac{a^2}{n^{2\alpha}}}} \to e^{-i|b|},$$

$$\lim_{n\to\infty} P = \begin{pmatrix} 1 & 1 \\ \frac{-i|b|}{b} & \frac{i|b|}{b} \end{pmatrix} = P_\infty, \qquad \text{and} \qquad \lim_{n\to\infty} P^{-1} = \begin{pmatrix} \frac{1}{2} & \frac{ib}{2|b|} \\ \frac{1}{2} & \frac{-ib}{2|b|} \end{pmatrix} = P_\infty^{-1}.$$

It follows that

$$P_\infty \begin{pmatrix} \lambda_+^n & 0 \\ 0 & \lambda_-^n \end{pmatrix} P_\infty^{-1} = \begin{pmatrix} 1 & 1 \\ \frac{-i|b|}{b} & \frac{i|b|}{b} \end{pmatrix} \begin{pmatrix} e^{i|b|} & 0 \\ 0 & e^{-i|b|} \end{pmatrix} \begin{pmatrix} \frac{1}{2} & \frac{ib}{2|b|} \\ \frac{1}{2} & \frac{-ib}{2|b|} \end{pmatrix} =$$

$$= \begin{pmatrix} \cos b & \frac{-b}{|b|} \sin|b| \\ \frac{b}{|b|}\sin|b| & \cos b \end{pmatrix} = \begin{pmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{pmatrix}.$$

▲

Also solved by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria; Albert Stadler, Switzerland; Moubinool Omarjee, Paris, France; Moti Levy, Rehovot, Israel; Michel Bataille, Rouen, France; Leonard Giugiuc, Romania and the proposer.

154. *Proposed by Anastasios Kotronis, Athens, Greece and Haroun Meghaichi, student, the University of Science and Technology, Houari Boumediene, Algiers, Algeria.*
Let m be a positive integer and let

$$S_{n,m} = \sum_{k=1}^{n} (-1)^k \binom{n}{k} k^{-m}.$$

Show that:

(i) $S_{n,1} = -\ln n - \gamma - \frac{1}{2n} + \mathcal{O}(n^{-2})$ as $n \to +\infty$;

(ii) $S_{n,2} = -\dfrac{\ln^2 n}{2} - \gamma \ln n - \dfrac{\gamma^2}{2} - \dfrac{\pi^2}{12} - \dfrac{\ln n}{2n} + \dfrac{1-\gamma}{2n} + \mathcal{O}\left(n^{-2}\ln n\right)$ as $n \to +\infty$;

(iii) There exist real numbers $a_m, \ldots, a_0$ and $b_{m-1}, \ldots, b_0$ such that

$$S_{n,m} = \sum_{k=0}^{m} a_{m-k} \ln^{m-k} n + \sum_{k=0}^{m-1} b_{m-k-1} \frac{\ln^{m-k-1} n}{n} + \mathcal{O}\left(n^{-2}\ln^{m-1} n\right)$$

as $n \to +\infty$, and determine them.

Solution by Moti Levy, Rehovot, Israel. This problem has been treated and solved by Flajolet and Sedgewick in [1]. However, only asymptotics up to $O\left(\frac{\log^m n}{n}\right)$ were given. We follow the footsteps of Flajolet and Sedgewick and then complete the extra terms required for the $O\left(\frac{\log^{m-1} n}{n^2}\right)$ error term.
We begin by evaluating the complex integral over the contour C, which is a circle of radius R centered at the origin, namely (the radius R is greater than n)

$$\int_{\boldsymbol{C}} \frac{n!}{s^{m+1}(s-1)\cdots(s-n)} ds.$$

It can be shown that as the radius $R > n$, the integral value is zero.
By the residue theorem, we have

$$\int_{\boldsymbol{C}} \frac{n!}{s^{m+1}(s-1)\cdots(s-n)} ds = 2\pi i \sum_{s=0,1,2,\ldots,n} \operatorname{Res} \frac{n!}{s^{m+1}(s-1)\cdots(s-n)}.$$

The residue at $s=k\geq 1$ is

$$\begin{aligned}\operatorname*{Res}_{s=k}\frac{n!}{s^{m+1}\left(s-1\right)\cdots\left(s-n\right)} &= \lim_{s\to k}\frac{n!}{s\left(s-1\right)\cdots\left(s-k+1\right)\left(s-k-1\right)\cdots\left(s-n\right)}\frac{1}{s^{m}}\\ &= \frac{\left(-1\right)^{n-k}n!}{k!\left(n-k\right)!}\frac{1}{k^{m}}.\end{aligned}$$

The residue at $s=0$ is

$$\operatorname*{Res}_{s=0}\frac{n!}{s^{m+1}\left(s-1\right)\cdots\left(s-n\right)}=\frac{1}{m!}\lim_{s\to 0}\frac{d^{m}}{ds^{m}}\left(\frac{n!}{\left(s-1\right)\cdots\left(s-n\right)}\right).$$

Let $\omega_{n}\left(s\right)$ be defined as

$$\omega_{n}\left(s\right):=\frac{1}{\left(1-\frac{s}{1}\right)\left(1-\frac{s}{2}\right)\cdots\left(1-\frac{s}{n}\right)}.$$

Then $\frac{d^{m}}{ds^{m}}\left(\frac{n!}{(s-1)\cdots(s-n)}\right)=\frac{d^{m}}{ds^{m}}\left(\left(-1\right)^{n}\omega_{n}\left(s\right)\right).$
The residue at $s=0$ is the coefficient of s^{m} in the series expansion of $\omega_{n}\left(s\right)$ multiplied by $\left(-1\right)^{n}$, that is

$$\operatorname*{Res}_{s=0}\frac{n!}{s^{m+1}\left(s-1\right)\cdots\left(s-n\right)}=\left(-1\right)^{n}\left[s^{m}\right]\omega_{n}\left(s\right).$$

Summing all residues,

$$0=\int_{\boldsymbol{C}}\frac{n!}{s^{m+1}\left(s-1\right)\cdots\left(s-n\right)}ds=2\pi i\left(\sum_{k=1}^{n}\frac{\left(-1\right)^{n-k}n!}{k!\left(n-k\right)!}\frac{1}{k^{m}}+\left(-1\right)^{n}\left[s^{m}\right]\omega_{n}\left(s\right)\right).$$

It follows that

$$S_{n,m}:=\sum_{k=1}^{n}\left(-1\right)^{k}\binom{n}{k}\frac{1}{k^{m}}=-\left[s^{m}\right]\omega_{n}\left(s\right).$$

The following two equations (1) and (2) (which I present here without proofs) appear in [1]:

$$\omega_{n}\left(s\right)=\exp\left(\sum_{k=1}^{\infty}\zeta_{n}\left(k\right)\frac{s^{k}}{k}\right),\qquad \zeta_{n}\left(r\right):=\sum_{k=1}^{n}\frac{1}{k^{r}}. \tag{1}$$

$$-S_{n,m}=\sum_{m_{1}+2m_{2}+m_{3}+\cdots=m}^{n}\frac{1}{m_{1}!m_{2}!m_{3}!\cdots}\left(\frac{\zeta_{n}\left(1\right)}{1}\right)^{m_{1}}\left(\frac{\zeta_{n}\left(2\right)}{2}\right)^{m_{2}}\left(\frac{\zeta_{n}\left(3\right)}{3}\right)^{m_{3}}\cdots. \tag{2}$$

Using the asymptotic approximations

$$\zeta\left(r\right)\sim\zeta_{n}\left(r\right)+\frac{1}{r-1}\frac{1}{n^{r-1}}+O\left(\frac{1}{n^{r}}\right),\quad r>1 \tag{3}$$

and

$$\zeta_{n}\left(1\right)=H_{n}=\ln n+\gamma+\frac{1}{2n}+O\left(\frac{1}{n^{2}}\right), \tag{4}$$

we get:

$$-S_{n,m} \sim \tag{5}$$
$$\sum_{m_1+2m_2+3m_3+\cdots=m} \frac{1}{m_1!m_2!m_3!\cdots} \left(\ln n + \gamma + \frac{1}{2n}\right)^{m_1} \left(\frac{\zeta(2)}{2} - \frac{1}{2n}\right)^{m_2} \left(\frac{\zeta(3)}{3}\right)^{m_3} \left(\frac{\zeta(4)}{4}\right)^{m_4} \cdots$$
$$+ O\left(\frac{\ln^{m-1} n}{n^2}\right).$$

Setting $m = 1$ and $m = 2$ in (5) gives,

$$-S_{n,1} = \ln n + \gamma + \frac{1}{2n} + O\left(\frac{1}{n^2}\right),$$

and

$$\begin{aligned} -S_{n,2} &= \sum_{m_1+2m_2=2} \frac{1}{m_1!m_2!} \left(\ln n + \gamma + \frac{1}{2n}\right)^{m_1} \left(\frac{\zeta(2)}{2} - \frac{1}{2n}\right)^{m_2} \\ &= \frac{\ln^2 n}{2} + \frac{\gamma^2}{2} + \gamma \ln n + \frac{\ln n}{2n} + \frac{\gamma}{2n} + \frac{\pi^2}{12} - \frac{1}{2n} + O\left(\frac{\ln n}{n^2}\right). \end{aligned}$$

By equation (1) and the approximations (3) and (4),

$$\begin{aligned} \omega_n(x) &\sim \exp\left(\left(\ln n + \gamma + \frac{1}{2n}\right)x + \left(\zeta(2) - \frac{1}{n}\right)\frac{x^2}{2} + \sum_{k=3}^{\infty} \zeta(k) \frac{x^k}{k}\right) \qquad (6) \\ &= e^{(\ln n)x + \frac{1}{2n}\left(x - x^2\right)} \exp\left(\gamma x + \sum_{k=2}^{\infty} \zeta(k) \frac{x^k}{k}\right). \end{aligned}$$

From [2] page 276, we have

$$\Gamma(1-z) = \left(\gamma z + \sum_{k=2}^{\infty} \frac{\zeta(k)}{k} z^k\right). \tag{7}$$

Hence, we have

$$\omega_n(x) \sim e^{(\ln n)x} e^{\frac{1}{2n}\left(x-x^2\right)} \Gamma(1-x), \tag{8}$$

$$e^{\frac{1}{2n}\left(x-x^2\right)} = 1 + \frac{1}{2n}\left(x - x^2\right) + O\left(\frac{1}{n^2}\right),$$

$$\omega_n(x) \sim \left(1 + \frac{1}{2n}\left(x - x^2\right)\right) e^{(\ln n)x} \Gamma(1-x).$$

By the Taylor series expansion of the Gamma function at $x = 1$, we have

$$\Gamma(1-x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!} \Gamma^{(k)}(1) x^k,$$
$$e^{(\ln n)x} = \sum_{k=0}^{\infty} \frac{\ln^k n}{k!} x^k.$$

By the Cauchy product (which is the discrete convolution of the sequences $\left((-1)^k \frac{1}{k!}\Gamma^{(k)}(1)\right)_{k\geq 0}$ and $\left(\frac{\ln^k n}{k!}\right)_{k\geq 0}$), we get

$$[x^m]\,\omega_n(x) \sim \frac{1}{m!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\Gamma^{(k)}(1)\ln^{m-k} n + O\left(\frac{\ln^{m-1}}{n}\right).$$

Now let $A_m := \frac{1}{m!}\sum_{k=0}^{m}(-1)^k\binom{m}{k}\Gamma^{(k)}(1)\ln^{m-k} n$. We have

$$[x^m]\left(1+\frac{1}{2n}x-\frac{1}{2n}x^2\right)\sum_{k=0}^{\infty}A_k x^k = A_m + \frac{1}{2n}\left(A_{m-1}-A_{m-2}\right).$$

Then

$$\begin{aligned}
&A_{m-1}-A_{m-2}\\
&=\frac{1}{(m-1)!}\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}\Gamma^{(k)}(1)\ln^{m-k-1} n-\frac{1}{(m-2)!}\sum_{k=0}^{m-2}(-1)^k\binom{m-2}{k}\Gamma^{(k)}(1)\ln^{m-k-2} n\\
&=\frac{1}{(m-1)!}\sum_{k=0}^{m-1}(-1)^k\binom{m-1}{k}\Gamma^{(k)}(1)\ln^{m-k-1} n+\frac{1}{(m-2)!}\sum_{k=1}^{m-1}(-1)^k\binom{m-2}{k-1}\Gamma^{(k-1)}(1)\ln^{m-k-1} n\\
&=\sum_{k=0}^{m-1}(-1)^k\left(\frac{\binom{m-1}{k}\Gamma^{(k)}(1)}{(m-1)!}+\frac{\binom{m-2}{k-1}\Gamma^{(k-1)}(1)}{(m-2)!}\right)\ln^{m-k-1} n.
\end{aligned}$$

Finally, we arrive at the required result:

$$S_{n,m}=\sum_{k=0}^{m}a_{m-k}\ln^{m-k} n+\sum_{k=0}^{m-1}b_{m-k-1}\frac{\ln^{m-k-1} n}{n}+O\left(\frac{\ln^{m-1} n}{n^2}\right) \qquad (9)$$

$$a_{m-k}=(-1)^{k+1}\frac{1}{m!}\binom{m}{k}\Gamma^{(k)}(1)$$

$$b_{m-k}=(-1)^{k+1}\frac{\binom{m-1}{k}}{2(m-1)!}\Gamma^{(k)}(1)+(-1)^{k+1}\frac{\binom{m-2}{k-1}}{2(m-2)!}\Gamma^{(k-1)}(1).$$

▲

Remark: The derivatives of the Gamma function at 1 can be evaluated by the following recurrence which appeared in [3]:

$$\Gamma^{(n+1)}(1)=-\gamma\Gamma^{(n)}(1)+n!\sum_{k=1}^{n}\frac{(-1)^{k+1}}{(n-k)!}\zeta(k+1)\,\Gamma^{(n-k)}(1),\qquad n\geq 0.$$

References:

[1] Philippe Flajolet, Robert Sedgewick, *Mellin transform and asymptotics: Finite differences and Rice's integrals*, Theoretical Computer Science, 144 (1995) pp. 101-124.

[2] E. T. Whittaker, G. N. Watson, *A Course of Modern Analysis*, Cambridge University Press, Fourth Edition.

[3] Junesang Choi, H. M. Srivastava, *Evaluation of Higher-Order Derivatives of the Gamma function*, Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat. 11 (2000), 9–18.

Also solved by Albert Stadler, Switzerland and the proposers.

155. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*
Let a be a positive real number, let $(L_n)_{n\geq 0}$ be Lucas sequence and let $(a_n)_{n\geq 0}$ be a positive real sequence such that $\lim_{n\to\infty} \frac{a_{n+1}}{n^2 a_n} = a$. Find

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}L_{n+1}}{(2n+1)!!}} - \sqrt[n]{\frac{a_n L_n}{(2n-1)!!}}\right).$$

Solution by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy. We will use the fact that $L_{n+1}/L_n \to (1+\sqrt{5})/2 = L$. We have

$$\sqrt[n+1]{\frac{a_{n+1}L_{n+1}}{(2n+1)!!}} - \sqrt[n]{\frac{a_n L_n}{(2n-1)!!}} =$$
$$= \sqrt[n]{\frac{a_n L_n}{n^n(2n-1)!!}} \frac{\sqrt[n+1]{\frac{a_{n+1}L_{n+1}}{(2n+1)!!}}\sqrt[n]{\frac{(2n-1)!!}{a_nL_n}} - 1}{\ln\left(\sqrt[n+1]{\frac{a_{n+1}L_{n+1}}{(2n+1)!!}}\sqrt[n]{\frac{(2n-1)!!}{a_nL_n}}\right)} \ln\left(\left(\frac{a_{n+1}L_{n+1}}{(2n+1)!!}\right)^{\frac{n}{n+1}} \frac{(2n-1)!!}{a_nL_n}\right).$$

We repeatedly use Cesaro–Stolz's result

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n}$$

provided that the second limit does exist. Based on this we compute

$$\lim_{n\to\infty} \frac{a_{n+1}L_{n+1}}{(n+1)^{n+1}(2n+1)!!} \frac{n^n(2n-1)!!}{a_nL_n} =$$
$$= \lim_{n\to\infty} \frac{a_{n+1}}{n^2a_n}\frac{L_{n+1}}{L_n}\frac{n^n}{(n+1)^n}\frac{n^2}{(n+1)(2n+1)} = \frac{aL}{2e}$$

and then

$$\lim_{n\to\infty}\sqrt[n]{\frac{a_nL_n}{n^n(2n-1)!!}} = \frac{aL}{2e}.$$

Moreover

$$\sqrt[n+1]{\frac{a_{n+1}L_{n+1}}{(2n+1)!!}}\sqrt[n]{\frac{(2n-1)!!}{a_nL_n}} = \sqrt[n+1]{\frac{a_{n+1}L_{n+1}}{(n+1)^{n+1}(2n+1)!!}}\sqrt[n]{\frac{n^n(2n-1)!!}{a_nL_n}}\frac{n}{n+1} \to$$
$$\to \frac{aL}{2e}\cdot\frac{2e}{aL}\cdot 1 = 1$$

This means that

$$\frac{\sqrt[n+1]{\frac{a_{n+1}L_{n+1}}{(2n+1)!!}}\sqrt[n]{\frac{(2n-1)!!}{a_nL_n}} - 1}{\ln\left(\sqrt[n+1]{\frac{a_{n+1}L_{n+1}}{(2n+1)!!}}\sqrt[n]{\frac{(2n-1)!!}{a_nL_n}}\right)} \to 1.$$

The last step is finding the limit of

$$\left(\frac{a_{n+1}L_{n+1}}{(2n+1)!!}\right)^{\frac{n}{n+1}}\frac{(2n-1)!!}{a_nL_n}=\frac{a_{n+1}L_{n+1}}{(2n+1)!!}\left(\frac{a_{n+1}L_{n+1}}{(2n+1)!!}\right)^{\frac{-1}{n+1}}\frac{(2n-1)!!}{a_nL_n}=$$

$$=\frac{a_{n+1}}{n^2a_n}\frac{L_{n+1}}{L_n}\frac{n}{2n+1}\left(\frac{(n+1)^{n+1}(2n+1)!!}{a_{n+1}L_{n+1}}\right)^{\frac{1}{n+1}}\to\frac{aL}{2}\frac{2e}{aL}=e.$$

Finally the limit is

$$\frac{aL}{2e}\cdot 1\cdot\ln(e)=\frac{aL}{2e}.$$

▲

Also solved by Arkady Alt, San Jose, California, USA; Albert Stadler, Switzerland; Michel Bataille, Rouen, France; Moubinool Omarjee, Paris, France and the proposers.

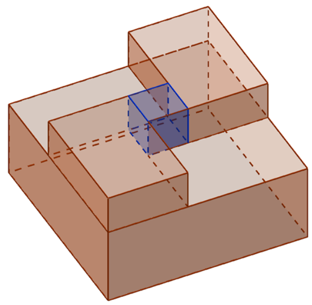
156. *Proposed by Dorlir Ahmeti, University of Prishtina, Republic of Kosovo and Alexander Gunning, Australia.*
With $(2n-1)^3$ identical unit cubes we build a bigger cube. We say any unit cube which is still in the big cube is *removable* if at least three faces of the unit cube are not shared with any other unit cube. We begin by removing a unit cube (by doing this we will cause other unit cubes to become removable), and we may repeat this procedure, removing further unit cubes. What is the minimum number of moves required to remove the unit cube which is in the centre of the big cube?

Solution by the authors. The construction is fairly straightforward: one can sequentially remove all the small cubes, and only those cubes in the cube of side length n in one corner. We will now show that we can remove no further cubes.
We define an n-cube to be a cube within the larger one of side length n. That is, if we index each small cube with (i,j,k) where $-n+1\leq i,j,k\leq n-1$ in the obvious way, an n-cube is the set of cubes

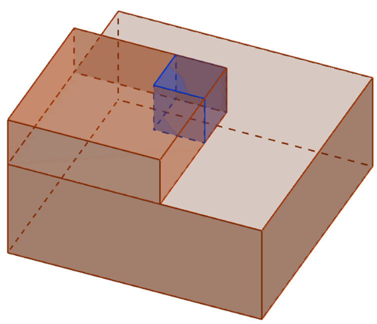
$$\{(i,j,k): x-n\leq i\leq x-1, y-n\leq j\leq y-1, z-n\leq k\leq z-1\}$$

for some choice of $1\leq x,y,z\leq n$. At any point during our removal of cubes we define an *intact* n-cube to be an n-cube all of whose small cubes have not yet been removed. Note that the central cube at $(0,0,0)$ is a component of all n^3 overlapping n-cubes. We show that if any small cube is removed, the total number of intact n-cubes decreases by at most 1, that is, the number of intact n-cubes is a monovariant. There are two cases.

Case 1: The cube removed has 2 exposed faces on opposite sides (and perhaps a number elsewhere). In this case the cube is not part of any intact n-cube.



Case 2: The cube removed has 3 exposed faces on mutually adjacent sides surrounding a corner. In this case if the cube is part of an intact n-cube it is the unique n-cube which this small cube lies in the corresponding corner of; thus there is at most 1 intact n-cube removed by this operation.



Thus the number of intact n-cubes decreases by at most 1 every time a cube is removed. After the centre cube is removed there are no remaining intact n-cubes. Thus at least n^3 cubes must be removed. ▲

157. *Proposed by Cornel Ioan Vălean, Timiş, Rumania.*
Prove that

$$2\sum_{n=1}^{\infty}\left(\zeta(3)\zeta(6)-H_n^{(3)}H_n^{(6)}\right)+7\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^6}=10\zeta(3)\zeta(5)-2\zeta(3)\zeta(6)-\frac{23}{12}\zeta(8).$$

where $H_n^{(m)}=1+\frac{1}{2^m}+\cdots+\frac{1}{n^m}$ denotes the nth harmonic number.

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria. First, note that

$$\zeta(3)\zeta(6)-H_n^{(3)}H_n^{(6)}=\zeta(6)(\zeta(3)-H_n^{(3)})+H_n^{(3)}(\zeta(6)-H_n^{(6)}).$$

Thus

$$\sum_{n=1}^{\infty}\left(\zeta(3)\zeta(6)-H_n^{(3)}H_n^{(6)}\right)=\zeta(6)\sum_{n=1}^{\infty}\left(\sum_{k=n+1}^{\infty}\frac{1}{k^3}\right)+\sum_{n=1}^{\infty}\left(\sum_{j=1}^{n}\frac{1}{j^3}\right)\left(\sum_{k=n+1}^{\infty}\frac{1}{k^6}\right)$$
$$=\zeta(6)\sum_{k=2}^{\infty}\frac{k-1}{k^3}+\sum_{1\le j\le n<k}\frac{1}{j^3k^6}$$
$$=\zeta(6)(\zeta(2)-\zeta(3))+\sum_{1\le j<k}\frac{k-j}{j^3k^6}$$
$$=\zeta(6)(\zeta(2)-\zeta(3))+\sum_{1\le j\le k}\frac{1}{j^3k^5}-\sum_{1\le j\le k}\frac{1}{j^2k^6}$$
$$=\zeta(6)(\zeta(2)-\zeta(3))+\sum_{k=1}^{\infty}\frac{H_k^{(3)}}{k^5}-\sum_{k=1}^{\infty}\frac{H_k^{(2)}}{k^6}.$$

This yields that

$$2\sum_{n=1}^{\infty}\left(\zeta(3)\zeta(6)-H_n^{(3)}H_n^{(6)}\right)+7\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^6}=2\zeta(6)(\zeta(2)-\zeta(3))+2S_{3,5}+5S_{2,6} \quad (1)$$

where

$$S_{p,q}=\sum_{n=1}^{\infty}\frac{H_n^{(p)}}{n^q}.$$

Now clearly we have

$$H_n^{(3)}=\sum_{j=1}^{\infty}\left(\frac{1}{j^3}-\frac{1}{(n+j)^3}\right)=\sum_{j=1}^{\infty}\frac{n^3+3nj(n+j)}{j^3(n+j)^3}.$$

So,

$$S_{3,5}=\sum_{n,j\ge1}\frac{n^3+3nj(n+j)}{n^5j^3(n+j)^3}=\sum_{n,j\ge1}\frac{n}{n^3j^3(n+j)^3}+3\sum_{n,j\ge1}\frac{j^2}{n^4j^4(n+j)^2}.$$

Now interchanging the roles of n and j in the sums and taking the half sum we get

$$S_{3,5}=\frac{1}{2}\sum_{n,j\ge1}\frac{n+j}{n^3j^3(n+j)^3}+\frac{3}{2}\sum_{n,j\ge1}\frac{n^2+j^2}{n^4j^4(n+j)^2}$$
$$=\frac{1}{2}\sum_{n,j\ge1}\frac{1}{n^3j^3(n+j)^2}+\frac{3}{2}\sum_{n,j\ge1}\frac{(n+j)^2-2nj}{n^4j^4(n+j)^2}$$
$$=\frac{1}{2}\sum_{n,j\ge1}\frac{1}{n^3j^3(n+j)^2}+\frac{3}{2}\sum_{n,j\ge1}\frac{1}{n^4j^4}-3\sum_{n,j\ge1}\frac{1}{n^3j^3(n+j)^2}$$
$$=\frac{3}{2}\zeta(4)^2-\frac{5}{2}\sum_{n,j\ge1}\frac{1}{n^3j^3(n+j)^2}.$$

Using the well-known values of $\zeta(4)$ and $\zeta(8)$ we conclude that

$$S_{3,5}=\frac{7}{4}\zeta(8)-\frac{5}{2}A,\quad\text{where}\quad A=\sum_{n,j\ge1}\frac{1}{n^3j^3(n+j)^2}. \quad (2)$$

On the other hand, because

$$H_n^{(2)} = \sum_{j=1}^{\infty} \left(\frac{1}{j^2} - \frac{1}{(n+j)^2} \right) = \sum_{j=1}^{\infty} \frac{n^2 + 2nj}{j^2(n+j)^2},$$

we have

$$S_{2,6} = \sum_{n,j \geq 1} \frac{n^2 + 2nj}{n^6 j^2 (n+j)^2} = \sum_{n,j \geq 1} \frac{j^2}{n^4 j^4 (n+j)^2} + 2 \sum_{n,j \geq 1} \frac{j^4}{n^5 j^5 (n+j)^2}.$$

Interchanging the roles of n and j and taking the half sum we get

$$\begin{aligned} S_{2,6} &= \frac{1}{2} \sum_{n,j \geq 1} \frac{j^2 + n^2}{n^4 j^4 (n+j)^2} + \sum_{n,j \geq 1} \frac{j^4 + n^4}{n^5 j^5 (n+j)^2} \\ &= \frac{1}{2} \sum_{n,j \geq 1} \frac{(n+j)^2 - 2nj}{n^4 j^4 (n+j)^2} + \sum_{n,j \geq 1} \frac{(n^2 - j^2)^2 + 2n^2 j^2}{n^5 j^5 (n+j)^2} \\ &= \frac{1}{2} \zeta(4)^2 - A + \sum_{n,j \geq 1} \frac{(n-j)^2}{n^5 j^5} + 2A \\ &= A + \frac{1}{2} \zeta(4)^2 + 2 \sum_{n,j \geq 1} \frac{1}{n^3 j^5} - 2 \sum_{n,j \geq 1} \frac{1}{n^4 j^4} \\ &= A - \frac{3}{2} \zeta(4)^2 + 2\zeta(3)\zeta(5). \end{aligned}$$

That is

$$S_{2,6} = A - \frac{7}{4} \zeta(8) + 2\zeta(3)\zeta(5). \tag{3}$$

From (2) and (3) we get

$$2S_{3,5} + 5S_{2,6} = -\frac{21}{4} \zeta(8) + 10\zeta(3)\zeta(5). \tag{4}$$

Replacing back in (1) and using the fact that $2\zeta(6)\zeta(2) = (10/3)\zeta(8)$ we get the required formula

$$2 \sum_{n=1}^{\infty} \left(\zeta(3)\zeta(6) - H_n^{(3)} H_n^{(6)} \right) + 7 \sum_{n=1}^{\infty} \frac{H_n^{(2)}}{n^6} = 10\zeta(3)\zeta(5) - 2\zeta(6)\zeta(3) - \frac{23}{12} \zeta(8).$$

▲

Remark. Formula (4) appears without proof in "*Euler Sums and Contour Integral Representation*", by Philippe Flajolet and Bruno Salvy, Experimental Mathematics, Vol. 7 (1998), No. 1, pp. 15–35.

Solution 2 by Michael E. Hoffman, U. S. Naval Academy, USA.
Define double zeta values as

$$\zeta(p,q)=\sum_{m>n\geq 1}^{n}\frac{1}{m^p n^q}$$

for p,q positive with $p>1$ and $p+q>2$. From [2] we have the following:
Theorem. If $p,q>1$, then

$$\sum_{n=1}^{\infty}\left(\zeta(p)-\sum_{j=1}^{n}\frac{1}{j^p}\right)\left(\zeta(q)-\sum_{k=1}^{n}\frac{1}{k^q}\right)$$
$$=\zeta(p,q-1)+\zeta(q,p-1)+\zeta(p+q+1).$$

From this result we can deduce a corollary.
Corollary. For $p,q,>2$,

$$\sum_{n=1}^{\infty}\zeta(p)\zeta(q)-H_n^{(p)}H_n^{(q)}=\zeta(p-1,q)+\zeta(q-1,p)+\zeta(p+q+1)-\zeta(p)\zeta(q).$$

Proof. We have

$$\sum_{n=1}^{\infty}\zeta(p)\zeta(q)-H_n^{(p)}H_n^{(q)}$$
$$=\sum_{n=1}^{\infty}\left(\zeta(p)-H_n^{(p)}\right)\left(\zeta(q)-H_n^{(q)}\right)+\sum_{n=1}^{\infty}\left(\zeta(p)H_n^{(q)}+\zeta(q)H_n^{(p)}-2H_n^{(p)}H_n^{(q)}\right)$$
$$=\zeta(p,q-1)+\zeta(q,p-1)+\zeta(p+q+1)-\zeta(p)\zeta(q)$$
$$+\sum_{n=1}^{\infty}H_n^{(q)}\left(\zeta(p)-H_n^{(p)}\right)+\sum_{n=1}^{\infty}H_n^{(p)}\left(\zeta(q)-H_n^{(q)}\right).$$

Now

$$\sum_{n=1}^{\infty}H_n^{(q)}\left(\zeta(p)-H_n^{(p)}\right)=\sum_{n=1}^{\infty}\left(1+\frac{1}{2^q}+\cdots+\frac{1}{n^q}\right)\left(\frac{1}{(n+1)^p}+\frac{1}{(n+2)^p}+\cdots\right)$$
$$=\sum_{m>n\geq 1}^{n}\frac{m-n}{m^p n^q}=\zeta(p-1,q)-\zeta(p,q-1)$$

and similarly

$$\sum_{n=1}^{\infty}H_n^{(p)}\left(\zeta(q)-H_n^{(q)}\right)=\zeta(q-1,p)-\zeta(q,p-1);$$

the conclusion follows.
From the corollary

$$\sum_{n=1}^{\infty}\zeta(3)\zeta(6)-H_n^{(3)}H_n^{(6)}=\zeta(2,6)+\zeta(5,3)+\zeta(8)-\zeta(3)\zeta(6),$$

and the evident relation

$$\sum_{n=1}^{\infty}\frac{H_n^{(3)}}{n^6}=\zeta(6,2)+\zeta(8)$$

follows

$$2\sum_{n=1}^{\infty}\left(\zeta(3)\zeta(6)-H_n^{(3)}H_n^{(6)}\right)+7\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^6}$$
$$=9\zeta(8)+2\zeta(2,6)+2\zeta(5,3)-2\zeta(3)\zeta(6)+7\zeta(6,2).$$

It is immediate that $\zeta(6,2)+\zeta(2,6)=\zeta(6)\zeta(2)-\zeta(8)$, and

$$\zeta(6)\zeta(2)=\frac{\pi^8}{5670}=\frac{5}{3}\zeta(8),$$

so this is

$$7\zeta(8)+2\zeta(6)\zeta(2)+5\zeta(6,2)+2\zeta(5,3)-2\zeta(3)\zeta(6) \tag{10}$$
$$=\frac{31}{3}\zeta(8)+5\zeta(6,2)+2\zeta(5,3)-2\zeta(3)\zeta(6).$$

Now there is the well known relation of double zeta values

$$\zeta(n)=\sum_{i=p+1}^{n-1}\binom{i-1}{p-1}\zeta(i,n-i)+\sum_{i=p+1}^{n-1}\binom{i-1}{q-1}\zeta(i,n-i)$$

when $p+q=n$, which in the case $n=8$, $p=3$, $q=5$ gives

$$\zeta(8)=3\zeta(4,4)+6\zeta(5,3)+15\zeta(6,2)+30\zeta(7,1)$$

or

$$2\zeta(5,3)+5\zeta(6,2)=\frac{1}{3}\zeta(8)-\zeta(4,4)-10\zeta(7,1).$$

Since $\zeta^2(4)=2\zeta(4,4)+\zeta(8)$,

$$\frac{1}{3}\zeta(8)-\zeta(4,4)=\frac{1}{3}\zeta(8)-\frac{1}{2}\left[\zeta^2(4)-\zeta(8)\right]=\frac{5}{6}\zeta(8)-\frac{1}{2}\zeta^2(4)=\frac{1}{4}\zeta(8),$$

and from the relation

$$\zeta(p,1)=\frac{p}{2}\zeta(p+1)-\frac{1}{2}\sum_{r=2}^{p-1}\zeta(r)\zeta(p+1-r)$$

going back to Euler [1] we have

$$10\zeta(7,1)=35\zeta(8)-10\zeta(2)\zeta(6)-5\zeta^2(4)-10\zeta(3)\zeta(5)$$
$$=\frac{25}{2}\zeta(8)-10\zeta(3)\zeta(5).$$

Hence the sum (10) is

$$\frac{31}{3}\zeta(8)+\frac{1}{4}\zeta(8)-\frac{25}{2}\zeta(8)+10\zeta(3)\zeta(5)-2\zeta(3)\zeta(6)=-\frac{23}{12}\zeta(8)+10\zeta(3)\zeta(5)-2\zeta(3)\zeta(6).$$

References:

[1] L. Euler, *Meditationes circa singulare serierum genus*, Opra Omnia, vol. I.15, B. G. Teubner, Berlin, 1927, pp. 217-267.
[2] M. E. Hoffman, *Sums of Products of Riemann Zeta Tails*, Mediterr. J. Math. 13 (2016), 2771–2781.

Also solved by Albert Stadler, Switzerland, Moti Levy, Rehovot, Israel, and the proposer.

158. *Proposed by Sava Grozdev, VUZF University of Finance, Business and Entrepreneurship, Bulgaria, Hiroshi Okumura, Department of Mathematics, Yamato University, Osaka, Japan and Deko Dekov, Stara Zagora, Bulgaria.*
Let ABC be a triangle with side lengths $BC = a, CA = b$ and $AB = c$. Prove that the pedal triangle of the inverse of the orthocenter of the triangle ABC in the circumcircle of the triangle ABC is similar to the orthic triangle of the triangle ABC. Find the similitude ratio as function of a, b, c.

Solution by Albert Stadler, Switzerland. We may assume that the vertices of triangle ABC are given by the complex numbers $A(Re^{iu})$, $B(Re^{iv})$, $C(Re^{iw})$ where R denotes the circumradius. By the law of sines, we have

$$a = BC = 2R\sin(A), \quad b = CA = 2R\sin(B), \quad c = AB = 2R\sin(C).$$

The orthocenter H equals $H(Re^{iu} + Re^{iv} + Re^{iw})$ (see for instance Wikipedia: Altitude (triangle)). Then the inverse H' of the orthocenter H with respect to the circumcircle equals

$$H'\left(R\frac{e^{iu} + e^{iv} + e^{iw}}{|e^{iu} + e^{iv} + e^{iw}|^2}\right) = H'\left(\frac{R}{e^{-iu} + e^{-iv} + e^{-iw}}\right).$$

The orthic triangle has side lengths $a|\cos A|$, $b|\cos B|$, $c|\cos C|$ (see for instance ProofWiki: Sides of Orthic Triangle of Acute Triangle).
The pedal triangle of point P with respect to a given triangle ABC is formed by the feet of the perpendiculars from P to the sides of triangle ABC. Let D, E, F be the feet of the perpendiculars from P to the sides BC, CA, AB, respectively. It is known (see for instance Cut-the-knot: Pedal Triangle) that

$$EF = \frac{AP \cdot BC}{2R}, \quad FD = \frac{BP \cdot CA}{2R}, \quad DE = \frac{CP \cdot AB}{2R}.$$

We have $P = H'$. So the pedal triangle has side lengths

$$\frac{AH' \cdot a}{2R}, \quad \frac{BH' \cdot b}{2R}, \quad \frac{CH' \cdot c}{2R}.$$

We have

$$AH' = \left|\frac{R}{e^{-iu} + e^{-iv} + e^{-iw}} - Re^{iu}\right| = \left|\frac{\mathrm{R}e^{iv} + \mathrm{R}e^{iw}}{e^{-iu} + e^{-iv} + e^{-iw}}\right|$$

$$= \frac{2R\left|\cos\left(\frac{v-w}{2}\right)\right|}{|e^{-iu} + e^{-iv} + e^{-iw}|} = \frac{2R|\cos A|}{|e^{-iu} + e^{-iv} + e^{-iw}|},$$

by the central angle theorem (see for instance Wikipedia: Inscribed angle).
Analogously,

$$BH' = \left|\frac{R}{e^{-iu} + e^{-iv} + e^{-iw}} - Re^{iv}\right| = \frac{2R|\cos B|}{|e^{-iu} + e^{-iv} + e^{-iw}|},$$

$$CH' = \left|\frac{R}{e^{-iu} + e^{-iv} + e^{-iw}} - Re^{iw}\right| = \frac{2R|\cos C|}{|e^{-iu} + e^{-iv} + e^{-iw}|}.$$

So the side lengths of the pedal triangle are

$$\frac{a|\cos A|}{|e^{-iu} + e^{-iv} + e^{-iw}|}, \frac{b|\cos B|}{|e^{-iu} + e^{-iv} + e^{-iw}|}, \frac{c|\cos C|}{|e^{-iu} + e^{-iv} + e^{-iw}|},$$

which proves that the pedal triangle of the inverse of the orthocenter of triangle ABC with respect to the circumcircle of triangle ABC is similar to the orthic triangle of triangle ABC and the similitude ratio equals

$$\left|\frac{1}{e^{-iu}+e^{-iv}+e^{-iw}}\right| = \left|\frac{1}{e^{iu}+e^{iv}+e^{iw}}\right| = \frac{R}{OH},$$

where OH denotes the distance from the circumcenter to the orthocenter. It is known (see for instance MathWorld: Circumcenter & WolframAlpha: Circumradius triangle) that

$$R = \frac{abc}{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}},$$

$$OH^2 = 9R^2 - a^2 - b^2 - c^2 = \frac{a^2b^2c^2 - (-a^2+b^2+c^2)(a^2+b^2-c^2)(a^2-b^2+c^2)}{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}.$$

We conclude that

$$\frac{R}{OH} = \frac{abc}{\sqrt{a^2b^2c^2 - (-a^2+b^2+c^2)(a^2+b^2-c^2)(a^2-b^2+c^2)}} = \frac{1}{\sqrt{1-8\cos A\cos B\cos C}}.$$

▲

Also solved by Michel Bataille, Rouen, France, Leonard Giugiuc, Romania, and the proposers.

MATHCONTEST SECTION

This section of the Journal offers readers an opportunity to solve interesting and elegant mathematical problems that have appeared in Math Contests around the world and that are most appropriate for undergraduate Math Olympiad training. Proposals are always welcome. The source of the proposals will appear when the solutions are published.

Proposals

110. Let n be a positive integer and let A be a real $n\times n$ matrix such that $A^{2023} = I$, where I is the $n \times n$ identity matrix. Suppose there exists a real number λ such that $(A - \lambda I)^2 = 0$. Prove that $A = I$.

111. Show that the sequence $x_n = \sin^2(n)$ is divergent.

112. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that for any zero x_0 of f, there exists $n \in \mathbb{N}$ and a continuous function $g : \mathbb{R} \to \mathbb{R}$ such that $g(x_0) \neq 0$ and

$$f(x) = (x - x_0)^n g(x)$$

for all $x \in \mathbb{R}$.

(a) Prove that f has a finite number of zeros in the interval $[-M, M]$ for any $M > 0$.
(b) Give an example of such a function with infinitely many zeros.
(c) Is it possible for f to have uncountably many zeros?

113. Let k be a positive integer and let $\varphi_k : \mathbb{N} \to \mathbb{R}$ be the multiplicative function defined by $\varphi_k(p^\alpha) = p^{(\alpha-1)k}(p^k - 1)$ on prime powers. Find all real β such that the limit

$$\lim_{n\to\infty} \frac{\frac{\varphi_k(1)}{1} + \frac{\varphi_k(2)}{2} + \cdots + \frac{\varphi_k(n)}{n}}{n^\beta}$$

exists and is finite. What is the value of the limit in each case?

114. (a) Find a constant $c > 0$ satisfying the following property: there exists $A \geq 1$ such that for all integers $n \geq A$ and all configurations of n points inside a unit square, there are two points at distance at most $\dfrac{c}{\sqrt{n}}$.

(b) Find a constant $d > 0$ satisfying the following property: there exists $B \geq 1$ such that for all integers $n \geq B$, there exists a configuration of n points inside a unit square such that all mutual distances between pairs of points are at least $\dfrac{d}{\sqrt{n}}$.

Solutions

105. Let $f : [0;+\infty) \to \mathbb{R}$ be a continuous function such that $\lim\limits_{x\to+\infty} f(x) = L$ exists (it may be finite or infinite). Prove that

$$\lim_{n\to\infty} \int_0^1 f(nx)\,\mathrm{d}x = L.$$

(IMC 2017)

Solution 1 by Michel Bataille, Rouen, France.

Let $I_n = \int_0^1 f(nx)\,dx = \frac{1}{n}\int_0^n f(u)\,du$. We first suppose that $L = 0$ and show that $\lim\limits_{n\to\infty} I_n = 0$.

Let $\varepsilon > 0$; since $\lim\limits_{x\to\infty} f(x) = 0$, we may fix $x_0 > 0$ such that $|f(x)| \le \varepsilon$ whenever $x \ge x_0$. Then, for any integer n with $n > x_0$ we have

$$\begin{aligned}
|I_n| &= \frac{1}{n}\left|\int_0^{x_0} f(u)\,du + \int_{x_0}^{n} f(u)\,du\right| \\
&\le \frac{1}{n}\left|\int_0^{x_0} f(u)\,du\right| + \frac{1}{n}\left|\int_{x_0}^{n} f(u)\,du\right| \\
&\le \frac{1}{n}\int_0^{x_0} |f(u)|\,du + \frac{1}{n}\int_{x_0}^{n} |f(u)|\,du \\
&\le \frac{1}{n}\int_0^{x_0} |f(u)|\,du + \frac{\varepsilon(n-x_0)}{n}.
\end{aligned}$$

Since $\lim\limits_{n\to\infty} \frac{1}{n}\int_0^{x_0} |f(u)|\,du = 0$ and $\lim\limits_{n\to\infty} \frac{\varepsilon(n-x_0)}{n} = \varepsilon$, it follows that $\limsup\limits_{n\to\infty} |I_n| \le \varepsilon$. Since this holds for any positive ε, we must have $\limsup\limits_{n\to\infty} |I_n| = 0$ and so $\lim\limits_{n\to\infty} I_n = 0$.

If $L \in \mathbb{R}$ and $L \ne 0$, we apply the above result to the function $x \mapsto f(x) - L$. Observing that $\int_0^1 L\,dx = L$, we obtain

$$0 = \lim_{n\to\infty} \int_0^1 (f(nx) - L)\,dx = \lim_{n\to\infty} \left(\int_0^1 f(nx)\,dx - L\right)$$

and so $\lim\limits_{n\to\infty} I_n = L$.

Suppose that $L = \infty$. Let A be any positive real number; we can fix $x_0 > 0$ such that $f(x) \ge A$ whenever $x > x_0$ and for $n > x_0$ write

$$I_n = \frac{1}{n}\int_0^{x_0} f(u)\,du + \frac{1}{n}\int_{x_0}^{n} f(u)\,du \ge \frac{1}{n}\int_0^{x_0} f(u)\,du + \frac{n-x_0}{n}\cdot A.$$

It follows that $\liminf\limits_{n\to\infty} I_n \ge A$ and since this holds for all positive A, we must have $\liminf\limits_{n\to\infty} I_n = \infty$ and so $\lim\limits_{n\to\infty} I_n = \infty = L$.

Lastly, if $L = -\infty$, we apply the previous result to the function $-f$ and obtain $\lim\limits_{n\to\infty} (-I_n) = \infty$ so that $\lim\limits_{n\to\infty} I_n = -\infty = L$.

Solution 2 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria. Let $F(x) = \int_0^x f(t)dt$. Then for each integer $k \geq 0$ there exists $\xi_k \in (k, k+1)$ such that $F(k+1) - F(k) = f(\xi_k)$ by the intermediate value theorem. The assumption on f implies that $\lim_{k\to\infty} f(\xi_k) = L$ and consequently

$$\lim_{k\to\infty} \int_k^{k+1} f(x)dx = L.$$

Now, using Stolz–Cesàro Theorem we conclude that

$$\lim_{k\to\infty} \frac{1}{n} \sum_{k=0}^{n} \int_k^{k+1} f(x)dx = L.$$

The required conclusion follows because

$$\frac{1}{n} \sum_{k=0}^{n} \int_k^{k+1} f(x)dx = \frac{1}{n} \int_0^n f(t)dt = \int_0^1 f(nx)dx.$$

Also solved by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy and Arkady Alt, San Jose, California, USA.

106. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that f has infinitely many zeros, but there is no $x \in (a, b)$ with $f(x) = f'(x) = 0$.

(a) Prove that $f(a)f(b) = 0$.
(b) Give an example of such a function on $[0, 1]$.

(IMC 2016)

Solution by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy. Let Z be the set of zeroes of f. Since Z is clearly bounded, via the Weierstrass theorem, there exists at least an accumulation point p of Z. Let $\{x_k\}_{k=1}^{\infty}$ be a sequence converging to p. The continuity of F also implies

$$0 = f(x_k) \to f(p) \implies f(p) = 0$$

Now the differentiability of f implies

$$f'(p) = \lim_{k\to\infty} \frac{f(x_k) - f(p)}{x_k - p} = \lim_{k\to\infty} \frac{0 - 0}{x_k - p} = 0$$

This means that $f(p) = f'(p) = 0$ and this is possible only if $p = a$ or $p = b$ whence $f(a)f(b) = 0$.

(b) $f[0, 1] \to \mathbb{R}$, $f(x) = x^2 \sin(1/x)$, if $x \neq 0$, $f(0) = 0$.

Clearly f annihilates for $x = 0$ and $x = 1/(k\pi)$, $k \in \mathbb{Z}\backslash\{0\}$. Indeed

$$f'(0) = \lim_{h\to 0} \frac{h^2 \sin(1/h)}{h} = 0, \qquad f'(x) = 2x\sin(1/x) - \cos(1/x),\ x \neq 0$$

By observing that

$$2x \sin\frac{1}{x} - \cos\frac{1}{x}\Big|_{x=\frac{1}{k\pi}} = -\cos(k\pi) = (-1)^{k+1} \neq 0$$

we conclude the proof of part (b).

107. Let A be a $n \times n$ complex matrix whose eigenvalues have absolute value at most 1. Prove that

$$\|A^n\| \leq \frac{n}{\ln 2}\|A\|^{n-1}.$$

(Here $\|B\| = \sup\limits_{\|x\|\leq 1} \|Bx\|$ for every $n \times n$ matrix B and $\|x\| = \sqrt{\sum\limits_{i=1}^{n} |x_i|^2}$ for every complex vector $x \in \mathbb{C}^n$.)

(IMC 2016)

Official solution.
Let $\chi(t) = (t-\lambda_1)(t-\lambda_2)\dots(t-\lambda_n) = t^n + c_1 t^{n-1} + \dots + c_n$ be the characteristic polynomial of A. From Vieta's formulas we get

$$|c_k| = \left|\sum_{1\leq i_1<\ldots<i_k\leq n} \lambda_{i_1}\cdots\lambda_{i_k}\right| \leq \sum_{1\leq i_1<\ldots<i_k\leq n} \left|\lambda_{i_1}\cdots\lambda_{i_k}\right| \leq \binom{n}{k} \quad (k=1,2,\ldots,n)$$

By the Cayley-Hamilton theorem we have $\chi(A) = 0$, so

$$\|A^n\| = \|c_1 A^{n-1} + \dots + c_n\| \leq \sum_{k=1}^{n} \binom{n}{k} \|A^k\| \leq \sum_{k=1}^{n} \binom{n}{k} r^k = (1+r)^n - r^n.$$

Combining this with the trivial estimate $\|A^n\| \leq r^n$, we have

$$\|A^n\| \leq \min\big(r^n, (1+r)^n - r^n\big).$$

Let $r_0 = \frac{1}{\sqrt[n]{2}-1}$; it is easy to check that the two bounds are equal if $r = r_0$, moreover

$$r_0 = \frac{1}{e^{\ln 2/n} - 1} < \frac{n}{\ln 2}.$$

For $r \leq r_0$ apply the trivial bound:

$$\|A^n\| \leq r^n \leq r_0 \cdot r^{n-1} < \frac{n}{\ln 2} r^{n-1}.$$

For $r > r_0$ we have

$$\|A^n\| \leq (1+r)^n - r^n = r^{n-1} \cdot \frac{(1+r)^n - r^n}{r^{n-1}}.$$

Notice that the function $f(r) = \frac{(1+r)^n - r^n}{r^{n-1}}$ is decreasing because the numerator has degree $n-1$ and all coefficients are positive, so

$$\frac{(1+r)^n - r^n}{r^{n-1}} < \frac{(1+r_0)^n - r_0^n}{r_0^{n-1}} = r_0\big((1+1/r_0)^n - 1\big) = r_0 < \frac{n}{\ln 2},$$

so $\|A^n\| < \frac{n}{\ln 2} r^{n-1}$.

108. Let k and n be positive integers with $n \geq k^2 - 3k + 4$, and let

$$f(z) = z^{n-1} + c_{n-2} z^{n-2} + \dots + c_0$$

be a polynomial with complex coefficients such that

$$c_0 c_{n-2} = c_1 c_{n-3} = \dots = c_{n-2} c_0 = 0.$$

Prove that $f(z)$ and $z^n - 1$ have at most $n-k$ common roots.

(IMC 2017)

Official solution Let $M = \{z : z^n = 1\}$, $A = \{z \in M : f(z) \neq 0\}$ and $A^{-1} = \{z^{-1} : z \in A\}$. We have to prove $|A| \geq k$.
Claim.

$$A \cdot A^{-1} = M.$$

That is, for any $n \in M$, there exist some elements $a, b \in A$ such that $ab^{-1} = \eta$.
Proof. As is well-known, for every integer m,

$$\sum_{z\in M} z^m = \begin{cases} n & \text{if } n|m \\ 0 & \text{otherwise.} \end{cases}$$

Define $c_{n-1} = 1$ and consider

$$\sum_{z\in M} z^2 f(z) f(\eta z) = \sum_{z\in M} z^2 \sum_{j=0}^{n-1} c_j z^j \sum_{\ell=0}^{n-1} c_\ell (\eta z)^\ell = \sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} c_j c_\ell \eta^\ell \sum_{z\in M} z^{j+\ell+2}$$

$$= \sum_{j=0}^{n-1}\sum_{\ell=0}^{n-1} c_j c_\ell \eta^\ell \begin{cases} n & \text{if } n|(j+\ell+2) \\ 0 & \text{otherwise} \end{cases} = c_{n-1}^2 + \sum_{j=0}^{n-2} c_j c_{n-2-j} \eta^{2-j} n \neq 0.$$

Therefore there exists some $b \in M$ such that $f(b) \neq 0$ and $f(\eta b) \neq 0$, i.e. $b \in A$, and $a = \eta b \in A$, satisfying $ab^{-1} = \eta$.
By double-counting the elements of M, from the Claim we conclude

$$|A|(|A| - 1) \geq |M \setminus \{1\}| = n - 1 \geq k^2 - 3k + 3 > (k-1)(k-2)$$

which shows $|A| > k - 1$

109. Let n be a positive integer, and let $p(x)$ be a polynomial of degree n with integer coefficients. Prove that

$$\max_{0\leq x\leq 1} |p(x)| > \frac{1}{e^n}.$$

(IMC 2015)

Solution by Omri Nisan Solan, Tel Aviv University, Isreal. It suffices to prove that $\int_0^1 \log|p(x)|dx \geq -n$. Denote $p(x) = x^k(1-x)^m q(x)$, where $q(0) \neq 0$, $q(1) \neq 0$. We have to prove $\int_0^1 \log|q(x)|dx \geq k + m - n = -\deg q(x)$. We know that $|q(0)| \geq 1$, $|q(1)| \geq 1$. General inequality (which holds for any complex polynomial q) is $\int_0^1 \log|q(x)|dx \geq \frac{\log|q(0)\cdot q(1)|}{2} - \deg q(x)$. Factorising $q(x) = C\prod(x - x_i)$ we reduce this to polynomial $q(x) = x - \alpha$ for some complex α. Without loss of generality $\alpha = u + iv$ for $u \leq 1/2$, $v \geq 0$. It follows from invariance of necessary fact under transforms $\alpha \to \bar{\alpha}$, $\alpha \to 1 - \alpha$. For the antiderivative of $\log|x - \alpha|$ on $[0, 1]$ we have the following formula: $\int \log|x - \alpha|d\alpha = \mathrm{Re}\,(x - \alpha)\log(x - \alpha) - x$, where $\log(x - \alpha)$ must vary continuously on $[0, 1]$ (this is possible unless $\alpha \in [0, 1]$, in this latter case we do not care on imaginary part of $\log(x - \alpha)$ at all.)

So, our desired inequality $\int_0^1 \log|x-\alpha|dx \geq (\log|1-\alpha| + \log|\alpha|)/2 - 1$ may be rewritten as

$$\mathrm{Re}\,(1-\alpha)\log(1-\alpha) - \alpha\log(-\alpha) \geq \frac{1}{2}\mathrm{Re}\,\left(\log(1-\alpha) + \log\alpha\right).$$

This in turn may be rewritten as

$$\mathrm{Re}\,(1-2\alpha)(\log(1-\alpha) - \log(-\alpha) \geq 0.$$

This follows immediately from formulae $1-2\alpha = (1-2u) - 2iv$, $1-2u \geq 0$, $v \geq 0$ $\log(1-\alpha) - \log(-\alpha) = \log|\frac{1-\alpha}{\alpha}| + i\varphi$, where $\varphi \in [0,\pi]$ is an angle between vectors $-\alpha$ and $1-\alpha$, i.e. both real and imaginary parts of $\log(1-\alpha) - \log(-\alpha)$ are non-negative. This is very unusual that two such different approaches work for exactly the same constant e, while this constant is not sharp.

MATHNOTES SECTION

Analogues of the established Landen-type identities in the form of series and some related Cauchy products

Cornel Ioan Vălean
Romania
cornel2001_ro@yahoo.com
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1. Introduction

Let's recall first some of the Landen-type identities in the form of series like $\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}H_n = -\frac{1}{x}\operatorname{Li}_2\left(\frac{x}{x-1}\right)$, $\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}(H_n^2+H_n^{(2)}) = -2\frac{1}{x}\operatorname{Li}_3\left(\frac{x}{x-1}\right)$ and, say, $\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}(H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}) = -6\frac{1}{x}\operatorname{Li}_4\left(\frac{x}{x-1}\right)$, which appear in *(Almost) Impossible Integrals, Sums, and Series (2019)*. Then how could the analogues of such established Landen-type identities be defined and constructed? For example, one could check how the solutions to the previous results are built and observe that $\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}H_n = -\sum_{n=1}^{\infty}\int_0^1 (xt)^{n-1}\log(1-t)\mathrm{d}t = -\int_0^1\frac{\log(1-t)}{1-xt}\mathrm{d}t$, $\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}(H_n^2+H_n^{(2)}) = \sum_{n=1}^{\infty}\int_0^1 (xt)^{n-1}\log^2(1-t)\mathrm{d}t = \int_0^1\frac{\log^2(1-t)}{1-xt}\mathrm{d}t$, $\sum_{n=1}^{\infty}\frac{x^{n-1}}{n}(H_n^3+3H_nH_n^{(2)}+2H_n^{(3)}) = -\sum_{n=1}^{\infty}\int_0^1 (xt)^{n-1}\log^3(1-t)\mathrm{d}t = -\int_0^1\frac{\log^3(1-t)}{1-xt}\mathrm{d}t$, and here we readily notice these logarithmic integrals, leading to results in terms of harmonic numbers, play a key part, that is, $\int_0^1 t^{n-1}\log(1-t)\mathrm{d}t$, $\int_0^1 t^{n-1}\log^2(1-t)\mathrm{d}t$, and $\int_0^1 t^{n-1}\log^3(1-t)\mathrm{d}t$. In a similar style, but this time considering the integrals $\int_0^1 t^{n-1}\operatorname{arctanh}(t)\mathrm{d}t$ and $\int_0^1 t^{n-1}\operatorname{arctanh}^2(t)\mathrm{d}t$, we'll be able to extract power series related to $\operatorname{Li}_2\left(-\frac{1+x}{1-x}\right)$ and $\operatorname{Li}_3\left(-\frac{1+x}{1-x}\right)$. Moreover, series representations involving the polylogarithmic parts $\operatorname{Li}_2\left(\frac{1-x}{2}\right)$, $\operatorname{Li}_3\left(\frac{1-x}{2}\right)$, $\operatorname{Li}_2\left(\frac{2x}{x-1}\right)$, and

$\mathrm{Li}_3\left(\frac{2x}{x-1}\right)$ are presented and extracted in the current work. In the last part of the paper, a bunch of curious Cauchy products are shown and proved, some of them serving as valuable supporting tools while proving results of the first two main theorems.

During the calculations, we'll consider results both from *(Almost) Impossible Integrals, Sums, and Series (2019)* and *More (Almost) Impossible Integrals, Sums, and Series (2023)*.

2. The main theorems and their proofs

Theorem 1. *(Two special series representations) For $|x|<1$, the following equalities hold:*

$$i)\ \frac{\pi^2}{12}+\mathrm{Li}_2\left(-\frac{1+x}{1-x}\right)$$
$$=\sum_{n=1}^{\infty}x^n\left((-1)^{n-1}\frac{\overline{H}_n}{n}-\frac{H_n}{n}-\log(2)\frac{1}{n}-\log(2)(-1)^{n-1}\frac{1}{n}\right);$$

$$ii)\ -\frac{3}{4}\zeta(3)-\mathrm{Li}_3\left(-\frac{1+x}{1-x}\right)$$
$$=\sum_{n=1}^{\infty}x^n\left(\frac{\pi^2}{12}\frac{1}{n}+\frac{\pi^2}{12}(-1)^{n-1}\frac{1}{n}+\log(2)\frac{H_n}{n}-\log(2)(-1)^{n-1}\frac{H_n}{n}+\frac{1}{2}\frac{H_n^2}{n}\right.$$
$$+(-1)^{n-1}\frac{H_n\overline{H}_n}{n}+\log(2)\frac{\overline{H}_n}{n}-\log(2)(-1)^{n-1}\frac{\overline{H}_n}{n}-\frac{1}{2}\frac{\overline{H}_n^2}{n}+(-1)^{n-1}\frac{\overline{H}_n^{(2)}}{n}$$
$$\left.-2(-1)^{n-1}\frac{1}{n}\sum_{k=1}^{n}(-1)^{k-1}\frac{H_k}{k}\right)$$
$$=\sum_{n=1}^{\infty}x^n\left(\frac{\pi^2}{12}\frac{1}{n}+\frac{\pi^2}{12}(-1)^{n-1}\frac{1}{n}+\log(2)\frac{H_n}{n}-\log(2)(-1)^{n-1}\frac{H_n}{n}+\frac{1}{2}\frac{H_n^2}{n}\right.$$
$$-(-1)^{n-1}\frac{H_n\overline{H}_n}{n}+\log(2)\frac{\overline{H}_n}{n}-\log(2)(-1)^{n-1}\frac{\overline{H}_n}{n}-\frac{1}{2}\frac{\overline{H}_n^2}{n}-(-1)^{n-1}\frac{\overline{H}_n^{(2)}}{n}$$
$$\left.+2(-1)^{n-1}\frac{1}{n}\sum_{k=1}^{n}\frac{\overline{H}_k}{k}\right),$$

where $H_n^{(m)}=1+\frac{1}{2^m}+\cdots+\frac{1}{n^m},\ m\geq 1$, is the nth generalized harmonic number of order m, $\overline{H}_n^{(m)}=1-\frac{1}{2^m}+\cdots+(-1)^{n-1}\frac{1}{n^m},\ m\geq 1$, represents the nth generalized skew-harmonic number of order m, ζ denotes the Riemann zeta function, and Li_n designates the Polylogarithm function.

Proof. The solution to the first point is straightforward if we consider the first result from Lemma 1, then multiply it by x^n and consider the summation from $n=1$ to ∞, which give

$$\sum_{n=1}^{\infty}x^n\left(\frac{1}{2}\log(2)\frac{1}{n}+\frac{1}{2}\log(2)(-1)^{n-1}\frac{1}{n}+\frac{1}{2}\frac{H_n}{n}-\frac{1}{2}(-1)^{n-1}\frac{\overline{H}_n}{n}\right)$$
$$=\sum_{n=1}^{\infty}x\int_0^1(xt)^{n-1}\operatorname{arctanh}(t)\mathrm{d}t=x\int_0^1\sum_{n=1}^{\infty}(xt)^{n-1}\operatorname{arctanh}(t)\mathrm{d}t=x\int_0^1\frac{\operatorname{arctanh}(t)}{1-xt}\mathrm{d}t$$

$$= -\frac{\pi^2}{24} - \frac{1}{2}\operatorname{Li}_2\left(-\frac{1+x}{1-x}\right),$$

whence the desired result follows. Note that for the last equality I also employed the case $n = 1$ of the point *iii*) of Lemma 1. The point *ii*) of the theorem is finalized in a similar style, except that now we use the second point and the case $n = 2$ of the third point of Lemma 1.
The two series representations also maintain their validity when x takes the value of -1. $\square$

Theorem 2. *(More special series representations) For $|x| < 1$, the following equalities hold:*

$$i)\ \frac{1}{2}\log^2(2) - \frac{\pi^2}{12} + \operatorname{Li}_2\left(\frac{1-x}{2}\right)$$

$$= \sum_{n=1}^{\infty} x^n \left(\frac{\overline{H}_n}{n} - (-1)^{n-1}\frac{1}{n^2} - \log(2)\frac{1}{n}\right);$$

$$ii)\ \log(2)\frac{\pi^2}{12} - \frac{7}{8}\zeta(3) - \frac{1}{6}\log^3(2) + \operatorname{Li}_3\left(\frac{1-x}{2}\right)$$

$$= \sum_{n=1}^{\infty} x^n \left(\left(\frac{1}{2}\log^2(2) - \frac{\pi^2}{12}\right)\frac{1}{n} - \log(2)\frac{1}{n^2} - (-1)^{n-1}\frac{1}{n^3} + \log(2)\frac{H_n}{n} - \frac{H_n\overline{H}_n}{n} + \frac{\overline{H}_n}{n^2} + \frac{1}{n}\sum_{k=1}^{n}(-1)^{k-1}\frac{H_k}{k}\right)$$

$$= \sum_{n=1}^{\infty} x^n \left(\left(\frac{1}{2}\log^2(2) - \frac{\pi^2}{12}\right)\frac{1}{n} - \log(2)\frac{1}{n^2} - (-1)^{n-1}\frac{1}{n^3} + \log(2)\frac{H_n}{n} + \frac{\overline{H}_n}{n^2} + \frac{\overline{H}_n^{(2)}}{n} - \frac{1}{n}\sum_{k=1}^{n}\frac{\overline{H}_k}{k}\right).$$

A curious dilogarithmic series representation related to a result by Srinivasa Ramanujan:

$$iii)\ -\operatorname{Li}_2\left(\frac{2x}{x-1}\right) = \sum_{n=1}^{\infty} x^n \left(\frac{H_n}{n} + \frac{\overline{H}_n}{n}\right).$$

$$iv)\ -\operatorname{Li}_3\left(\frac{2x}{x-1}\right)$$

$$= \sum_{n=1}^{\infty} x^n \left(\frac{1}{2}\frac{H_n^2}{n} + \frac{1}{2}\frac{H_n^{(2)}}{n} + \frac{1}{n}\sum_{k=1}^{n}\frac{\overline{H}_k}{k}\right)$$

$$= \sum_{n=1}^{\infty} x^n \left(\frac{1}{2}\frac{H_n^2}{n} + \frac{1}{2}\frac{H_n^{(2)}}{n} + \frac{H_n\overline{H}_n}{n} + \frac{\overline{H}_n^{(2)}}{n} - \frac{1}{n}\sum_{k=1}^{n}(-1)^{k-1}\frac{H_k}{k}\right),$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, is the nth generalized harmonic number of order m, $\overline{H}_n^{(m)} = 1 - \frac{1}{2^m} + \cdots + (-1)^{n-1}\frac{1}{n^m}$, $m \geq 1$, represents the nth generalized skew-harmonic number of order m, ζ denotes the Riemann zeta function, and Li_n *designates the Polylogarithm function.*

Proof. One solution to the point *i*) is immediate if we combine the result from the first point of Theorem 1, where we replace x by $-x$, with the Landen's dilogarithmic identity, $\operatorname{Li}_2(x) + \operatorname{Li}_2\left(\frac{x}{x-1}\right) = -\frac{1}{2}\log^2(1-x)$, and the fact that $-\frac{1}{2}\frac{\log^2(1+x)}{x} = \sum_{n=1}^{\infty}(-1)^{n-1}x^{n-1}\left(\frac{H_n}{n} - \frac{1}{n^2}\right)$, which is easily extracted based on the first result, the case $m = 1$, appearing in [3, Chapter 4, Section 4.6, p.398].

For a second solution to the point i), we exploit like in [1] the fact that $\sum_{n=1}^{\infty} t^n \overline{H}_n = \frac{\log(1+t)}{1-t}$, which is found in a more general form in [3, Chapter 4, Section 4.10, p.406], where if we further divide both sides by t, then apply partial fraction decomposition, and rearrange, we get $\sum_{n=1}^{\infty} t^{n-1}\overline{H}_n = \frac{\log(1+t)}{t(1-t)} = \frac{\log(1-(1-t)/2)}{1-t} + \frac{\log(2)}{1-t} + \frac{\log(1+t)}{t}$. Next, integrating both sides of the previous result from $t=0$ to $t=x$, and using the power series of $1/(1-t)$ and $\log(1+t)$, we arrive at the stated result from the point i).

Passing to the point ii), we want to multiply both sides of the result from the point i) by $1/(1-x)$, and then replacing x by t, integrating from $t=0$ to $t=x$, and exploiting that $\int_0^x \frac{1}{1-t}\mathrm{d}t = \int_0^x \sum_{n=1}^{\infty} t^{n-1}\mathrm{d}t = \sum_{n=1}^{\infty}\int_0^x t^{n-1}\mathrm{d}t = \sum_{n=1}^{\infty}\frac{x^n}{n}$, $|x|<1$, next $\frac{d}{dx}\left(\operatorname{Li}_3\left(\frac{1-x}{2}\right)\right) = -\frac{1}{1-x}\operatorname{Li}_2\left(\frac{1-x}{2}\right)$, together with the use of the special value of the Trilogarithm, $\operatorname{Li}_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) - \log(2)\frac{\pi^2}{12} + \frac{1}{6}\log^3(2)$, and further employing the Cauchy product of two series, and rearranging and reindexing, we get that

$$\log(2)\frac{\pi^2}{12} - \frac{7}{8}\zeta(3) - \frac{1}{6}\log^3(2) + \operatorname{Li}_3\left(\frac{1-x}{2}\right)$$

$$= \sum_{n=1}^{\infty}\frac{x^n}{n}\left(\left(\frac{1}{2}\log^2(2) - \frac{\pi^2}{12}\right) + \log(2)\sum_{k=1}^{n-1}\frac{1}{k} + \sum_{k=1}^{n-1}(-1)^{k-1}\frac{1}{k^2} - \sum_{k=1}^{n-1}\frac{\overline{H}_k}{k}\right)$$

$$= \sum_{n=1}^{\infty} x^n\left(\left(\frac{1}{2}\log^2(2) - \frac{\pi^2}{12}\right)\frac{1}{n} - \log(2)\frac{1}{n^2} - (-1)^{n-1}\frac{1}{n^3} + \log(2)\frac{H_n}{n} + \frac{\overline{H}_n}{n^2} + \frac{\overline{H}_n^{(2)}}{n} - \frac{1}{n}\sum_{k=1}^{n}\frac{\overline{H}_k}{k}\right)$$

$$= \sum_{n=1}^{\infty} x^n\left(\left(\frac{1}{2}\log^2(2) - \frac{\pi^2}{12}\right)\frac{1}{n} - \log(2)\frac{1}{n^2} - (-1)^{n-1}\frac{1}{n^3} + \log(2)\frac{H_n}{n} - \frac{H_n\overline{H}_n}{n} + \frac{\overline{H}_n}{n^2} + \frac{1}{n}\sum_{k=1}^{n}(-1)^{k-1}\frac{H_k}{k}\right),$$

where the last equality is achieved by the exploiting the point iii) of Lemma 2.

As regards the result from the third point, it is worth mentioning the following form obtained in [1], $\operatorname{Li}_2\left(\frac{2x}{1+x}\right) + \operatorname{arctanh}^2(x) = 2\sum_{n=1}^{\infty}\frac{x^{2n-1}}{2n-1}\left(1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}\right)$, where the focus is on the dilogarithmic part; and this last result, as shown at the given reference, is extracted from a result by Srinivasa Ramanujan, where the latter one is presented in *Ramanujan's Notebooks, Part I* by Bruce C. Berndt.

To prove the result, we differentiate the dilogarithmic part, rearrange, and then integrate back, which give that

$$-\operatorname{Li}_2\left(\frac{2x}{x-1}\right) = \int_0^x \left(-\operatorname{Li}_2\left(\frac{2y}{y-1}\right)\right)'\mathrm{d}y = \int_0^x \frac{\log(1+y)}{y(1-y)}\mathrm{d}y - \int_0^x \frac{\log(1-y)}{y(1-y)}\mathrm{d}y$$

$$= \sum_{n=1}^{\infty}\int_0^x y^{n-1}\overline{H}_n\mathrm{d}y + \sum_{n=1}^{\infty}\int_0^x y^{n-1}H_n\mathrm{d}y = \sum_{n=1}^{\infty} x^n\left(\frac{H_n}{n} + \frac{\overline{H}_n}{n}\right),$$

where in the calcualtions I used that $\sum_{n=1}^{\infty} t^n H_n = -\frac{\log(1-t)}{1-t}$ and $\sum_{n=1}^{\infty} t^n \overline{H}_n = \frac{\log(1+t)}{1-t}$, which are the cases $m=1$ of the generalizations in [3, Chapter 4, Section 4.6, p.398] and [3, Chapter 4, Section 4.10, p.406].

The result from the last point is pretty straightforward if start with the result from the previous point where we multiply both sides by $1/(x(1-x))$, then replace x by t, next integrate from $t=0$ to $t=x$, and finally apply the Cauchy product of two series and use the points $i)$ and $iii)$ of Lemma 2. So, we get that

$$-\int_0^x \frac{1}{t(1-t)} \operatorname{Li}_2\left(\frac{2t}{t-1}\right) \mathrm{d}t = -\operatorname{Li}_3\left(\frac{2x}{x-1}\right)$$

$$= \int_0^x \sum_{n=1}^{\infty} t^{n-1} \left(\sum_{k=1}^{n} \frac{H_k}{k} + \sum_{k=1}^{n} \frac{\overline{H}_k}{k}\right) \mathrm{d}t = \sum_{n=1}^{\infty} \frac{x^n}{n} \left(\sum_{k=1}^{n} \frac{H_k}{k} + \sum_{k=1}^{n} \frac{\overline{H}_k}{k}\right)$$

$$= \sum_{n=1}^{\infty} x^n \left(\frac{1}{2}\frac{H_n^2}{n} + \frac{1}{2}\frac{H_n^{(2)}}{n} + \frac{1}{n}\sum_{k=1}^{n} \frac{\overline{H}_k}{k}\right)$$

$$= \sum_{n=1}^{\infty} x^n \left(\frac{1}{2}\frac{H_n^2}{n} + \frac{1}{2}\frac{H_n^{(2)}}{n} + \frac{H_n \overline{H}_n}{n} + \frac{\overline{H}_n^{(2)}}{n} - \frac{1}{n}\sum_{k=1}^{n} (-1)^{k-1}\frac{H_k}{k}\right),$$

and the desired results have been proved.

Further investigations reveal that the initial two series representations remain valid when $x = \pm 1$, and the final two series representations remain valid for $x = -1$. □

Theorem 3. *(A bunch of curious Cauchy products) For $|x| < 1$, the following equalities hold:*

$$i)\ \operatorname{arctanh}^2(x)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} x^n \left(\frac{H_n}{n} - (-1)^{n-1}\frac{H_n}{n} + \frac{\overline{H}_n}{n} - (-1)^{n-1}\frac{\overline{H}_n}{n}\right);$$

$$ii)\ \frac{\operatorname{arctanh}^2(x)}{1-x}$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} x^n \left(\frac{1}{2}H_n^2 + H_n\overline{H}_n - \frac{1}{2}\overline{H}_n^2 + \overline{H}_n^{(2)} - 2\sum_{k=1}^{n}(-1)^{k-1}\frac{H_k}{k}\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} x^n \left(\frac{1}{2}H_n^2 - H_n\overline{H}_n - \frac{1}{2}\overline{H}_n^2 - \overline{H}_n^{(2)} + 2\sum_{k=1}^{n}\frac{\overline{H}_k}{k}\right);$$

$$iii)\ \frac{\pi^2}{12}\frac{1}{1-x} + \frac{1}{1-x}\operatorname{Li}_2\left(-\frac{1+x}{1-x}\right)$$

$$= \sum_{n=1}^{\infty} x^n \left(\frac{1}{2}\overline{H}_n^2 - \log(2)\overline{H}_n - \frac{1}{2}H_n^2 - \log(2)H_n\right);$$

$$iv)\ \frac{\pi^2}{12}\frac{1}{1-x} + \frac{1}{1-x}\operatorname{Li}_2\left(-\frac{1-x}{1+x}\right)$$

$$= \sum_{n=1}^{\infty} x^n \left(\log(2)H_n - H_n\overline{H}_n + \log(2)\overline{H}_n - \overline{H}_n^{(2)} + 2\sum_{k=1}^{n}(-1)^{k-1}\frac{H_k}{k}\right)$$

$$= \sum_{n=1}^{\infty} x^n \left(\log(2) H_n + H_n \overline{H}_n + \log(2) \overline{H}_n + \overline{H}_n^{(2)} - 2 \sum_{k=1}^{n} \frac{\overline{H}_k}{k} \right);$$

$$v) \ \left(\frac{\pi^2}{12} - \frac{1}{2} \log^2(2) \right) \frac{1}{1-x} - \frac{1}{1-x} \operatorname{Li}_2 \left(\frac{1-x}{2} \right)$$

$$= \sum_{n=1}^{\infty} x^n \left(\log(2) H_n - H_n \overline{H}_n + \sum_{k=1}^{n} (-1)^{k-1} \frac{H_k}{k} \right)$$

$$= \sum_{n=1}^{\infty} x^n \left(\log(2) H_n + \overline{H}_n^{(2)} - \sum_{k=1}^{n} \frac{\overline{H}_k}{k} \right);$$

$$vi) \ \left(\frac{1}{2} \log^2(2) - \frac{\pi^2}{12} \right) \frac{1}{1-x} + \frac{1}{1-x} \operatorname{Li}_2 \left(\frac{1+x}{2} \right)$$

$$= \sum_{n=1}^{\infty} x^n \left(\frac{1}{2} H_n^{(2)} + \log(2) \overline{H}_n - \frac{1}{2} \overline{H}_n^2 \right);$$

$$vii) \ \frac{1}{1-x} \operatorname{Li}_2 \left(\frac{2x}{x-1} \right)$$

$$= \sum_{n=1}^{\infty} x^n \left(-\frac{1}{2} H_n^2 - \frac{1}{2} H_n^{(2)} - H_n \overline{H}_n - \overline{H}_n^{(2)} + \sum_{k=1}^{n} (-1)^{k-1} \frac{H_k}{k} \right)$$

$$= \sum_{n=1}^{\infty} x^n \left(-\frac{1}{2} H_n^2 - \frac{1}{2} H_n^{(2)} - \sum_{k=1}^{n} \frac{\overline{H}_k}{k} \right);$$

$$viii) \ \frac{1}{1-x} \operatorname{Li}_2 \left(\frac{2x}{1+x} \right)$$

$$= \sum_{n=1}^{\infty} x^n \left(\frac{1}{2} \overline{H}_n^2 + \frac{1}{2} H_n^{(2)} + \sum_{k=1}^{n} (-1)^{k-1} \frac{H_k}{k} \right)$$

$$= \sum_{n=1}^{\infty} x^n \left(\frac{1}{2} \overline{H}_n^2 + \overline{H}_n^{(2)} + H_n \overline{H}_n + \frac{1}{2} H_n^{(2)} - \sum_{k=1}^{n} \frac{\overline{H}_k}{k} \right),$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m}$, $m \geq 1$, *is the nth generalized harmonic number of order m,* $\overline{H}_n^{(m)} = 1 - \frac{1}{2^m} + \cdots + (-1)^{n-1} \frac{1}{n^m}$, $m \geq 1$, *represents the nth generalized skew-harmonic number of order m, and* Li_n *denotes the Polylogarithm function.*

Proof. Recall that according to the Cauchy product of two series, if $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are absolutely convergent, then we have $\left(\sum_{n=1}^{\infty} a_n \right) \left(\sum_{n=1}^{\infty} b_n \right) = \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} a_k b_{n-k+1} \right)$. Since we know that $\operatorname{arctanh}(x) = \frac{1}{2} \sum_{n=1}^{\infty} x^n \left(\frac{1}{n} + (-1)^{n-1} \frac{1}{n} \right)$, if we apply the Cauchy product of two series, we get that

$$\operatorname{arctanh}^2(x) = \frac{1}{4} \sum_{n=1}^{\infty} x^{n+1} \sum_{k=1}^{n} \left(\frac{1}{k} + (-1)^{k-1} \frac{1}{k} \right) \left(\frac{1}{n-k+1} + (-1)^{n-k} \frac{1}{n-k+1} \right)$$

$$= \frac{1}{4}\sum_{n=1}^{\infty} x^{n+1}\frac{1}{n+1}\Bigg(\sum_{k=1}^{n}\frac{1}{k}+\sum_{k=1}^{n}\frac{1}{n-k+1}+(-1)^{n-1}\sum_{k=1}^{n}\frac{1}{k}+(-1)^{n-1}\sum_{k=1}^{n}\frac{1}{n-k+1}$$

$$+\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}+\sum_{k=1}^{n}(-1)^{n-k}\frac{1}{n-k+1}+(-1)^{n-1}\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}+(-1)^{n-1}\sum_{k=1}^{n}(-1)^{n-k}\frac{1}{n-k+1}\Bigg)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} x^{n+1}\frac{1}{n+1}\left(H_n + (-1)^{n-1}H_n + \overline{H}_n + (-1)^{n-1}\overline{H}_n\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} x^{n}\left(\frac{H_n}{n} - (-1)^{n-1}\frac{H_n}{n} + \frac{\overline{H}_n}{n} - (-1)^{n-1}\frac{\overline{H}_n}{n}\right).$$

To get the result from the point *ii*), we want to exploit the result from the point *i*) together with the Cauchy product of two series and all points of Lemma 2, and then we write

$$\frac{\operatorname{arctanh}^2(x)}{1-x} = \frac{1}{2}\sum_{n=1}^{\infty} x^{n}\left(\sum_{k=1}^{n}\frac{H_k}{k} - \sum_{k=1}^{n}(-1)^{k-1}\frac{H_k}{k} + \sum_{k=1}^{n}\frac{\overline{H}_k}{k} - \sum_{k=1}^{n}(-1)^{k-1}\frac{\overline{H}_k}{k}\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} x^{n}\left(\frac{1}{2}H_n^2 + H_n\overline{H}_n - \frac{1}{2}\overline{H}_n^2 + \overline{H}_n^{(2)} - 2\sum_{k=1}^{n}(-1)^{k-1}\frac{H_k}{k}\right)$$

$$= \frac{1}{2}\sum_{n=1}^{\infty} x^{n}\left(\frac{1}{2}H_n^2 - H_n\overline{H}_n - \frac{1}{2}\overline{H}_n^2 - \overline{H}_n^{(2)} + 2\sum_{k=1}^{n}\frac{\overline{H}_k}{k}\right).$$

Next, to prove the result from the point *iii*), we combine the series representation from the first point of Theorem 1 together with the Cauchy product of two series and the points *i*) and *ii*) of Lemma 2,

$$\frac{\pi^2}{12}\frac{1}{1-x} + \frac{1}{1-x}\operatorname{Li}_2\left(-\frac{1+x}{1-x}\right)$$

$$= \sum_{n=1}^{\infty} x^{n}\left(\sum_{k=1}^{n}(-1)^{k-1}\frac{\overline{H}_k}{k} - \sum_{k=1}^{n}\frac{H_k}{k} - \log(2)\sum_{k=1}^{n}\frac{1}{k} - \log(2)\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}\right)$$

$$= \sum_{n=1}^{\infty} x^{n}\left(\frac{1}{2}\overline{H}_n^2 - \log(2)\overline{H}_n - \frac{1}{2}H_n^2 - \log(2)H_n\right).$$

Moving on to the point *iv*) of the problem, we exploit the previous point together with the dilogarithm function identity, $\operatorname{Li}_2(-x)+\operatorname{Li}_2(-1/x) = -\pi^2/6-1/2\log^2(x)$, which show that

$$\sum_{n=1}^{\infty} x^{n}\left(\frac{1}{2}\overline{H}_n^2 - \log(2)\overline{H}_n - \frac{1}{2}H_n^2 - \log(2)H_n\right)$$

$$= \frac{\pi^2}{12}\frac{1}{1-x} + \frac{1}{1-x}\operatorname{Li}_2\left(-\frac{1+x}{1-x}\right)$$

$$= -\frac{\pi^2}{12}\frac{1}{1-x} - \frac{1}{1-x}\operatorname{Li}_2\left(-\frac{1-x}{1+x}\right) - 2\frac{\operatorname{arctanh}^2(x)}{1-x},$$

and since the Cauchy product of $\dfrac{\operatorname{arctanh}^2(x)}{1-x}$ is derived at the point *ii*), the desired result is immediately extracted.

Further, in order to get the result from the point v), we combine the series representation from the point i) of Theorem 2 together with the Cauchy product of two series and the third point of Lemma 2,

$$\left(\frac{\pi^2}{12}-\frac{1}{2}\log^2(2)\right)\frac{1}{1-x}-\frac{1}{1-x}\operatorname{Li}_2\left(\frac{1-x}{2}\right)$$

$$=\sum_{n=1}^{\infty}x^n\left(\log(2)\sum_{k=1}^{n}\frac{1}{k}+\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k^2}-\sum_{k=1}^{n}\frac{\overline{H}_k}{k}\right)$$

$$=\sum_{n=1}^{\infty}x^n\left(\log(2)H_n+\overline{H}_n^{(2)}-\sum_{k=1}^{n}\frac{\overline{H}_k}{k}\right)$$

$$=\sum_{n=1}^{\infty}x^n\left(\log(2)H_n-H_n\overline{H}_n+\sum_{k=1}^{n}(-1)^{k-1}\frac{H_k}{k}\right).$$

□

Jumping to the the point vi), we combine the series representation from the point i) of Theorem 2, with x replaced by $-x$, together with the Cauchy product of two series and the second point of Lemma 2,

$$\left(\frac{1}{2}\log^2(2)-\frac{\pi^2}{12}\right)\frac{1}{1-x}+\frac{1}{1-x}\operatorname{Li}_2\left(\frac{1+x}{2}\right)$$

$$=\sum_{n=1}^{\infty}x^n\left(\sum_{k=1}^{n}\frac{1}{k^2}+\log(2)\sum_{k=1}^{n}(-1)^{k-1}\frac{1}{k}-\sum_{k=1}^{n}(-1)^{k-1}\frac{\overline{H}_k}{k}\right)$$

$$=\sum_{n=1}^{\infty}x^n\left(\frac{1}{2}H_n^{(2)}+\log(2)\overline{H}_n-\frac{1}{2}\overline{H}_n^2\right).$$

With respect to the point vii), we want to exploit the series representation given at the third point of Theorem 2, the Cauchy product of two series, and the first and third points of Lemma 2,

$$\frac{1}{1-x}\operatorname{Li}_2\left(\frac{2x}{x-1}\right)$$

$$=-\sum_{n=1}^{\infty}x^n\left(\sum_{k=1}^{n}\frac{H_k}{k}+\sum_{k=1}^{n}\frac{\overline{H}_k}{k}\right)=\sum_{n=1}^{\infty}x^n\left(-\frac{1}{2}H_n^2-\frac{1}{2}H_n^{(2)}-\sum_{k=1}^{n}\frac{\overline{H}_k}{k}\right)$$

$$=\sum_{n=1}^{\infty}x^n\left(-\frac{1}{2}H_n^2-\frac{1}{2}H_n^{(2)}-H_n\overline{H}_n-\overline{H}_n^{(2)}+\sum_{k=1}^{n}(-1)^{k-1}\frac{H_k}{k}\right).$$

Finally, to attack the last point, we'll employ again the series representation given at the third point of Theorem 2, the Cauchy product of two series, and the second and third points of Lemma 2,

$$\frac{1}{1-x}\operatorname{Li}_2\left(\frac{2x}{1+x}\right)$$

$$=\sum_{n=1}^{\infty}x^n\left(\sum_{k=1}^{n}(-1)^{k-1}\frac{H_k}{k}+\sum_{k=1}^{n}(-1)^{k-1}\frac{\overline{H}_k}{k}\right)$$

$$= \sum_{n=1}^{\infty} x^n \left(\frac{1}{2}\overline{H}_n^2 + \frac{1}{2}H_n^{(2)} + \sum_{k=1}^{n}(-1)^{k-1}\frac{H_k}{k}\right)$$

$$= \sum_{n=1}^{\infty} x^n \left(\frac{1}{2}\overline{H}_n^2 + \overline{H}_n^{(2)} + H_n\overline{H}_n + \frac{1}{2}H_n^{(2)} - \sum_{k=1}^{n}\frac{\overline{H}_k}{k}\right).$$

3. The lemmas and their proofs

Lemma 1. *(Three parametric integrals with inverse hyperbolic tangent) Given n a positive integer and $a < 1,\ a \in \mathbb{R}$, the following equalities hold:*

$$i)\ \int_0^1 t^{n-1}\operatorname{arctanh}(t)\,dt$$

$$= \frac{1}{2}\log(2)\frac{1}{n} + \frac{1}{2}\log(2)(-1)^{n-1}\frac{1}{n} + \frac{1}{2}\frac{H_n}{n} - \frac{1}{2}(-1)^{n-1}\frac{\overline{H}_n}{n};$$

$$ii)\ \int_0^1 t^{n-1}\operatorname{arctanh}^2(t)\,dt$$

$$= \frac{\pi^2}{24}\frac{1}{n} + \frac{\pi^2}{24}(-1)^{n-1}\frac{1}{n} + \frac{1}{2}\log(2)\frac{H_n}{n} - \frac{1}{2}\log(2)(-1)^{n-1}\frac{H_n}{n} + \frac{1}{4}\frac{H_n^2}{n}$$
$$+\frac{1}{2}(-1)^{n-1}\frac{H_n\overline{H}_n}{n} + \frac{1}{2}\log(2)\frac{\overline{H}_n}{n} - \frac{1}{2}\log(2)(-1)^{n-1}\frac{\overline{H}_n}{n} - \frac{1}{4}\frac{\overline{H}_n^2}{n} + \frac{1}{2}(-1)^{n-1}\frac{\overline{H}_n^{(2)}}{n}$$
$$-(-1)^{n-1}\frac{1}{n}\sum_{k=1}^{n}(-1)^{k-1}\frac{H_k}{k}$$

$$= \frac{\pi^2}{24}\frac{1}{n} + \frac{\pi^2}{24}(-1)^{n-1}\frac{1}{n} + \frac{1}{2}\log(2)\frac{H_n}{n} - \frac{1}{2}\log(2)(-1)^{n-1}\frac{H_n}{n} + \frac{1}{4}\frac{H_n^2}{n}$$
$$-\frac{1}{2}(-1)^{n-1}\frac{H_n\overline{H}_n}{n} + \frac{1}{2}\log(2)\frac{\overline{H}_n}{n} - \frac{1}{2}\log(2)(-1)^{n-1}\frac{\overline{H}_n}{n} - \frac{1}{4}\frac{\overline{H}_n^2}{n} - \frac{1}{2}(-1)^{n-1}\frac{\overline{H}_n^{(2)}}{n}$$
$$+(-1)^{n-1}\frac{1}{n}\sum_{k=1}^{n}\frac{\overline{H}_k}{k},$$

$$iii)\ \int_0^1 \frac{\operatorname{arctanh}^n(t)}{1-at}\,dt$$

$$= -\frac{n!}{a2^n}(1-2^{-n})\zeta(n+1) - \frac{n!}{a2^n}\operatorname{Li}_{n+1}\left(-\frac{1+a}{1-a}\right),$$

where $H_n^{(m)} = 1 + \frac{1}{2^m} + \cdots + \frac{1}{n^m},\ m \ge 1$, is the nth generalized harmonic number of order m, $\overline{H}_n^{(m)} = 1 - \frac{1}{2^m} + \cdots + (-1)^{n-1}\frac{1}{n^m},\ m \ge 1$, represents the nth generalized skew-harmonic number of order m, ζ denotes the Riemann zeta function, and Li_n *designates the Polylogarithm function.*

Proof. The first two integrals have already appeared in [3, Chapter 1, pp.28–29]. As regards the third one, we can exploit a similar strategy to the one give to the fourth integral in [3, Chapter 1, Section 1.10, p.10].
Making the variable change $(1-t)/(1+t) = y$ in the integral from the point *iii*), we get

$$\int_0^1 \frac{\operatorname{arctanh}^n(t)}{1-at}\mathrm{d}t = (-1)^n\frac{1}{(1-a)2^{n-1}}\int_0^1 \frac{\log^n(y)}{(1+y)(1+(1+a)/(1-a)y)}\mathrm{d}y$$

$$= (-1)^{n-1}\frac{1}{a2^n}\int_0^1 \frac{\log^n(y)}{1+y}\mathrm{d}y - (-1)^{n-1}\frac{1}{a2^n}\int_0^1 \frac{(1+a)/(1-a)\log^n(y)}{1+(1+a)/(1-a)y}\mathrm{d}y$$

$$= -\frac{n!}{a2^n}\eta(n+1) - \frac{n!}{a2^n}\operatorname{Li}_{n+1}\left(-\frac{1+a}{1-a}\right)$$

$$= -\frac{n!}{a2^n}(1-2^{-n})\zeta(n+1) - \frac{n!}{a2^n}\operatorname{Li}_{n+1}\left(-\frac{1+a}{1-a}\right),$$

where I used that $\displaystyle\int_0^1 \frac{y\log^n(x)}{1-yx}\mathrm{d}x = (-1)^n n!\operatorname{Li}_{n+1}(y),\ y \in (-\infty, 1]$, which is found in [2, Chapter 1, p.4], and the facts that $\operatorname{Li}_n(-1) = -\eta(n)$ and $\eta(s) = (1-2^{1-s})\zeta(s)$. □

Lemma 2. *(Useful finite sums with the generalized (skew-)harmonic numbers) The following equalities hold:*

$$i)\ \sum_{k=1}^{n}\frac{H_k}{k} = \frac{1}{2}\left(H_n^2 + H_n^{(2)}\right);$$

$$ii)\ \sum_{k=1}^{n}(-1)^{k-1}\frac{\overline{H}_k}{k} = \frac{1}{2}\left(\overline{H}_n^2 + H_n^{(2)}\right);$$

$$iii)\ \sum_{k=1}^{n}(-1)^{k-1}\frac{H_k}{k} + \sum_{k=1}^{n}\frac{\overline{H}_k}{k} = H_n\overline{H}_n + \overline{H}_n^{(2)},$$

Proof. The sums from the points *i)* and *ii)* are straightforward by exploiting the change of summation order of the type, $\displaystyle\sum_{i=1}^{n}\sum_{j=1}^{i}a_{i,j} = \sum_{j=1}^{n}\sum_{i=j}^{n}a_{i,j}$, or via Abel's summation. Derivation details may be found in [2, Chapter 3, p.60] and [3, Chapter 3, pp.218–219]. The result from the last point is also straightforward if we apply Abel's summation where we set $a_k = 1/k$ and $b_k = \overline{H}_k$. □

Curious readers who enjoy establishing exotic relations with integrals and series might combine the results in the present paper with the ones in [4].

References

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JUNIOR PROBLEMS

Solutions to the problems in this issue should arrive before February 25, 2024

Proposals

71. *Proposed by Mihaly Bencze, Braşov, Romania, and Neculai Stanciu, Buzău, Romania.*
Let $a_1, a_2, \ldots, a_n$ be positive real numbers and define $a_{n+1} = a_1$. Prove

$$\sum_{k=1}^{n} \frac{a_k^2}{a_k + a_{k+1}} \geq \frac{1}{2}\sum_{k=1}^{n} a_k \geq \sum_{k=1}^{n} \frac{a_k a_{k+1}^2}{a_k^2 + a_{k+1}^2}.$$

72. *Proposed by George-Florin Şerban, National Pedagocial College "D. P. Perpessicius", Braila, Romania.*
A 4-digit positive integer $\overline{abcd}$ is called *special* if it satisfies

$$\overline{abcd} = (a+b+c+d)(a^2+b^2+c^2+d^2)^2.$$

One easily verifies that 2023 is a special number. Find all special numbers.

73. *Proposed by Leonard Giugiuc, Romania.*
Let x, y and z be non-negative real numbers, no two of which are simultaneously zero, such that $x + y + z = 3$. Find the least upper bound of the expression

$$\frac{(x+y)(y+z)(z+x)}{xy+yz+zx} \cdot \left(\frac{1}{x+3} + \frac{1}{y+3} + \frac{1}{z+3}\right).$$

74. *Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam.*
Given the triangle ABC. The internal angle bisectors from A, B, C meet sides BC, CA, AB at A_1, B_1, C_1 respectively. Prove

$$\tan\frac{A}{2} + \tan\frac{B}{2} + \tan\frac{C}{2} + \frac{\cos(\overrightarrow{BB_1}, \overrightarrow{CC_1})}{\cos\frac{A}{2}} + \frac{\cos(\overrightarrow{CC_1}, \overrightarrow{AA_1})}{\cos\frac{B}{2}} + \frac{\cos(\overrightarrow{AA_1}, \overrightarrow{BB_1})}{\cos\frac{C}{2}} = 0.$$

75. *Proposed by Besfort Shala, University of Bristol, Bristol, United Kingdom.*
Let n be a positive integer. Find the smallest value of k (in terms of n) such that the system of equations

$$x_1 + x_2 + \cdots + x_n = x_1^2 + x_2^2 + \cdots + x_n^2 = \cdots = x_1^k + x_2^k + \cdots + x_n^k$$

has a unique solution over the positive real numbers.

No problem is ever permanently closed. We will be very pleased to consider publishing new solutions or comments on previous problems.

Solutions

66. *Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam.*
Let x, y, z be real numbers in the interval $\left[\frac{1}{2}, 2\right]$. Find the minimum and maximum possible value of

$$f(x,y,z) = \frac{x}{yz+1} + \frac{y}{zx+1} + \frac{z}{xy+1}.$$

Solution by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy. The function is symmetric respect to any permutation of (x, y, z).
Let us first find the maximum. We have

$$\frac{\partial^2 f}{\partial x^2} = \frac{yz^2}{(xz+1)^2} + \frac{zy^2}{(xy+1)^2} > 0.$$

By symmetry the same happens with the second derivative with respect to y and z, hence the maximum occurs when each variable assumes one of the extreme values. We need to check the value of the function when (x, y, z) takes the values

$$(2,2,2),\ (1/2,1/2,1/2),\ (2,2,1/2),\ (2,1/2,1/2).$$

There are 6 other possible permutations but by symmetry it suffices to consider just the third and fourth above. We have

$$f(2,2,2) = 6/5,\ f(1/2,1/2,1/2) = 6/5,\ f(2,2,1/2) = 21/10,\ f(2,1/2,1/2) = 21/10,$$

therefore the maximum is $21/10$.
Now we find the minimum. We prove that $f(x, y, z) \geq 6/5$. The inequality to be proved is

$$\frac{x^2}{x(yz+1)} + \frac{y^2}{y(zx+1)} + \frac{z^2}{z(xy+1)} - \frac{6}{5} \geq 0.$$

Cauchy-Schwarz yields

$$\frac{x^2}{x(yz+1)} + \frac{y^2}{y(zx+1)} + \frac{z^2}{z(xy+1)} - \frac{6}{5} \geq \frac{(x+y+z)^2}{x+y+z+3xyz} - \frac{6}{5} \geq 0,$$

that is,

$$G(xyz) \doteq -18xyz + 5(x+y+z)^2 - 6(x+y+z) \geq 0, \quad xyz \leq (x+y+z)^3/27$$

by AM-GM. G is a linear decreasing function hence we must check $G((xyz)_{\max}) \geq 0$, that is

$$-18\frac{(x+y+z)^3}{27} + 5(x+y+z)^2 - 6(x+y+z) \geq 0 \underbrace{\iff}_{S=x+y+z} \frac{S}{3}((S-6)(3-2S) \geq 0.$$

But clearly $3/2 \leq x + y + z \leq 6$, so this concludes the proof. ▲

Also solved by Leonard Giugiuc, Romania, and the proposer.

67. *Proposed by Daniel Sitaru, Mathematics Department, Colegiul National Economic Theodor Costescu, Drobeta Turnu - Severin, Mehedinti, Romania.*
Let $n \in \mathbb{N}$ such that $n \geq 2$. Prove that in any triangle ABC the following inequality holds:

$$\sum\left(\frac{\sqrt[n]{b}+\sqrt[n]{c}-2\sqrt[n]{a}}{\sqrt[n]{b}+\sqrt[n]{c}}\right)^2+\frac{3}{\sqrt[n]{abc}}\prod(\sqrt[n]{b}+\sqrt[n]{c}-\sqrt[n]{a})\leq 3.$$

Solution by Leonard Giugiuc, Romania. Without loss of generality, we may assume that $a \geq b \geq c$. We have

$$\sqrt[n]{a}<\sqrt[n]{b+c}<\sqrt[n]{b}+\sqrt[n]{c},$$

which means that the numbers $\sqrt[n]{a}$, $\sqrt[n]{b}$ and $\sqrt[n]{c}$ are lengths of the sides of a triangle. Denote $\sqrt[n]{b}+\sqrt[n]{c}-\sqrt[n]{a}=2x, -\sqrt[n]{b}+\sqrt[n]{c}+\sqrt[n]{a}=2y$ and $\sqrt[n]{b}-\sqrt[n]{c}+\sqrt[n]{a}=2z$. Then x, y, z are positive and the required inequality is equivalent to

$$\sum\left(\frac{y+z-2x}{y+z+2x}\right)^2+\frac{24xyz}{(x+y)(y+z)(z+x)}\leq 3.$$

But

$$\left(\frac{y+z-2x}{y+z+2x}\right)^2=1-\frac{8x(y+z)}{(y+z+2x)^2},$$

therefore the required inequality is equivalent to

$$\frac{3xyz}{(x+y)(y+z)(z+x)}\leq\sum\frac{x(y+z)}{(y+z+2x)^2}.$$

We will prove that

$$\frac{xyz}{(x+y)(y+z)(z+x)}\leq\frac{x(y+z)}{(y+z+2x)^2},$$

which is equivalent to

$$4x^2yz+4xyz(y+z)\leq x^2(y+z)^2+z(y+z)^3.$$

But by AM-GM, we have $4x^2yz \leq x^2(y+z)^2$ and $4xyz(y+z) \leq x(y+z)^3$. Therefore we have

$$\sum\frac{xyz}{(x+y)(y+z)(z+x)}\leq\sum\frac{x(y+z)}{(y+z+2x)^2},$$

which is exactly what we wanted to prove. ▲

Also solved by Michel Bataille, Rouen, France, and the proposers.

68. *Proposed by Michael Rozenberg, Tel Aviv, Israel and Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Romania.*
Let a, b, c and d be non-negative real numbers, no three of which are all 0, and such that $a+b+c+d=4$. Prove that

$$\frac{a^2+b^2+c^2+d^2}{ab+bc+cd+da+ac+bd}+\frac{12abcd}{(ab+bc+cd+da+ac+bd)^2}\geq 1.$$

When does equality occur?

Solution by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy. If $d=0$ we get

$$\frac{a^2+b^2+c^2}{ab+bc+ca}\geq 1, \quad a+b+c=4,$$

which is clearly true with $a = b = c = 4/3$ being the only equality case. Clearly there are the other 3 analogous equality cases where $c = 0$ or $b = 0$ or $a = 0$.
Now assume $abcd \neq 0$. Define $a+b = 2u$, $ab = v^2$, $c+d = 2t$, $cd = s^2$ and $u+t = 2$ by $a+b+c+d = 4$. The inequality now is

$$\frac{-3v^4 + 6s^2v^2 - 48uv^2 + 24u^2v^2 + 16v^2 + 16s^2 + 128u - 256u^2 - 3s^4}{(-v^2 - s^2 - 8u + 4u^2)^2} + \\ + \frac{-48s^2u + 24s^2u^2 + 192u^3 - 48u^4}{(-v^2 - s^2 - 8u + 4u^2)^2} \geq 0$$

with $0 \leq v \leq u$ and $0 \leq s \leq t$ by AM-GM. The numerator of the above ratio is a concave polynomial of v^2 thus it is greater than or equal to its values for $v = 0$ and $v = u$. The case $v = 0$ has been already considered while $v = u$ yields the numerator

$$-(27u^4 - 30s^2u^2 - 144u^3 + 240u^2 - 16s^2 - 128u + 3s^4 + 48s^2u) = f(s^2) \geq 0$$

which is a concave polynomial of s^2 thus $f(s^2) \geq \min\{f(0), f(s^2_{\max}\})$. By AM-GM we have $s \leq t = 2 - u$, thus

$$f((2-u)^2) = 16(u-1)^2 \geq 0$$

with $u = 1$ being the only equality case, yielding $a = b = c = d = 1$. The proof is complete. ▲

Also solved by Nicuşor Zlota "Traian Vuia" Technical College, Focşani, Romania; Dion Aliu, Republic of Kosova and the proposers.

69. *Proposed by Mohammed Aassila, Strasbourg, France.*
Let N be a positive fixed integer. Determine the number of integers $1 \leq n \leq N$ such that

$$11 \cdot 2^{n-1} \equiv 4n + 6 \pmod{13}.$$

Solution by Edi Berisha, Republic of Kosova. We claim that there are 12 solutions $\pmod{156}$ that satisfy the equation. We have

$$4n + 6 \equiv 11 \times 2^{n-1} \equiv -2 \times 2^{n-1} \equiv -2^n \pmod{13}.$$

It is trivial that left hand side of this equation is dependent only on the remainder of $n \pmod{13}$. On the other hand, since $\text{ord}_{13}(2) = 12$, the right hand side is dependent only on $n \pmod{12}$. We need to find a modulus M such that both sides are dependent only on $n \pmod{M}$. Clearly, $M = \text{lcm}(12, 13) = 156$ is the least such modulus.
Among all possible values of $n \pmod{13}$, the left hand side takes 13 distinct values, each of which may be written as a power of 2, except 0. Each of these gives us an equation $\pmod{12}$ since $2^n \equiv 2^x \pmod{13}$ implies $n \equiv x \pmod{12}$. Since $\gcd(13, 12) = 1$, we will get a unique solution $\pmod{12 \cdot 13}$ for each $n \pmod{13}$ except 0. In conclusion, the equation has 12 solutions $\pmod{156}$ that satisfy the equation.
From here, it is clear that the answer to the problem is $12 \cdot \lfloor \frac{N}{156} \rfloor + f(N)$, where $f(N)$ is the number of the solutions up to the remainder of N when divided by 156, which could be calculated explicitly knowing the solutions $\pmod{156}$ are $3, 22, 32, 33, 37, 40, 54, 60, 101, 103, 134$ and 143. ▲

Also solved by Klesta Qehaja, Republic of Kosova; Dion Aliu, Republic of Kosova and the proposer.

70. *Proposed by Dorlir Ahmeti, University of Prishtina, Department of Mathematics, Republic of Kosova.*
The following numbers are written on a board along a straight line, as shown.

$$1 \qquad \tfrac{1}{2} \qquad \tfrac{1}{3} \quad \dots \quad \tfrac{1}{n-1} \qquad \tfrac{1}{n}$$

Now we add the k-th number with the $(k+1)$-st for each $k = 1, 2, \dots, n-1$, and write the sum down below in the middle of the two numbers. As such, we create a new line with $n-1$ new numbers and we repeat the same procedure with new line and keep going until we are left with only one number.
For example, if $n = 3$ then:

$$\begin{array}{ccccc} 1 & & \frac{1}{2} & & \frac{1}{3} \\ & \frac{3}{2} & & \frac{5}{6} & \\ & & \frac{7}{3} & & \end{array}$$

Find the last number which is written on the board.

Solution by the author. First we find the general sum when we start with any sequence $a_1, a_2, ..., a_n$. We have

$$\begin{array}{ccccccc} a_1 & & a_2 & \dots & a_{n-1} & & a_n \\ & a_1 + a_2 & & \dots & & a_{n-1} + a_n & \\ & . & & . & & . & \\ & & . & & . & & \\ & & & d_n & & & \end{array}$$

Let us denote the last number in this triangle by $[a_1, a_2, ..., a_n] = d_n$.
We find that $d_1 = a_1$, $d_2 = a_1 + a_2$, $d_3 = a_1 + 2a_2 + a_3$ and $d_4 = a_1 + 3a_2 + 3a_3 + a_4$.
So we can guess that

$$d_n = \sum_{k=1}^{n} \binom{n-1}{k-1} a_k$$

for $n \geq 2$.
Now we use induction to prove this. The cases $n = 2, 3, 4$ are already done. Suppose that it is true for n and we prove the statement for $n+1$. We have

$$\begin{aligned}
d_{n+1} &= [a_1, a_2, ..., a_n, a_{n+1}] = [a_1, a_2, ..., a_n] + [a_2, a_3, ..., a_{n+1}] \\
&= \sum_{k=1}^{n} \binom{n-1}{k-1} a_k + \sum_{k=1}^{n} \binom{n-1}{k-1} a_{k+1} \\
&= a_1 + \sum_{k=2}^{n} \binom{n-1}{k-1} a_k + \sum_{k=1}^{n-1} \binom{n-1}{k-1} a_{k+1} + a_{n+1} \\
&= a_1 + \sum_{k=1}^{n-1} \binom{n-1}{k} a_{k+1} + \sum_{k=1}^{n-1} \binom{n-1}{k-1} a_{k+1} + a_{n+1} \\
&= a_1 + \sum_{k=1}^{n-1} \binom{n}{k} a_{k+1} + a_{n+1} = a_1 + \sum_{k=2}^{n} \binom{n}{k-1} a_k + a_{n+1} \\
&= \sum_{k=1}^{n+1} \binom{n}{k-1} a_k.
\end{aligned}$$

(Here we used the identity $\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$.) Now in our case we have $a_k = \frac{1}{k}$ so we have

$$d_n = \sum_{k=1}^{n} \binom{n-1}{k-1} \cdot \frac{1}{k} = \sum_{k=1}^{n} \frac{1}{n} \cdot \binom{n}{k} = \frac{1}{n} \cdot \sum_{k=1}^{n} \binom{n}{k} = \frac{1}{n} \cdot (2^n - 1) = \frac{2^n - 1}{n}.$$

(Here we used the identity $\binom{n-1}{k-1} \cdot \frac{1}{k} = \frac{1}{n} \cdot \binom{n}{k}$.) So the last number is written on the board is $\frac{2^n-1}{n}$. ▲