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PROBLEMS AND SOLUTIONS

Proposals and solutions must be legible and should appear on separate sheets, each indicating the name of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editors have supplied a solution. The editors encourage undergraduate and pre-college students to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the class and school name. Solutions will be evaluated for publication by a committee of professors according to a combination of criteria. Questions concerning proposals and/or solutions can be sent by e-mail to: *mathproblems-ks@hotmail.com*

Solutions to the problems stated in this issue should arrive before June 19, 2017

Problems

145. *Proposed by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy.* Let $0 \le x \le 1$. Prove that $x^x \le x^2 - x + 1 - x^2(1-x)^4$.

146. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let $n \ge 1$ be an integer. Solve in $\mathcal{M}_2(\mathbb{Z})$ the equation $X^{2n+1} - X = I_2$.

147. *Proposed by Anastasios Kotronis, Athens, Greece.* Let a_n be the sequence defined by the relations

$$a_{n+3} - \left(1 + \frac{b-p-1}{n+3}\right) a_{n+2} + \frac{b-2a}{n+3} a_{n+1} + \frac{2a}{n+3} a_n = 0$$

and

$$a_0 = 1,\ a_1 = b - p \wedge\ a_2 = a + \frac{(b-p)^2 - p}{2},$$

where $a, b \in \mathbb{R}$ and $\mathbb{R} \ni p \notin \{-2, -1, 0, 1, \dots\}$.

(1) Show that $\lim_{n\to+\infty} n^{p+1} a_n = \frac{e^{a+b}}{\Gamma(-p)}$ and
(2) Find $\lim_{n\to+\infty} n \left(n^{p+1} a_n - \frac{e^{a+b}}{\Gamma(-p)}\right)$ if it exists.

148. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Find

$$\lim_{x\to\infty}\left(x^{\cosh^2(t)}\left((\Gamma(x+1))^{\frac{-\sinh^2(t)}{x}}-(\Gamma(x+2))^{\frac{-\sinh^2(t)}{x+1}}\right)\right),$$

where $t\in\mathbb{R}$ and Γ is the Gamma function.

149. *Proposed by Arkady Alt, San Jose, California, USA.* Let D be set of strictly decreasing sequences of positive real numbers with first term equal to 1. For given real positive p,r and any $x_{\mathbb{N}}=(x_1,x_2,...,x_n,...)\in D$. Let $S\left(x_{\mathbb{N}}\right)=\sum_{n=1}^{\infty}\dfrac{x_n^{p+q}}{x_{n+1}^{p}}$ if series $\sum_{n=1}^{\infty}\dfrac{x_n^{p+r}}{x_{n+1}^{p}}$ converges and $S\left(x_{\mathbb{N}}\right)=\infty$ if it diverges. Find $\inf\left\{S\left(x_{\mathbb{N}}\right)\mid x_{\mathbb{N}}\in D\right\}.$

150. *Proposed by Cornel Ioan Vălean, Timiş, Rumania.* Find

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{k+n}\frac{H_{k+n}^3}{k+n},$$

where $H_n=1+\dfrac{1}{2}+\cdots+\dfrac{1}{n}$ denotes the nth harmonic number.

151. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Prove that

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=n}^{\infty}\frac{1}{1+k^2}=\left(\frac{3}{2}+\frac{\pi}{2}\coth\pi\right)\sum_{k=1}^{\infty}\frac{1}{k(1+k^2)}-\sum_{k=1}^{\infty}\frac{1}{k(1+k^2)^2}.$$

Solutions

No problem is ever permanently closed. We will be very pleased considering for publication new solutions or comments on the past problems.

138. *Proposed by Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Romania.* Let a, b, c, x, y and z be real numbers such that $a+b+c+x+y+z=3$ and $a^2+b^2+c^2+x^2+y^2+z^2=9$. Prove that $abcxyz \geq -2$.

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria. The variables a, b ,c, x, y, z will be denoted by $x_1, x_2, x_3, x_4, x_5, x_6$ for simplicity!. The set

$$K = \left\{ (x_1, \ldots, x_6) \in \mathbb{R}^6, \sum_{i=1}^{6} x_i = 3, \sum_{i=1}^{6} x_i^2 = 9 \right\}$$

is a compact subset of $\mathbb{R}^6$, so, the continuous function $(x_1, \ldots, x_6) \mapsto \prod_{i=1}^{6} x_i$ attains its minimum μ on K, So, let $(a_1, \ldots, a_6) \in K$ such that $a_1 \leq a_2 \leq \ldots \leq a_6$ with

$$\mu = a_1 a_2 \cdots a_6 = \min \left\{ \prod_{i=1}^{6} x_i : (x_1, x_2, x_3, x_4, x_5, x_6) \in \mathbb{R}^6 \right\}$$

Since $(-2, 1, 1, 1, 1, 1) \in K$ we conclude that $\mu \leq -2$. This proves that none of the a_i's is zero, and that the number of negative a_i's is odd. Therefore, we have three cases:

(a) $a_5 < 0 < a_6$. In this case we have $a_6 = 3 - \sum_{k=1}^{5} a_i > 3$ and consequently $9 < a_6^2 < \sum_{i=1}^{6} a_i^2 = 9$ which is absurd.

(b) $a_3 < 0 < a_4$. In this case we have

$$9 = \left(\sum_{i=1}^{3} (a_i + a_{i+3}) \right)^2 \leq 3 \sum_{i=1}^{3} (a_i + a_{i+3})^2$$

so,

$$3 \leq \sum_{i=1}^{6} a_i^2 + 2(a_1 a_4 + a_2 a_5 + a_3 a_6) = 9 - 2(|a_1|\, a_4 + |a_2|\, a_5 + |a_3|\, a_6)$$

or

$$\frac{1}{3} \left(|a_1|\, a_4 + |a_2|\, a_5 + |a_3|\, a_6 \right) \leq 1.$$

The arithmetic mean- geometric mean inequality proves then that $|\mu| = |a_1|\,|a_2|\,|a_3|\, a_4 a_5 a_6 \leq 1$ which is also absurd.

(c) $a_1 < 0 < a_2$. Here we have

$$(3 - a_1)^2 = \left(\sum_{i=2}^{6} a_i \right)^2 \leq 5 \sum_{i=2}^{6} a_i^2 = 5(9 - a_1^2)$$

this is equivalent to $a_1^2 - a_1 - 6 \leq 0$ and consequently $-2 \leq a_1 < 0$. It follows that

$$0 < \prod_{i=2}^{6} a_i \leq \left(\frac{a_2 + a_3 + a_4 + a_5 + a_6}{5} \right)^5 = \left(\frac{3 - a_1}{5} \right)^5 \leq 1$$

hence $\mu = \prod_{i=1}^{6} a_i \geq -2$. Consequently $\mu = -2$ which is the desired conclusion.

Solution 2 by S.C. Locke, Department of Mathematical Sciences, Florida Atlantic University and B. Reinhart (student), Oxbridge Academy.

We may assume that none of the variables a, b, c, x, y or z is zero, since then $abcxyz = 0 \geqslant -2$.

We use the method of Lagrange multipliers. Let

$$F(a,b,c,x,y,z,\lambda,\mu) = abcxyz + \lambda(a+b+c+x+y+z-3) + \mu\left(a^2+b^2+c^2+x^2+y^2+z^2-9\right).$$

Then,

$$\nabla F = \begin{bmatrix} \frac{\partial F}{\partial a} \\ \frac{\partial F}{\partial b} \\ \frac{\partial F}{\partial c} \\ \frac{\partial F}{\partial x} \\ \frac{\partial F}{\partial y} \\ \frac{\partial F}{\partial z} \\ \frac{\partial F}{\partial \lambda} \\ \frac{\partial F}{\partial \mu} \end{bmatrix} = \begin{bmatrix} bcxyz + \lambda + 2\mu a \\ acxyz + \lambda + 2\mu b \\ abxyz + \lambda + 2\mu c \\ abcyz + \lambda + 2\mu x \\ abcxz + \lambda + 2\mu y \\ abcxy + \lambda + 2\mu y \\ a+b+c+x+y+z-3 \\ a^2+b^2+c^2+x^2+y^2+z^2-9 \end{bmatrix}.$$

We want $\nabla F = \overrightarrow{0}$. Thus, for any $w \in \{a,b,c,x,y,z\}$,

$$w\frac{\partial F}{\partial w} = 0 \Longrightarrow 0 = abcxyz + \lambda w + 2\mu w^2.$$

Hence,
$\lambda a + 2\mu a^2 = \lambda b + 2\mu b^2$, $\lambda(a-b) + 2\mu\left(a^2-b^2\right) = 0$, $(a-b)(\lambda + 2\mu(a+b)) = 0$. Either $a = b$, or $\lambda = -2\mu(a+b)$. Suppose that $\mu = 0$. We have already stated that we may assume $a \neq 0$. Thus, $abcxyz + \lambda a = 0$ and $bcxyz = \lambda$. Similarly, $bcxyz = \lambda = acxyz = abxyz = abcyz = abcxz = abcxy$, and $a = b = c = x = y = z = \frac{1}{2}$, which is impossible, since then $a^2+b^2+c^2+x^2+y^2+z^2 = \frac{3}{2} \neq 9$. Hence, $\mu \neq 0$. Now, we have $a = b$ or $a + b = \frac{-\lambda}{2\mu}$. Thus, $|\{a,b,c,x,y,z\}| \in \{1,2\}$, and we've already ruled out $|\{a,b,c,x,y,z\}| = 1$. The remaining cases are, without loss of generality,

(i) $a = b = c = x = y$, and then $5a + z = 3$, $5a^2 + z^2 = 9$, $5a^2 + (3-5a)^2 = 9$, $a = 1$, $z = -2$, and $abcxyz = -2$.

(ii) $a = b = c = x$, $y = z$, and then $abcxyz = a^4z^2 > 0$.

(ii) $a = b = c$, $x = y = z$, $3a + 3z = 3$, $3a^2 + 3z^2 = 9$, $a + z = 1$, $a^2 + z^2 = 3$, $az = \frac{1}{2}\left((a+z)^2 - \left(a^2+z^2\right)\right) = -1$, and $abcxyz = (az)^3 = -1 > -2$.

Therefore, $abcxyz \geqslant -2$, with equality if five of the variables have value one and one of the variables has value negative two.

Solution 3 by Moti Levy, Rehovot, Israel. The following notation and results will be used in this solution:
1) The elementary symmetric *polynomials* in n variables are defined as:

$$e_k := \sum_{1\le j_1<j_2<\cdots<j_k\le n} x_{j_1}\cdots x_{j_k}, \qquad k=1,2,\ldots,n.$$

2) The elementary symmetric *means* in n variables are defined as:

$$E_k := \frac{e_k}{\binom{n}{k}}, \qquad E_0 = 1.$$

3) The power sums are defined as:

$$p_k := \sum_{i=1}^{n} x_i^k.$$

4) From Newton's identities,

$$p_1 = e_1, \tag{1}$$

$$p_2 = e_1 p_1 - 2e_2. \tag{2}$$

5) The Newton's inequalities are:

$$E_{k-1}E_{k+1} \le E_k^2. \tag{3}$$

Let us denote, for convenience, $x_1 = a$, $x_2 = b$, $x_3 = c$, $x_4 = x$, $x_5 = y$ and $x_6 = z$.
The first constraint can be re-written as $x_6 = 3 - \sum_{i=1}^{5} x_i = 3 - e_1$.
We will use the first constraint to eliminate x_6, so we are dealing with elementary symmetric polynomials in 5 variables.
The second constraint becomes $\left(3 - \sum_{i=1}^{5} x_i\right)^2 + \sum_{i=1}^{5} x_i^2 = 9$,
or

$$(3-p_1)^2 + p_2 = 9,$$

which is equivalent to $p_2 - 6p_1 + p_1^2 = 0$.
We reformulate the problem, in terms of elementary symmetric polynomials and power sums as:
Show that

$$e_5\left(e_1 - 3\right) \le 2, \tag{4}$$

subjected to the constraint,

$$p_2 - 6p_1 + p_1^2 = 0. \tag{5}$$

Substitution of (2) in (5) gives

$$e_2 = p_1\left(p_1 - 3\right), \tag{6}$$

and substitution of (2) in (6) gives

$$p_2 = p_1\left(6 - p_1\right). \tag{7}$$

By Cauchy-Schwarz inequality,

$$p_1^2 \le 5p_2. \tag{8}$$

By (7) and (8)

$$p_1^2 \le 5p_1\left(6 - p_1\right),$$

hence

$$e_1 = p_1 \leq 5, \tag{9}$$

or

$$E_1 \leq 1. \tag{10}$$

Since by definition, $p_2 \geq 0$, it follows from (7) and (9) that

$$e_1 = p_1 \geq 0.$$

It follows from (4) and (9) that

$$e_5\left(e_1 - 3\right) \leq e_5\left(5 - 3\right) \leq 2,$$

or

$$E_5 \leq 1.$$

Thus, showing that $E_5 \leq 1$ subjected to the constraint (5) is equivalent to solving our original inequality.
To show that $E_5 \leq 1$ we will use the Newton's inequalities, which are:

$$E_0 E_2 = E_2 \leq E_1^2, \tag{11}$$

$$E_1 E_3 \leq E_2^2, \tag{12}$$

$$E_2 E_4 \leq E_3^2, \tag{13}$$

$$E_3 E_5 \leq E_4^2. \tag{14}$$

Clearly $E_5 = x_1 x_2 x_3 x_4 x_5 \leq |x_1|\,|x_2|\,|x_3|\,|x_4|\,|x_5|$, therefore, if we show that $E_5 \leq 1$ for positive values of x_i , we are done.
For positive values of x_i, we have: $E_1, E_2, E_3 > 0$.
Using the Newton's inequalities (11) to (14), we obtain:

$$E_5 \leq \frac{E_4^2}{E_3} \leq \frac{\left(\frac{E_3^2}{E_2}\right)^2}{E_3} = \frac{E_3^3}{E_2^2} \leq \frac{\left(\frac{E_2^2}{E_1}\right)^3}{E_2^2} = \frac{E_2^4}{E_1^3} \leq \frac{E_1^8}{E_1^3} = E_1^5.$$

But we have shown in (10), that for variables x_i, $i = 1, \ldots, 5$, which meet the constraint, we have $E_1 \leq 1$, hence $E_5 \leq 1$.

Also solved by Richdad Phuc, Vietnam; Albert Stadler, Herrliberg, Switzerland and the proposer.

139. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let $a, b \in \mathbb{R}$, $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Calculate

$$\lim_{n\to\infty} \int_a^b \frac{f(x)}{1 + \sin^2 x \sin^2(x+1) \cdots \sin^2(x+n)} dx.$$

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria. Note that,for real x

$$\begin{aligned}
\sum_{k=0}^{n} \sin^2(x+k) &= \frac{n+1}{2} - \frac{1}{2}\sum_{k=0}^{n} \cos(2x+2k) \\
&= \frac{n+1}{2} - \frac{1}{2}\Re\left(e^{2ix}\sum_{k=0}^{n} e^{2ik}\right) \\
&= \frac{n+1}{2} - \frac{\cos(2x+n)\sin(n+1)}{2\sin 1} < \frac{n+3}{2}
\end{aligned}$$

In particular, for all $n \geq 3$ and $x \in \mathbb{R}$ we have

$$\frac{1}{n+1}\sum_{k=0}^{n}\sin^2(x+k) \leq \frac{3}{4}$$

and by the arithmetic mean-geometric mean inequality we conclude that for all $n \geq 3$ and $x \in \mathbb{R}$ we have

$$\prod_{k=0}^{n}\sin^2(x+k) \leq \left(\frac{3}{4}\right)^{n+1}$$

It follows that, for $n \geq 3$ we have

$$\begin{aligned}\left|\int_a^b f(x)dx - \int_a^b \frac{f(x)}{1+\prod_{k=0}^{n}\sin^2(x+k)}dx\right| &\leq \int_a^b |f(x)| \prod_{k=0}^{n}\sin^2(x+k)dx \\ &\leq \left(\frac{3}{4}\right)^{n+1}\int_a^b |f(x)|\, dx\end{aligned}$$

Thus,

$$\lim_{n\to\infty}\int_a^b \frac{f(x)}{1+\sin^2 x\sin^2(x+1)\cdots\sin^2(x+n)}dx = \int_a^b f(x)dx.$$

Solution 2 by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia. $\forall x \in \mathbb{R}$ we have

$$\frac{1}{1+\sin^2 x\sin^2(x+1)\cdot\ldots\cdot\sin^2(x+n)} \leq 1 \qquad (1)$$

Using AM-GM inequality we have:

$$\sin^2 x\cdot\sin^2(x+1)\cdot\ldots\cdot\sin^2(x+n) \leq \left(\frac{\sin^2 x+\sin^2(x+1)+\ldots+\sin^2(x+n)}{n+1}\right)^{n+1}.$$

Other hand, we get:

$$\sin^2 x+\sin^2(x+1)+\ldots+\sin^2(x+n) = \frac{n+1}{2} - \frac{\sin(n+1)\cos(n+2x)}{2\sin 1}.$$

Hence we have:

$$\frac{1}{1+\left(\frac{1}{2}-\frac{\sin(n+1)\cos(n+2x)}{2(n+1)\sin 1}\right)^{n+1}} \leq \frac{1}{1+\sin^2 x\sin^2(x+1)\ldots\sin^2(x+n)} \qquad (2)$$

Let

$$g_n(x) = \frac{1}{1+\left(\frac{1}{2}-\frac{\sin(n+1)\cos(n+2x)}{2(n+1)\sin 1}\right)^{n+1}}.$$

$$0 < \left|\left(\frac{1}{2}-\frac{\sin(n+1)\cos(n+2x)}{2(n+1)\sin 1}\right)^{n+1}\right| \leq \left(\frac{1}{2}+\frac{1}{2(n+1)\sin 1}\right)^{n+1} \to 0$$

hence

$$\forall x \in [a,b]: \lim_{n\to\infty} g_n(x) = 1.$$

From (1) and (2), we have $\forall x \in [a,b]$:

$$\lim_{n\to\infty}\frac{1}{1+\sin^2 x\sin^2(x+1)\ldots\sin^2(x+n)} = 1$$

Using the Bounded Convergence Theorem, we get that

$$\begin{aligned}\lim_{n\to\infty}\int_a^b &\frac{f(x)}{1+\sin^2 x\sin^2(x+1)\cdot\ldots\cdot\sin^2(x+n)}dx\\ &=\int_a^b\left(\lim_{n\to\infty}\frac{1}{1+\sin^2 x\sin^2(x+1)\cdot\ldots\cdot\sin^2(x+n)}\right)f(x)dx\\ &=\int_a^b f(x)dx.\end{aligned}$$

Solution 3 by Perfetti Paolo, Dipartimento di Matematica, Università degli studi di Tor Vergata Roma. Let consider on the unit circle C the three subsets: $I_1=\left[-\frac{\pi}{6},\frac{\pi}{6}\right]\bigcup\left[\frac{5}{6}\pi,\frac{7}{6}\pi\right],\quad I_2=\left(\frac{\pi}{6},\frac{5}{6}\pi\right),\quad I_3=\left(\frac{7}{6}\pi,\frac{11}{6}\pi\right).$
Clearly $C=I_1\cup I_2\cup I_3$ and let's observe that the length of I_1 is $2\pi/3$.
Now consider the following set

$$\sin^2(x+n+1),\quad \sin^2(x+n+2),\ldots,\sin^2(x+n+7))$$

Out of the seven terms written, at least $\dfrac{2\pi}{2\pi/3}=3$ of them belong to I_1 and this means that

$$\sin^2(x+n+1)\cdot\sin^2(x+n+2)\cdots\sin^2(x+n+7))\le\frac{1}{4^3}$$

regardless the value of x. Now let's divide the first N integers (0 included) in blocks of length 7 getting $\frac{N}{7}$ blocks plus the rest which is a set made of some integers up 6. It follows

$$\sin^2 x\sin^2(x+1)\cdots\sin^2(x+N)\le\frac{1}{4^{3N/7}}$$

and this implies that

$$\lim_{n\to\infty}\sin^2 x\sin^2(x+1)\cdots\sin^2(x+n)=0$$

uniformly on $[a,b]$. It follows that

$$\lim_{n\to\infty}\int_a^b\frac{f(x)}{1+\sin^2 x\sin^2(x+1)\cdots\sin^2(x+n)}dx=\int_a^b f(x)dx$$

Solution 4 by Michel Bataille, Rouen, France.
Let $K_n(x)=\frac{1}{1+\sin^2 x\sin^2(x+1)\cdots\sin^2(x+n)}$. We show that

$$\lim_{n\to\infty}\int_a^b K_n(x)f(x)\,dx=\int_a^b f(x)\,dx.$$

From a known result, we have $\lim\limits_{n\to\infty}\int_a^b K_n(x)\,dx=b-a$ (see O. Furdui, *Limits, Series, and Fractional Part Integrals*, Springer, 2013, Problem 1.36, pp. 55-6). Thus, if $(\alpha,\beta)\subset[a,b]$ and $\xi_{(\alpha,\beta)}$ is the characteristic function of (α,β), we have $\lim\limits_{n\to\infty}\int_a^b K_n\cdot\xi_{(\alpha,\beta)}=\lim\limits_{n\to\infty}\int_\alpha^\beta K_n=\beta-\alpha=\int_a^b\xi_{(\alpha,\beta)}$.
Now, consider the functional $L:f\mapsto L(f)=\lim_{n\to\infty}\int_a^b K_n(x)f(x)\,dx$. Clearly, L is a linear functional on the linear space of Riemann integrable functions on $[a,b]$. Since $L(\xi_{(\alpha,\beta)})=\int_a^b\xi_{(\alpha,\beta)}$, by linearity we also have $L(\phi)=\int_a^b\phi$ whenever ϕ is a step function from $[a,b]$ to $\mathbb{R}$.

Now, let f be any Riemann integrable function on $[a,b]$ and let ε be any positive real number. There exists a step function ϕ such that $\int_a^b |f-\phi| \le \varepsilon$. Then, from $K_n f - f = K_n(f-\phi) + K_n\phi - \phi + (\phi - f)$ and $0 \le K_n(x) \le 1$ for all $x \in [a,b]$ we deduce

$$\begin{aligned}
\left|\int_a^b K_n f - \int_a^b f\right| &= \left|\int_a^b (K_n f - f)\right| \\
&= \left|\int_a^b K_n(f-\phi) + \int_a^b K_n\phi - \int_a^b \phi + \int_a^b (\phi - f)\right| \\
&\le \left|\int_a^b K_n(f-\phi)\right| + \left|\int_a^b K_n\phi - \int_a^b \phi\right| + \left|\int_a^b (\phi - f)\right| \\
&\le \int_a^b K_n|f-\phi| + \left|\int_a^b K_n\phi - \int_a^b \phi\right| + \int_a^b |\phi - f| \\
&\le 2\int_a^b |f-\phi| + \left|\int_a^b K_n\phi - \int_a^b \phi\right|.
\end{aligned}$$

Since $L(\phi) = \lim_{n\to\infty} \int_a^b K_n\phi = \int_a^b \phi$, we deduce

$$\limsup_{n\to\infty} \left|\int_a^b K_n f - \int_a^b f\right| \le 2\int_a^b |f-\phi| + 0 \le 2\varepsilon$$

and since this holds for any $\varepsilon > 0$, we must have $\limsup_{n\to\infty} \left|\int_a^b K_n f - \int_a^b f\right| = 0$. Thus, $\lim_{n\to\infty} \left|\int_a^b K_n f - \int_a^b f\right| = 0$ so that

$$L(f) = \lim_{n\to\infty} \int_a^b K_n f = \int_a^b f \qquad \text{and we are done.}$$

Also solved by Moubinool Omarjee, Lycée Henri IV, Paris, France; Moti Levy, Rehovot, Israel; Albert Stadler, Herrliberg, Switzerland; Ramya Dutta, Chennai Mathematical Institute (student), India and the proposer.

140. *Proposed by Cornel Ioan Vălean, Timiş, Romania.* Find

$$\sum_{n=1}^{\infty} \frac{H_n^3}{(n+1)2^n}$$

Where $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ denotes the nth harmonic number.

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria.

We will write $H_0 = 0$ for convenience. Since $H_n = \mathcal{O}(\log n)$, the following power series

$$F(x) = \sum_{n=1}^{\infty} H_n x^n, \quad G(x) = \sum_{n=1}^{\infty} H_n^2 x^n, \quad H(x) = \sum_{n=1}^{\infty} H_n^3 x^n, \quad I(x) = \sum_{n=1}^{\infty} \frac{H_n^3}{n+1} x^{n+1},$$

converge for $x \in (-1,1)$.

(1) For all $x \in (-1,1)$ we have

$$(1-x)F(x) = \sum_{n=1}^{\infty}(H_n - H_{n-1})x^n = \sum_{n=1}^{\infty}\frac{x^n}{n} = -\log(1-x).$$

thus

$$F(x) = -\frac{\log(1-x)}{1-x} \tag{1}$$

(2) For all $x \in (-1,1)$ we have

$$\begin{aligned}(1-x)G(x) &= \sum_{n=1}^{\infty}(H_n^2 - H_{n-1}^2)x^n = \sum_{n=1}^{\infty}\left(\left(H_{n-1}+\frac{1}{n}\right)^2 - H_{n-1}^2\right)x^n \\ &= \sum_{n=1}^{\infty}\frac{2H_{n-1}}{n}x^n + \sum_{n=1}^{\infty}\frac{x^n}{n^2} = 2\sum_{n=1}^{\infty}\frac{H_n}{n+1}x^{n+1} + \mathrm{Li}_2(x) \\ &= 2\int_0^x\left(\sum_{n=1}^{\infty}H_n t^n\right)dt + \mathrm{Li}_2(x) = -2\int_0^x\frac{\log(1-t)}{1-t}dt + \mathrm{Li}_2(x) \\ &= \log^2(1-x) + \mathrm{Li}_2(x).\end{aligned}$$

where $\mathrm{Li}_2(x) = \sum_{n=1}^{\infty} x^n/n^2$ is the Dilogarithm. Thus

$$G(x) = \frac{\log^2(1-x)}{1-x} + \frac{\mathrm{Li}_2(x)}{1-x} \tag{2}$$

(3) For all $x \in (-1,1)$ we have

$$\begin{aligned}(1-x)H(x) &= \sum_{n=1}^{\infty}(H_n^3 - H_{n-1}^3)x^n = \sum_{n=1}^{\infty}\left(\left(H_{n-1}+\frac{1}{n}\right)^3 - H_{n-1}^3\right)x^n \\ &= \sum_{n=1}^{\infty}\frac{3H_{n-1}^2}{n}x^n + \sum_{n=1}^{\infty}\frac{3H_{n-1}}{n^2}x^n + \sum_{n=1}^{\infty}\frac{x^n}{n^3} \\ &= 3\sum_{n=1}^{\infty}\frac{H_n^2}{n+1}x^{n+1} + 3\sum_{n=1}^{\infty}\frac{H_n}{(n+1)^2}x^{n+1} + \mathrm{Li}_3(x) \\ &= 3\int_0^x\left(\sum_{n=1}^{\infty}H_n^2 t^n\right)dt + 3\int_0^x\frac{1}{t}\left(\sum_{n=1}^{\infty}\frac{H_n}{n+1}t^{n+1}\right)dt + \mathrm{Li}_3(x)\end{aligned}$$

where $\mathrm{Li}_3(x) = \sum_{n=1}^{\infty} x^n/n^3$ is the Trilogarithm. Using the results of **(2)** we see that

$$\begin{aligned}(1-x)H(x) &= 3\int_0^x G(t)dt + \frac{3}{2}\int_0^x\frac{\log^2(1-t)}{t}dt + \mathrm{Li}_3(x) \\ &= -\log^3(1-x) + 3\int_0^x\frac{\mathrm{Li}_2(t)}{1-t}dt + \frac{3}{2}\int_0^x\frac{\log^2(1-t)}{t}dt + \mathrm{Li}_3(x)\end{aligned}$$

Now, recall that for $0 < x < 1$ we have

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = \frac{\pi^2}{6} - \log(x)\log(1-x)$$

So

$$\int_0^x \frac{\mathrm{Li}_2(t)}{1-t}dt = \frac{\pi^2}{6}\int_0^x \frac{dt}{1-t} - \int_0^x \frac{\log(t)\log(1-t)}{1-t}dt - \int_0^x \frac{{}_2(1-t)}{1-t}dt$$
$$= -\frac{\pi^2}{6}\log(1-x) - \int_0^x \frac{\log(t)\log(1-t)}{1-t}dt - \int_{1-x}^1 \frac{\mathrm{Li}_2(t)}{t}dt$$
$$= -\frac{\pi^2}{6}\log(1-x) - \int_0^x \frac{\log(t)\log(1-t)}{1-t}dt + \mathrm{Li}_3(1-x) - \mathrm{Li}_3(1)$$

But

$$\left(\frac{\log(t)\log^2(1-t)}{2}\right)' = -\frac{\log(t)\log(1-t)}{1-t} + \frac{1}{2}\cdot\frac{\log^2(1-t)}{t}$$

Hence

$$\int_0^x \frac{\mathrm{Li}_2(t)}{1-t}dt + \frac{1}{2}\int_0^x \frac{\log^2(1-t)}{t}dt$$
$$= -\frac{\pi^2}{6}\log(1-x) + \frac{1}{2}\log(x)\log^2(1-x) + \mathrm{Li}_3(1-x) - \mathrm{Li}_3(1)$$

Thus

$$H(x) = -\frac{\log^3(1-x)}{1-x} - \frac{\pi^2}{2}\frac{\log(1-x)}{1-x} + 3\frac{\mathrm{Li}_3(1-x)}{1-x} - \frac{3\,\mathrm{Li}_3(1)}{1-x}$$
$$+ \frac{3}{2}\frac{\log(x)\log^2(1-x)}{1-x} + \frac{\mathrm{Li}_3(x)}{1-x} \qquad (3)$$

(4) Integrating (3) we obtain

$$I(x) = \frac{1}{4}\log^4(1-x) + \frac{\pi^2}{4}\log^2(1-x) + 3\zeta(3)\log(1-x) - 3\,\mathrm{Li}_4(1-x) + 3\zeta(4)$$
$$+ \frac{3}{2}\underbrace{\int_0^x \frac{\log(t)\log^2(1-t)}{1-t}dt}_{J(x)} + K(x) + \underbrace{\int_0^x \frac{\mathrm{Li}_3(t)}{1-t}dt}_{K(x)}$$

Now,

$$J(1-x) = \int_0^{1-x} \frac{\log(t)\log^2(1-t)}{1-t}dt = \int_x^1 \frac{\log(1-t)}{t}\log^2(t)dt$$
$$= \Big[-\mathrm{Li}_2(t)\log^2(t)\Big]_x^1 + 2\int_x^1 \frac{\mathrm{Li}_2(t)}{t}\log(t)dt$$
$$= \mathrm{Li}_2(x)\log^2(x) + 2\Big[\mathrm{Li}_3(t)\log(t)\Big]_x^1 - 2\int_x^1 \frac{\mathrm{Li}_3(t)}{t}dt$$
$$= \mathrm{Li}_2(x)\log^2(x) - 2\,\mathrm{Li}_3(x)\log(x) + 2\,\mathrm{Li}_4(x) - 2\zeta(4)$$

$$K(x) = \Big[-\mathrm{Li}_3(t)\log(1-t)\Big]_0^x - \int_0^x \frac{-\log(1-t)}{t}\mathrm{Li}_2(t)dt$$
$$= -\mathrm{Li}_3(x)\log(1-x) - \int_0^x \mathrm{Li}_2'(t)\,\mathrm{Li}_2(t)dt$$
$$= -\mathrm{Li}_3(x)\log(1-x) - \frac{1}{2}\mathrm{Li}_2^2(x)$$

It follows that,

$$I(x) = \frac{1}{4}\log^4(1-x) + \frac{\pi^2}{4}\log^2(1-x) + 3\zeta(3)\log(1-x) + \frac{3}{2}\operatorname{Li}_2(1-x)\log^2(1-x)$$
$$- 3\operatorname{Li}_3(1-x)\log(1-x)) - \operatorname{Li}_3(x)\log(1-x) - \frac{1}{2}\operatorname{Li}_2^2(x) \tag{4}$$

(5) Fortunately the values of $\operatorname{Li}_2(1/2)$ and $\operatorname{Li}_3(1/2)$ are known:

$$\operatorname{Li}_2\left(\frac{1}{2}\right) = \frac{\pi^2}{12} - \frac{1}{2}\log^2 2$$
$$\operatorname{Li}_3\left(\frac{1}{2}\right) = \frac{7}{8}\zeta(3) + \frac{1}{6}\log^3(2) - \frac{\pi^2}{12}\log(2).$$

We conclude immediately that

$$I\left(\frac{1}{2}\right) = \frac{1}{2}\zeta(3)\log(2) - \frac{\pi^4}{288} + \frac{\log^4(2)}{24} + \frac{\pi^2}{12}\log^2(2)$$

and the announced answer follows since the desired sum is $2I\left(\frac{1}{2}\right)$.

Solution 2 by Moti Levy, Rehovot, Israel. Let $f(z) := \sum_{n=1}^{\infty} H_n^3 z^n$ be the generating function of the sequence $\left(H_n^3\right)_{n\geq 1}$.

An expression for $f(z)$ can be found in a nice article by Professor István Mezõ, "*Nonlinear Euler Sums*", in the Pacific Journal of Mathematics, Vol.272, No. 1, 2014:

$$f(z) = \frac{1}{1-z}\left(-\frac{\pi^2}{2}\ln(1-z) - \ln^3(1-z) + \frac{3}{2}\ln^2(1-z)\ln z + 3\operatorname{Li}_3(1-z) + \operatorname{Li}_3(z) - 3\zeta(3)\right), \tag{15}$$

where $\operatorname{Li}_k(z) := \sum_{n=1}^{\infty}\frac{z^n}{n^k}$ is the polylogarithmic function.

$$\frac{1}{z}\int_0^z f(t)\,dt = \frac{1}{z}\int_0^z \sum_{n=1}^{\infty} H_n^3 t^n\,dt = \frac{1}{z}\sum_{n=1}^{\infty} H_n^3 \int_0^z t^n\,dt = \sum_{n=1}^{\infty}\frac{H_n^3}{n+1}z^n \tag{16}$$

Putting $z = \frac{1}{2}$ in (16),

$$\sum_{n=1}^{\infty}\frac{H_n^3}{(n+1)\,2^n} = 2\int_0^{\frac{1}{2}} f(t)\,dt. \tag{17}$$

$$\int_0^{\frac{1}{2}} f(t)\,dt = -\frac{\pi^2}{2}\int_0^{\frac{1}{2}}\frac{\ln(1-t)}{1-t}dt - \int_0^{\frac{1}{2}}\frac{\ln^3(1-t)}{1-t}dt + \frac{3}{2}\int_0^{\frac{1}{2}}\frac{\ln^2(1-t)\ln t}{1-t}dt$$
$$+ 3\int_0^{\frac{1}{2}}\frac{\operatorname{Li}_3(1-t)}{1-t}dt + \int_0^{\frac{1}{2}}\frac{\operatorname{Li}_3(t)}{1-t}dt - 3\zeta(3)\int_0^{\frac{1}{2}}\frac{1}{1-t}dt.$$

$$\int_0^{\frac{1}{2}} \frac{\ln(1-t)}{1-t}dt = -\frac{1}{2}\ln^2 2, \tag{18}$$

$$\int_0^{\frac{1}{2}} \frac{\ln^3(1-t)}{1-t}dt = -\frac{1}{4}\ln^4 2, \tag{19}$$

$$\int_0^{\frac{1}{2}} \frac{\ln^2(1-t)\ln t}{1-t}dt = -\frac{\pi^4}{45} - \frac{\pi^2}{12}\ln^2 2 - \frac{1}{6}\ln^4 2 + 2\mathrm{Li}_4\left(\frac{1}{2}\right) + \frac{7}{4}\zeta(3)\ln 2, \tag{20}$$

$$\int_0^{\frac{1}{2}} \frac{\mathrm{Li}_3(1-t)}{1-t}dt = \frac{\pi^4}{90} - \mathrm{Li}_4\left(\frac{1}{2}\right), \tag{21}$$

$$\int_0^{\frac{1}{2}} \frac{\mathrm{Li}_3(t)}{1-t}dt = -\frac{1}{288}\pi^4 - \frac{1}{24}\pi^2\ln^2 2 + \frac{1}{24}\ln^4 2 + \frac{7}{8}\zeta(3)\ln 2, \tag{22}$$

$$\int_0^{\frac{1}{2}} \frac{1}{1-t}dt = \ln 2. \tag{23}$$

$$\sum_{n=1}^{\infty}\frac{H_n^3}{(n+1)2^n} = 2\int_0^{\frac{1}{2}} f(t)\,dt = \frac{1}{6}\pi^2\ln^2 2 + \frac{1}{12}\ln^4 2 - \frac{1}{144}\pi^4 + \zeta(3)\ln 2 \cong 0.966\,3.$$

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Refik Zeraoulia, Algeria and the proposer.

141. *Proposed by Valmir Krasniqi, University of Prishtina, Republic of Kosova.* Let $p \in \mathbb{N}$, and let $A_p(x) = \frac{(p+1)!p^{x-1}}{x(x+1)\cdots(x+p)}$. Prove that $(-1)^n\frac{d^n}{dx^n}(\ln\phi(x)) > 0$, for all $n = 1, 2, 3\ldots$ and $x > 0$, where $\phi(x) = \frac{\sqrt[x]{A_p(x+1)}}{x}$.

Solution by Moti Levy, Rehovot, Israel. A positive function φ is said to be *logarithmically completely monotonic* on an interval I if its logarithm $\ln f$ satisfies

$$(-1)^n\frac{d^n}{dx^n}(\ln\varphi(x)) \geq 0$$

for $n = 1, 2, 3, \ldots$ on I.

So we are asked here to show that $\varphi(x) = \frac{\sqrt[x]{A_p(x+1)}}{x}$ is strictly logarithmically completely monotonic on the interval $(0, \infty)$.

$$\Gamma_p(x) := \frac{p!p^x}{x(x+1)\cdots(x+p)}. \tag{24}$$

$$A_p(x) = \frac{(p+1)!p^{x-1}}{x(x+1)\cdots(x+p)} = \frac{p+1}{p}\Gamma_p(x).$$

Feng Qi and Chao -Ping Chen actually showed in [1], that $\frac{\sqrt[x]{\Gamma(x+1)}}{x}$ is strictly logarithmically completely monotonic.

I will follow here their footsteps , for $\frac{\sqrt[x]{A_p(x+1)}}{x}$:

Let

$$f(x) := \ln\left(\frac{\sqrt[x]{A_p(x+1)}}{x}\right) = -\ln x + \frac{1}{x}\ln\left(\frac{p+1}{p}\Gamma_p(x+1)\right).$$

Using Leibnitz' rule $\frac{d^n}{dx^n}(u(x)v(x)) = \sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dx^k}(u(x))\frac{d^{n-k}}{dx^{n-k}}(v(x))$,

$$\begin{aligned}
&\frac{d^n}{dx^n}(f(x)) \\
&= -\frac{(-1)^{n-1}(n-1)!}{x^n} + \sum_{k=0}^{n}\binom{n}{k}\frac{d^k}{dx^k}\left(\ln\left(\frac{p+1}{p}\Gamma_p(x+1)\right)\right)\frac{d^{n-k}}{dx^{n-k}}\left(\frac{1}{x}\right) \\
&= \frac{(-1)^n n!}{nx^n} + \ln\left(\frac{p+1}{p}\Gamma_p(x+1)\right)\frac{d^n}{dx^n}\left(\frac{1}{x}\right) + \sum_{k=1}^{n}\binom{n}{k}\frac{d^{k-1}}{dx^{k-1}}\left(\frac{d\left(\ln\left(\frac{p+1}{p}\Gamma_p(x+1)\right)\right)}{dx}\right)\frac{d^{n-k}}{dx^{n-k}}\left(\frac{1}{x}\right) \\
&= \frac{(-1)^n n!}{nx^n} + \ln\left(\frac{p+1}{p}\Gamma_p(x+1)\right)\frac{d^n}{dx^n}\left(\frac{1}{x}\right) + \sum_{k=1}^{n}\binom{n}{k}\frac{d^{k-1}}{dx^{k-1}}(\psi_p(x+1))\frac{d^{n-k}}{dx^{n-k}}\left(\frac{1}{x}\right) \\
&= \frac{(-1)^n n!}{nx^n} + \frac{(-1)^n n!}{x^{n+1}}\ln\left(\frac{p+1}{p}\Gamma_p(x+1)\right) + \sum_{k=1}^{n}\binom{n}{k}\frac{d^{k-1}}{dx^{k-1}}(\psi_p(x+1))\frac{d^{n-k}}{dx^{n-k}}\left(\frac{1}{x}\right) \\
&= \frac{(-1)^n n!}{nx^n} + \frac{(-1)^n n!}{x^{n+1}}\ln\left(\frac{p+1}{p}\Gamma_p(x+1)\right) + \sum_{k=1}^{n}\frac{n!}{k!(n-k)!}\frac{(-1)^{n-k}(n-k)!}{x^{n-k+1}}\frac{d^{k-1}}{dx^{k-1}}(\psi_p(x+1)) \\
&= \frac{(-1)^n n!}{nx^n} + \frac{(-1)^n n!}{x^{n+1}}\ln\left(\frac{p+1}{p}\Gamma_p(x+1)\right) + \sum_{k=1}^{n}\frac{n!}{k!}\frac{(-1)^{n-k}}{x^{n-k+1}}\frac{d^{k-1}}{dx^{k-1}}(\psi_p(x+1)) \\
&= \frac{(-1)^n n!}{x^{n+1}}\left(\frac{x}{n} + \ln\left(\frac{p+1}{p}\Gamma_p(x+1)\right) + \sum_{k=1}^{n}(-1)^k\frac{x^k}{k!}\frac{d^{k-1}}{dx^{k-1}}(\psi_p(x+1))\right)
\end{aligned}$$

Let

$$g(x) := \frac{x}{n} + \ln\left(\frac{p+1}{p}\Gamma_p(x+1)\right) + \sum_{k=1}^{n}(-1)^k\frac{x^k}{k!}\frac{d^{k-1}}{dx^{k-1}}(\psi_p(x+1)),$$

so that

$$\frac{d^n}{dx^n}(f(x)) = \frac{(-1)^n n!}{x^{n+1}}g(x). \tag{25}$$

One can check that

$$\frac{d}{dx}(g(x)) = \frac{1}{n} + \frac{(-1)^n x^n}{n!}\frac{d^n}{dx^n}(\psi_p(x+1)).$$

Now we need the Laplace transforms of $\frac{d^n}{dx^n}(\psi_p(x))$ and of $\frac{1}{x^n}$, for $x > 0$ (can be found in [2]):

$$\frac{1}{x^n} = \frac{1}{(n-1)!}\int_0^\infty t^{n-1}e^{-xt}dt$$

$$\frac{d^n}{dx^n}(\psi_p(x)) = (-1)^{n+1}\int_0^\infty \frac{1-e^{-(p+1)t}}{1-e^{-t}}t^n e^{-xt}dt$$

$$\begin{aligned}
\frac{1}{x^n}\frac{d}{dx}(g(x)) &= \frac{1}{nx^n} + \frac{(-1)^n}{n!}\frac{d^n}{dx^n}(\psi_p(x+1)) \\
&= \frac{1}{n!}\int_0^\infty t^{n-1}e^{-xt}dt - \frac{1}{n!}\int_0^\infty \frac{e^{-t}}{1-e^{-t}}\left(1-e^{-(p+1)t}\right)t^n e^{-xt}dt \\
&= \frac{1}{n!}\int_0^\infty \left(1 - \frac{t}{e^t-1}\left(1-e^{-(p+1)t}\right)\right)t^{n-1}e^{-xt}dt.
\end{aligned}$$

Since $0 < \frac{t}{e^t-1} < 1$ and $0 < 1-e^{-(p+1)t} < 1$ for $t > 0$, then $1-\frac{t}{e^t-1}\left(1-e^{-(p+1)t}\right) > 0$ for $t > 0$, and $\int_0^\infty \left(1-\frac{t}{e^t-1}\left(1-e^{-(p+1)t}\right)\right) t^{n-1}e^{-xt}dt > 0$.
Thus $\frac{1}{x^n}\frac{d}{dx}\left(g\left(x\right)\right) > 0$ for $x > 0$, and $\frac{d}{dx}\left(g\left(x\right)\right) > 0$ for $x > 0$, which implies that

$$g\left(x\right) > g\left(0\right) = 0 \qquad \text{on} \qquad (0,\infty). \tag{26}$$

It follows from (25) and (26) that $(-1)^n \frac{d^n}{dx^n}\left(f\left(x\right)\right) = \frac{n!}{x^{n+1}}g\left(x\right) > 0$ for $x > 0$ and for all $n = 1, 2, 3...$, which implies that $\frac{\sqrt[x]{A_p(x+1)}}{x}$ is strictly logarithmically completely monotonic.

References:

[1] Feng Qi and Chao -Ping Chen, "A complete monotonicity property of the gamma function", J. Math. Anal. Appl. 296 (2004), pages 603-607.
[2] Valmir Krasniqi and Feng Qi, "Complete monotonicity of a function involving the *p*-psi function and alternative proofs", G. Jour. Math. Anal. 2 (2014), no. 3, 204–208.

Also solved by the proposer.

142. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Let $(a_n)_{n\geq 1}$ such that $a_1 = 1$ and $a_{n+1} = (n+1)!a_n$ for all $n \in \mathbb{N}^*$. Let $(b_n)_{n>1}$ be a positive sequence such that $\lim\limits_{n\to\infty}\frac{b_n}{n!} = b > 0$. Compute

$$\lim_{n\to\infty}\frac{\sqrt[2n]{b_n}}{\sqrt[n^2]{a_n}}.$$

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria. The answer is $e^{1/4}$. Let $u_n = \sqrt[2n]{n!}/\sqrt[n^2]{a_n}$. Since $a_n = \prod_{k=1}^{n} k! = \prod_{k=1}^{n} k^{n+1-k}$ we conclude that

$$\begin{aligned}\log u_n &= \frac{1}{2n}\sum_{k=1}^{n}\log k - \frac{1}{n^2}\sum_{k=1}^{n}(n+1-k)\log k\\ &= \frac{1}{2n}\sum_{k=1}^{n}\log\frac{k}{n} + \frac{1}{2}\log n - \frac{1}{n^2}\sum_{k=1}^{n}\log k - \frac{1}{n^2}\sum_{k=1}^{n}(n-k)\log\frac{k}{n} - \frac{n-1}{2n}\log n\\ &= \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}-\frac{1}{2}\right)\log\frac{k}{n} - \frac{\log n}{2n} - \frac{1}{n^2}\sum_{k=1}^{n}\log k\end{aligned}$$

So we have proved that

$$\log u_n = \frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}-\frac{1}{2}\right)\log\frac{k}{n} + \mathcal{O}\left(\frac{\log n}{n}\right)$$

But

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}-\frac{1}{2}\right)\log\frac{k}{n} &= \int_0^1\left(x-\frac{1}{2}\right)\log x\,dx\\ &= \left[\frac{x^2-x}{2}\log x\right]_0^1 + \int_0^1\frac{1-x}{2}dx = \frac{1}{4}\end{aligned}$$

Thus $\lim_{n\to\infty} u_n = e^{1/4}$. Finally, because $\lim_{n\to\infty} \frac{b_n}{n!} = b > 0$ we see that $\lim_{n\to\infty} \frac{\sqrt[2n]{b_n}}{\sqrt[2n]{n!}} = 1$, hence

$$\lim_{n\to\infty} \frac{\sqrt[2n]{b_n}}{\sqrt[n^2]{a_n}} = \lim_{n\to\infty} \frac{\sqrt[2n]{b_n}}{\sqrt[2n]{n!}} u_n = e^{1/4}$$

as announced.

Solution 2 by Angel Plaza, University of Las Palmas de Gran Canaria, Spain. The answer is $e^{1/4}$. Let $u_n = \sqrt[2n]{n!}/\sqrt[n^2]{a_n}$
Let L be the proposed limit. Then

$$\begin{aligned}
\ln L &= \lim_{n\to\infty} \ln \sqrt[n^2]{\frac{b_n^{n^2/2n}}{a_n}} = \lim_{n\to\infty} \frac{\ln \frac{b_n^{n/2}}{a_n}}{n^2} \\
&= \lim_{n\to\infty} \frac{\ln \frac{b_{n+1}^{(n+1)/2}}{a_{n+1}} - \ln \frac{b_n^{n/2}}{a_n}}{2n+1} \\
&= \lim_{n\to\infty} \frac{\ln \frac{b_{n+1}^{(n+1)/2}}{b_n^{n/2}} \cdot \frac{a_n}{a_{n+1}}}{2n+1} \\
&= \lim_{n\to\infty} \frac{\ln \frac{b_{n+1}^{(n+1)/2}}{b_n^{n/2}} \cdot \frac{1}{(n+1)!}}{2n+1} \\
&= \lim_{n\to\infty} \frac{\ln \frac{b_{n+2}^{(n+2)/2}}{b_{n+1}^{(n+1)/2}(n+2)!} - \ln \frac{b_{n+1}^{(n+1)/2}}{b_n^{n/2}(n+1)!}}{(2n+3)-(2n+1)} \\
&= \frac{1}{2} \lim_{n\to\infty} \ln \left(\frac{b_{n+2}^{(n+2)/2}}{b_{n+1}^{(n+1)/2}(n+2)!} \cdot \frac{b_n^{n/2}(n+1)!}{b_{n+1}^{(n+1)/2}} \right) \\
&= \frac{1}{2} \lim_{n\to\infty} \ln \frac{\left(\frac{b_{n+2}}{b_{n+1}}\right)^{(n+2)/2} \cdot \left(\frac{b_n}{b_{n+1}}\right)^{n/2}}{n+2} \\
&= \frac{1}{2} \lim_{n\to\infty} \ln \frac{\left(\frac{n+2}{n+1}\right)^{n/2} (n+2)}{n+2} \\
&= \frac{1}{2} \ln e^{1/2} = \frac{1}{4}.
\end{aligned}$$

Also solved by Arkady Alt, San Jose, California, USA; Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Romania; Moti Levy, Rehovot, Israel; George-Florin Serban, Pedagogical High School, Braila, Romania; Michel Bataille, Rouen, France and the proposers.

143. *Proposed by Florin Stanescu, Serban Cioculescu school, city Gaesti, jud. Dambovita, Romania.* We consider two matrices $A, B \in M_2(\mathbb{R})$, at least one of them is not invertible. If $A^2 + AB + B^2 = 2BA$, prove that $AB = BA = O_2$.

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria.
(a) $A + B$ is singular. indeed by assumption $\det(BA) = \det B \det A = 0$, so from $(A+B)^2 = 3BA$ we conclude that $\det(A+B) = 0$.

(b) Both A and B are singular. Indeed, from the fact that $A+B$ is singular and the two equalities:

$$(A-2B)(A+B) = A^2+AB-2BA-2B^2 = -3B^2$$
$$(B+A)(B-2A) = B^2+AB-2BA-2A^2 = -3A^2$$

(c) Let $\alpha = A$, $\beta = B$. Since $\det A = \det B = \det(A+B) = 0$ we conclude that $A^2 = \alpha A$, $B^2 = \beta B$ and $(A+B)^2 = (\alpha+\beta)(A+B)$. Taking traces of both sides of the equality $A^2+AB+BA+B^2 = (\alpha+\beta)(A+B)$ we conclude that $(AB) = (BA) = \alpha\beta$. Now, taking traces of both sides of the equality $A^2+AB+B^2 = 2BA$ yields $\alpha^2-\alpha\beta+\beta^2 = 0$, and consequently $\alpha = \beta = 0$. So, $A^2 = B^2 = (A+B)^2 = O_2$. This implies that $AB+BA = O_2$ and the $A^2+AB+B^2 = 2BA$ becomes $AB = 2BA$. These two equalities imply that $AB = BA = O_2$, which is the desired conclusion.

Solution 2 by Michel Bataille, Rouen, France. We shall use the following known result: if $M \in M_n(\mathbb{R})$ is not invertible, then $M^2 = \operatorname{tr}(M)M$ (this results from the Hamilton-Cayley Theorem since the characteristic polynomial of M is $x^2 - \operatorname{tr}(M)x + \det(M) = x^2 - \operatorname{tr}(M)x$).

We have $(A+B)^2 = A^2+AB+BA+B^2 = 3BA$, hence $\det((A+B)^2) = 9(\det A)(\det B) = 0$ (the latter equality because A or B is not invertible). Thus $A+B$ is not invertible and so

$$3BA = (A+B)^2 = \operatorname{tr}(A+B)(A+B) \qquad (1).$$

Let $m = \operatorname{tr}(A+B) = \operatorname{tr}(A)+\operatorname{tr}(B)$. We first show that $m = 0$.
If neither A nor B is invertible, then $A^2 = tA$ and $B^2 = (m-t)B$ where $t = \operatorname{tr}(A)$ so that $\operatorname{tr}(A^2) = t^2$ and $\operatorname{tr}(B^2) = (m-t)^2$. Since $\operatorname{tr}(AB) = \operatorname{tr}(BA)$, the hypothesis $A^2+AB+B^2 = 2BA$ gives $t^2+(m-t)^2 = \operatorname{tr}(BA)$. But, from (1), we have $m^2 = (\operatorname{tr}(A+B))^2 = \operatorname{tr}((A+B)^2) = 3\operatorname{tr}(BA)$ and so $t^2+(m-t)^2 = \frac{m^2}{3}$. This rewrites as $\left(t-\frac{m}{2}\right)^2 + \frac{m^2}{12} = 0$, which implies $m = 0$.
Now, suppose that only one of A and B is invertible and assume that $m \neq 0$. Since $A+B$ is not invertible, we have $(A+B)X = \mathbf{0}$ for some nonzero column vector X. Then, from (1), we also have $BAX = \mathbf{0}$ and so $B^2X = B(A+B)X = \mathbf{0}$. This cannot occur if B is invertible (since $X \neq \mathbf{0}$). If B is not invertible, then $BY = \mathbf{0}$ for some column vector $Y \neq 0$ and so $\frac{m}{3}$ is an eigenvalue of B (because $BAY = \frac{m}{3}(A+B)Y = \frac{m}{3}AY$ and $AY \neq \mathbf{0}$ since A is invertible). But X and Y are independent vectors ($X = \alpha Y$ implies $BX = \alpha BY = \mathbf{0}$, a contradiction since $(A+B)X = \mathbf{0}$ and $AX \neq \mathbf{0}$) and $B^2X = B^2Y = \mathbf{0}$, hence $B^2 = O_2$. This contradicts the fact that B (hence also B^2) has a nonzero eigenvalue (namely $\frac{m}{3}$). We conclude that $m = 0$.
Now, since $m = 0$, we already have $BA = O_2$ and we deduce that $A = O_2$ if B is invertible and that $B = O_2$ if A is invertible, in which cases $AB = BA = O_2$ is obvious. If neither A nor B is invertible, then $A^2 = \operatorname{tr}(A)A$ and $B^2 = \operatorname{tr}(B)B$ and so $\operatorname{tr}(A^2) = (\operatorname{tr}(A))^2$ and $\operatorname{tr}(B^2) = (\operatorname{tr}(B))^2$. Since $\operatorname{tr}(AB) = \operatorname{tr}(BA) = \operatorname{tr}(O_2) = 0$, the hypothesis $A^2+AB+B^2 = 2BA$ gives $\operatorname{tr}(A))^2 + \operatorname{tr}(B))^2 = 0$ so that $\operatorname{tr}(A) = \operatorname{tr}(B) = 0$ ($\operatorname{tr}(A)$ and $\operatorname{tr}(B)$ being real numbers). We deduce $A^2 = B^2 = O_2$ and so $AB = 2BA = O_2$, as desired.

Also solved by Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Romania; George-Florin Serban, Pedagogical High School, Braila, Romania and the proposer.

144. *Proposed by Anastasios Kotronis, Athens, Greece.* Show that, as $x \to \pi^-$,

$$\int_1^\infty \frac{y\cosh(xy)}{\sinh(\pi y)}\,dy = \frac{1}{(\pi-x)^2} + \frac{\mathrm{Li}_2(e^{-2\pi})}{2\pi^2} - \frac{\ln(1-e^{-2\pi})}{\pi} - \frac{1}{2} + \mathcal{O}(\pi - x),$$

where $\mathrm{Li}_2(x) = \sum_{k\geq 1} \frac{x^2}{k^2}$ denotes the Dilogarithm.

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria.
Let $\varepsilon = \pi - x > 0$. Clearly

$$\cosh(xy) = \cosh((\pi-\varepsilon)y) = \sinh(\pi y)e^{-\varepsilon y} + e^{-\pi y}\cosh(\varepsilon y)$$

It follows that

$$\frac{y\cosh(xy)}{\sinh(\pi y)} = ye^{-\varepsilon y} + \frac{2y\cosh(\varepsilon y)}{e^{2\pi y}-1}$$

Hence

$$\int_1^\infty \frac{y\cosh(xy)}{\sinh(\pi y)}\,dy = \frac{(1+\varepsilon)e^{-\varepsilon}}{\varepsilon^2} + \int_1^\infty \frac{2y}{e^{2\pi y}-1}\,dy + g(\varepsilon)$$

where

$$g(\varepsilon) = \int_1^\infty \frac{2y(\cosh(\varepsilon y - 1)}{e^{2\pi y}-1}\,dy$$

Noting that $0 \leq \cosh u - 1 \leq \frac{u^2}{2}e^u$ for $u \geq 0$ and that $e^{2\pi y} - 1 \geq e^{\pi y}$ for $y \geq 1$ we conclude that

$$0 < g(\varepsilon) \leq \varepsilon^2 \int_1^\infty \frac{y^3 e^{\varepsilon y}}{e^{\pi y}}dy \leq \varepsilon^2 \int_0^\infty y^3 e^{-(\pi-\varepsilon)y}dy = \frac{6\varepsilon^2}{(\pi-\varepsilon)^4}$$

In particular, $g(\varepsilon) = \mathcal{O}(\varepsilon^2)$. Also, clearly we have $\dfrac{(1+\varepsilon)e^{-\varepsilon}}{\varepsilon^2} = \dfrac{1}{\varepsilon^2} - \dfrac{1}{2} + \dfrac{\varepsilon}{3} + \mathcal{O}(\varepsilon^2)$.
It follows that

$$\int_1^\infty \frac{y\cosh(xy)}{\sinh(\pi y)}\,dy = \frac{1}{\varepsilon^2} - \frac{1}{2} + \int_1^\infty \frac{2y}{e^{2\pi y}-1}\,dy + \frac{\varepsilon}{3} + \mathcal{O}(\varepsilon^2)$$

Now it remains to evaluate the integral on the right:

$$\int_1^\infty \frac{2y}{e^{2\pi y}-1}dy = \int_1^\infty \frac{2ye^{-2\pi y}}{1-e^{-2\pi y}}dy = -\frac{1}{2\pi^2}\int_0^{e^{-2\pi}} \frac{\ln x}{1-x}dx \qquad : x \leftarrow e^{-2\pi y}$$

$$= \left[\frac{1}{2\pi^2}\ln(1-x)\ln x\right]_0^{e^{-2\pi}} - \frac{1}{2\pi^2}\int_0^{e^{-2\pi}} \frac{\ln(1-x)}{x}dx$$

$$= -\frac{1}{\pi}\ln(1-e^{-2\pi}) + \frac{\mathrm{Li}_2(e^{-2\pi})}{2\pi^2}$$

So, we have shown that as $x \to \pi^-$ we have

$$\int_1^\infty \frac{y\cosh(xy)}{\sinh(\pi y)}\,dy = \frac{1}{(\pi-x)^2} + \frac{\mathrm{Li}_2(e^{-2\pi})}{2\pi^2} - \frac{1}{2} - \frac{\ln(1-e^{-2\pi})}{\pi} + \frac{\pi - x}{3} + \mathcal{O}((\pi-x)^2)$$

which is stronger than the desired result.

Solution 2 by Ramya Dutta, Chennai Mathematical Institute (student), India.
We start with the integral $\displaystyle\int_0^\infty \frac{\sinh(xy)}{\sinh(\pi y)}\,dy = \frac{1}{2}\tan\left(\frac{x}{2}\right)$ for $x \in (0, \pi)$ $\qquad \cdots (1)$

$$\begin{aligned}\int_0^\infty \frac{\sinh(xy)}{\sinh(\pi y)}\,dy &= \int_0^\infty \frac{e^{-\pi y}(e^{xy}-e^{-xy})}{1-e^{-2\pi n y}}\,dy\\ &= \sum_{n=0}^\infty \int_0^\infty e^{-(2n+1)\pi y}(e^{xy}-e^{-xy})\,dy\\ &= \sum_{n=0}^\infty \left(\frac{1}{(2n+1)\pi - x} - \frac{1}{(2n+1)\pi + x}\right)\\ &= \frac{1}{2\pi}\sum_{n=0}^\infty \left(\frac{1}{n+\frac{1}{2}-\frac{x}{2\pi}} - \frac{1}{n+\frac{1}{2}+\frac{x}{2\pi}}\right)\\ &= \frac{1}{2\pi}\left(\psi\left(\frac{1}{2}+\frac{x}{2\pi}\right) - \psi\left(\frac{1}{2}-\frac{x}{2\pi}\right)\right) = \frac{1}{2}\tan\frac{x}{2}\end{aligned}$$

where, we used the reflection formula for Digamma function,

$$\psi(1-z)-\psi(z) = \pi\cot(\pi z) \qquad \text{for } z\in(0,1)$$

Differentiating under the integration sign in (1) we get,

$$\int_0^\infty \frac{y\cosh(xy)}{\sinh(\pi y)}\,dy = \frac{1}{4}\sec^2\frac{x}{2}$$

Now, denoting $x = (\pi - t)$, as $x \to \pi^-$ we have $t \to 0^+$, hence by the Laurent series expansion, $\sec^2\frac{x}{2} = \left(\csc\frac{t}{2}\right)^2 = \left(\frac{2}{t}+\frac{t}{12}+O\left(t^3\right)\right)^2 = \frac{4}{t^2}+\frac{1}{3}+O\left(t^2\right) = \frac{4}{(\pi-x)^2}+\frac{1}{3}+O\left((\pi-x)^2\right)$.

Now, from Mean Value theorem, $\cosh(\pi y)-\cosh(xy) = (\pi - x)\sinh(\theta_x y)$ for some $\theta_x \in (x,\pi)$ and as $y\to 0^+$ we have $\frac{\sinh(\theta_x y)}{\sinh(\pi y)} \to \frac{\theta_x}{\pi}$, i.e., $\frac{\sinh(\theta_x y)}{\sinh(\pi y)}$ is bounded on $y\in[0,1]$.

Hence,

$$\begin{aligned}\int_0^1 \frac{y\cosh(xy)}{\sinh(\pi y)}\,dy &= \int_0^1 \frac{y\cosh(\pi y)}{\sinh(\pi y)}\,dy - (\pi - x)\int_0^1 \frac{y\sinh(\theta_x y)}{\sinh(\pi y)}\,dy\\ &= \int_0^1 y\coth(\pi y)\,dy + O(\pi - x)\end{aligned}$$

Thus, combining the results,

$$\int_1^\infty \frac{y\cosh(xy)}{\sinh(\pi y)}\,dy = \frac{1}{(\pi-x)^2}+\frac{1}{12}-\int_0^1 y\coth(\pi y)\,dy + O\left((\pi - x)\right)$$

and since,

$$\begin{aligned}\int_0^1 y\coth(\pi y)\,dy &= \int_0^1 y + \frac{2ye^{-2\pi y}}{1-e^{-2\pi y}}\,dy \\ &= \frac{1}{2} + 2\sum_{n=1}^{\infty}\int_0^1 ye^{-2\pi n y}\,dy \\ &= \frac{1}{2} + 2\sum_{n=1}^{\infty}\left(\frac{1}{4\pi^2 n^2} - \frac{e^{-2\pi n}}{2\pi n} - \frac{e^{-2\pi n}}{4\pi^2 n^2}\right) \\ &= \frac{1}{2} + \frac{1}{12} + \frac{1}{\pi}\log(1-e^{2\pi}) - \frac{1}{2\pi^2}\,\mathrm{Li}_2(e^{-2\pi})\end{aligned}$$

we have the desired result.

Also solved by Albert Stadler, Herrliberg, Switzerland; Refik Zeraoulia, Algeria; Moti Levy, Rehovot, Israel and the proposer.

MATHCONTEST SECTION

This section of the Journal offers readers an opportunity to solve interesting and elegant mathematical problems mainly appeared in Math Contest around the world and most appropriate for training Math Olympiads. Proposals are always welcomed. The source of the proposals will appear when the solutions be published.

Proposals

100. Let $(a_n)_{n\in N}$ be a sequence of real numbers such that $\lim\limits_{n\to\infty} n\left(a_n-1\right)=l\in(-\infty,\infty)$ and let $p\geq 1$ be a natural number. Calculate $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n}\left(a_n+\frac{1}{\sqrt[p]{kn}}\right)$.

101. Let $f,g:[a,b]\to R$ be two nonnegative continuous functions. Assume that f attains its maximum at a unique point on $[a,b]$ and g attains its maximum at the same point as f and possibly at other points.

1) Prove that $\lim\limits_{n\to\infty}\frac{\int_a^b f^{n+1}(x)g(x)dx}{\int_a^b f^n(x)dx}=\|f\|_\infty\|g\|_\infty$.

2) Does the result hold under no assumption on f and g?

102. Let $f\in C^3\left(R^n,R\right)$ with $f\left(0\right)=f'\left(0\right)=0$. Prove that there exist $h\in C^3\left(R^m,S_n\left(R\right)\right)$, such that $f\left(x\right)=x^t h\left(x\right)x$, when $S_n\left(R\right)$, is the set of symmetric matrix, and x^t is the transpose of x.

103. Find the nature of the series $\sum\limits_{n\geq 1}\frac{e^{i\ln(p_n)}}{p_n}$ when $(p_n)_{n\geq 1}$ is the prime number increasing order, and i imaginary complex number.

104. Let a,b, and c be positive real numbers. Prove that

$$\left(\frac{(6n+1)a-b}{n(b+c)}\right)^2+\left(\frac{(6n+1)b-c}{n(c+a)}\right)^2+\left(\frac{(6n+1)c-a}{n(a+b)}\right)^2\geq 27$$

for any positive integer $n\geq 1$.

Solutions

95. Let $n \in \mathbb{N}$ and let $O_n = 1 + \frac{1}{3} + \cdots + \frac{1}{2n-1}$. Calculate $\lim_{n\to\infty} \frac{1}{n}\left(1 + \frac{2O_n}{n}\right)^n$.

(Jozsef Wildt IMC 2016)

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Let $H_n = \sum_{k=1}^{n} 1/k$ be the nth harmonic number. It is well-known that $H_n = \ln n + \gamma + \mathcal{O}\left(\frac{1}{n}\right)$ where γ is the Euler-Mascheroni constant. Clearly,

$$2O_n = 2H_{2n} - H_n = \ln n + 2\ln 2 + \gamma + \mathcal{O}\left(\frac{1}{n}\right)$$

It follows that

$$\ln\left(\frac{1}{n}\left(1 + \frac{2O_n}{n}\right)^n\right) = n\ln\left(1 + \frac{2O_n}{n}\right) - \ln n = 2O_n - \ln n + \mathcal{O}\left(\frac{\ln^2 n}{n}\right)$$
$$= 2\ln 2 + \gamma + \mathcal{O}\left(\frac{\ln^2 n}{n}\right)$$

Hence $\lim_{n\to\infty} \ln\left(\frac{1}{n}\left(1 + \frac{2O_n}{n}\right)^n\right) = 2\ln 2 + \gamma$, and consequently

$$\lim_{n\to\infty} \frac{1}{n}\left(1 + \frac{2O_n}{n}\right)^n = 4e^{\gamma}.$$

Solution 2 by Angel Plaza, University of Las Palmas de Gran Canaria, Spain. Let $H - n$ denote the nth harmonic number $H_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$, and also $\gamma_n = H_n - \ln n$. It is well-know that $\lim_{n\to\infty} \gamma_n = \gamma$, where γ is the Euler-Marcheroni constant. Then $O_n = H_{2n} - \frac{1}{2}H_n$ and therefore, the proposed limit may be written as

$$\lim_{n\to\infty} \frac{1}{n}\left(1 + \frac{2H_{2n} - H_n}{n}\right)^n.$$

It follows that

$$x_n = \frac{1}{n}\left(1 + \frac{2H_{2n} - H_n}{n}\right)^n = \frac{1}{n}\left(1 + a_n\right)^n$$

where $a_n = \dfrac{2H_{2n} - H_n}{n}$. Also, we note that $a_n = O\left(\dfrac{2\ln n - \ln n}{n}\right) = O\left(\dfrac{\ln(4n)}{n}\right)$. Thus,

$$\begin{aligned}
\ln x_n &= n\left(a_n - \frac{a_n^2}{2} + \ldots\right) - \ln n \\
&= na_n - \ln n - n\left(O\left(\frac{\ln(4n)}{n}\right)^2\right) \\
&= 2\gamma_{2n} + 2\ln(2n) - \gamma_n - \ln n - \ln n - n\left(O\left(\frac{\ln(4n)}{n}\right)^2\right) \\
&= 2\gamma_{2n} - \gamma_n + \ln 4 - n\left(O\left(\frac{\ln(4n)}{n}\right)^2\right)
\end{aligned}$$

and hence, $\lim_{n\to\infty} \ln x_n = \gamma + \ln 4$ and therefore the proposed limit equals $4e^{\gamma}$.

Also solved by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy; Albert Stadler, Herrliberg, Switzerland; Michel Bataille, Rouen, France; Arkady Alt, San Jose, California, USA.

96. Let p be a positive real number and let $(a_n)_{n\geq 1}$ be a sequence defined by $a_1 = 1$, $a_{n+1} = \frac{a_n}{1+a_n^p}$. Find those real values $q \neq 0$ such that following series converges $\sum_{n=1}^{\infty} \left|(pn)^{-1/p} - a_n\right|^q$.

(Jozsef Wildt IMC 2016)

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. First, the case $p = 1$ is easy since we have $a_n = 1/n$ for every $n \geq 1$ in this case, and the considered series converges for every $q > 0$ (all the terms are 0 in this case). In what follows we suppose that $p \neq 1$.
A simple proof by induction shows that $(a_n)_{n\geq 1}$ is positive decreasing. So it must converge and its limit ℓ must satisfy $\ell + \ell^{1+p} = \ell$ that is $\ell = 0$. Let $b_n = 1/a_n$. The sequence $(b_n)_{n\geq 1}$is monotone increasing and satisfy $\lim_{n\to\infty} b_n = +\infty$. Moreover,

$$\forall\, n \geq 1, \quad \left(\frac{b_{n+1}}{b_n} - 1\right) b_n^p = (b_{n+1} - b_n) b_n^{p-1} = 1 \qquad (*)$$

Now,

$$\begin{aligned} b_{n+1}^p - b_n^p &= \int_{b_n}^{b_{n+1}} pt^{p-1} dt = p(b_{n+1} - b_n) \int_0^1 (b_n + s(b_{n+1} - b_n))^{p-1} ds \\ &= \frac{p}{b_n^{p-1}} \int_0^1 \left(b_n + s b_n^{1-p}\right)^{p-1} ds = p \int_0^1 (1 + s a_n^p)^{p-1} ds \end{aligned}$$

The dominated convergence theorem shows that $\lim_{n\to\infty} (b_{n+1}^p - b_n^p) = p$. Using Cezáro's lemma we conclude that $\lim_{n\to\infty} (b_n^p/n) = p$. Again, from the above formula we have

$$b_n^p (b_{n+1}^p - b_n^p - p) = p \int_0^1 \frac{(1 + s a_n^p)^{p-1} - 1}{a_n^p} ds$$

and the dominated convergence theorem shows that

$$\lim_{n\to\infty} b_n^p (b_{n+1}^p - b_n^p - p) = p(p-1) \int_0^1 s\, ds = \frac{p(p-1)}{2}$$

Therefore, using $\lim_{n\to\infty} (b_n^p/n) = p$, we get

$$\lim_{n\to\infty} n(b_{n+1}^p - b_n^p - p) = \frac{p-1}{2}$$

or,

$$b_{n+1}^p - b_n^p = p + \frac{p-1}{2n} + o\left(\frac{1}{n}\right)$$

Recalling that $1 + \frac{1}{2} + \cdots + \frac{1}{n} = \log n + \mathcal{O}(1)$, it follows:

$$b_n^p = pn + \frac{p-1}{2} \log n + o(\log n) = pn \left(1 + \frac{p-1}{2p} \frac{\log n}{n} + o\left(\frac{\log n}{n}\right)\right)$$

So

$$\begin{aligned} a_n &= (pn)^{-1/p}\left(1+\frac{p-1}{2p}\frac{\log n}{n}+o\left(\frac{\log n}{n}\right)\right)^{-1/p} \\ &= (pn)^{-1/p}\left(1-\frac{p-1}{2p^2}\frac{\log n}{n}+o\left(\frac{\log n}{n}\right)\right) \\ &= \frac{1}{p^{1/p}n^{1/p}}-\frac{p-1}{2p^{2+1/p}}\frac{\log n}{n^{1+1/p}}+o\left(\frac{\log n}{n^{1+1/p}}\right) \end{aligned}$$

Thus

$$\left|a_n-(np)^{-1/p}\right|^q \sim \frac{\log^q n}{n^{q(1+1/p)}}$$

and the series $\sum_{n=1}^{\infty}\left|(pn)^{-1/p}-a_n\right|^q$ converges if and only if $q(1+1/p)>1$ or equivalently $q>p/(p+1)$.

Solution 2 by Michel Bataille, Rouen, France. By an easy induction, we obtain $a_n>0$ for all $n\in\mathbb{N}$. It follows that $a_{n+1}<a_n$ for all $n\in\mathbb{N}$ so that $\{a_n\}_{n\geq 1}$ is decreasing and bounded below, hence convergent. Its limit ℓ satisfies $\ell=\frac{\ell}{1+\ell^p}$, hence $\ell=0$. If $p=1$, then $a_n=\frac{1}{n}$ for all $n\in\mathbb{N}$ (by induction) and so $\left|(pn)^{-\frac{1}{p}}-a_n\right|^q$ is defined only for $q>0$ and vanishes for all $n\geq 1$ and all $q>0$. Thus, for $p=1$ the series $\sum_{n=1}^{\infty}\left|(pn)^{-\frac{1}{p}}-a_n\right|^q$ is convergent if and only if $q>0$. Now, we suppose that $p\neq 1$. Let $f(x)=\frac{x}{1+x^p}$ and let $b_n=a_n^{-p}$. Since $f(x)=x(1-x^p+x^{2p}+o(x^{2p}))$ as $x\to 0^+$, we have, as $n\to\infty$,

$$\begin{aligned} b_{n+1}=(f(a_n))^{-p} &= (a_n-a_n^{p+1}+a_n^{2p+1}+o(a_n^{2p+1}))^{-p} \\ &= a_n^{-p}(1+pa_n^p-pa_n^{2p}+\frac{p(p+1)}{2}a_n^{2p}+o(a_n^{2p})) \\ &= b_n+p+\frac{p(p-1)}{2}a_n^p+o(a_n^p). \end{aligned}$$

We deduce $b_{n+1}-b_n \sim p$ and $b_{n+1}-b_n-p \sim \frac{p(p-1)}{2}a_n^p$. At this point, we shall use the following form of Stolz's Theorem: if $u_n \sim v_n>0$ as $n\to\infty$ and the series $\sum_{n\geq 1}v_n$ is divergent, then $\sum_{k=1}^{n}u_k \sim \sum_{k=1}^{n}v_k$ as $n\to\infty$. From $b_{n+1}-b_n \sim p$, we then deduce $b_n \sim np$, that is, $a_n^p \sim \frac{1}{np}$ as $n\to\infty$. It follows that $b_{n+1}-b_n-p \sim \frac{p-1}{2}\cdot\frac{1}{n}$ and a second application of Stolz's Theorem gives $b_n-np \sim \frac{p-1}{2}\sum_{k=1}^{n}\frac{1}{k} \sim \frac{p-1}{2}\ln(n)$. It follows that

$$\begin{aligned} a_n=b_n^{-1/p} &= \left(np+\frac{p-1}{2}\ln(n)+o(\ln(n))\right)^{-1/p} \\ &= (np)^{-1/p}\left(1-\frac{p-1}{2p^2}\cdot\frac{\ln(n)}{n}+o((\ln(n))/n)\right) \\ &= (np)^{-1/p}-\alpha\cdot\frac{\ln(n)}{n^{1+\frac{1}{p}}}+o((\ln(n))/n^{1+\frac{1}{p}}) \end{aligned}$$

where we set $\alpha = \frac{p-1}{2p^{2+\frac{1}{p}}}$. As a result,

$$\left|(pn)^{-\frac{1}{p}} - a_n\right|^q \sim |\alpha| \frac{(\ln(n))^q}{n^{q(1+\frac{1}{p})}}$$

as $n \to \infty$. From known results about Bertrand's series, we deduce that the series $\sum_{n=1}^{\infty} \left|(pn)^{-\frac{1}{p}} - a_n\right|^q$ is convergent if and only if $q(1+\frac{1}{p}) > 1$ i.e. $q > \frac{p}{p+1}$. In conclusion, convergence occurs if and only if ($p = 1$ and $q > 0$) or ($p \neq 1$ and $q > \frac{p}{p+1}$).

97. Let $n \in \mathbb{N}^*$, and for an integer k such that $1 \le k \le n$ let n_k be the remainder on euclidean division of n by k. Finally, define p_n to be the probability that $n_k \ge \frac{k}{2}$. Calculate p_n and find $\lim_{n\to\infty} p_n$.

(Jozsef Wildt IMC 2016)

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

Let $q = \left\lfloor \frac{n}{k} \right\rfloor$ so that $n = qk + n_k$ with $0 \le n_k < k$. Now, if $n_k \ge k/2$ then $2n = (2q+1)k + 2n_k - k$ with $0 \le 2n_k - k < k$ so $2q + 1 = \left\lfloor \frac{2n}{k} \right\rfloor$ or equivalently $\left\lfloor \frac{2n}{k} \right\rfloor - 2\left\lfloor \frac{n}{k} \right\rfloor = 1$. On the other hand, if $n_k < k/2$ then $2n = 2qk + 2n_k$ with $0 \le 2n_k < k$ so $2q = \left\lfloor \frac{2n}{k} \right\rfloor$ or equivalently $\left\lfloor \frac{2n}{k} \right\rfloor - 2\left\lfloor \frac{n}{k} \right\rfloor = 0$. We have proved that

$$\left\lfloor \frac{2n}{k} \right\rfloor - 2\left\lfloor \frac{n}{k} \right\rfloor = \begin{cases} 1 & \text{if } n_k \le k/2 \\ 0 & \text{if } n_k < k/2 \end{cases}$$

It follows that

$$\text{Card}\{1 \le k \le n : n_k \ge k/2\} = \sum_{k=1}^{n} \left(\left\lfloor \frac{2n}{k} \right\rfloor - 2\left\lfloor \frac{n}{k} \right\rfloor \right) = \sum_{k=1}^{n} \left(2\left\{ \frac{n}{k} \right\} - \left\{ \frac{2n}{k} \right\} \right)$$

(where $\{x\}$ is the fractional part of x). Consequently,

$$p_n = \frac{1}{n}\sum_{k=1}^{n} \left(2\left\{ \frac{n}{k} \right\} - \left\{ \frac{2n}{k} \right\} \right) = \frac{1}{n}\sum_{k=1}^{n} f\left(\frac{k}{n}\right)$$

where $f(x) = 2\left\{\frac{1}{x}\right\} - \left\{\frac{2}{x}\right\}$. So, p_n is a Riemann sum of a Riemann integrable function, (because f is bounded and continuous on $(0,1) \setminus \{\frac{1}{j}, j \in \mathbb{N}^*\}$). It follows that

$$\begin{aligned} \lim_{n\to\infty} p_n &= \int_0^1 f(x)dx = \int_1^\infty \frac{2\{t\} - \{2t\}}{t^2}dt = 2\int_1^\infty \frac{\{t\}}{t^2}dt - \int_1^\infty \frac{\{2t\}}{t^2}dt \\ &= 2\int_1^\infty \frac{\{t\}}{t^2}dt - 2\int_2^\infty \frac{\{u\}}{u^2}du = 2\int_1^2 \frac{\{t\}}{t^2}dt = 2\int_1^2 \frac{t-1}{t^2}dt \\ &= \left[2\log t + \frac{2}{t}\right]_1^2 = 2\log 2 - 1 \approx 0.386294. \end{aligned}$$

Solution 2 by Michel Bataille, Rouen, France. Let $\lfloor a \rfloor$ denote the integer part of the real number a and $\{a\} = a - \lfloor a \rfloor$ its fractional part. Since $n_k = n - \left\lfloor \frac{n}{k} \right\rfloor k$, it is readily seen that the condition $n_k \ge \frac{k}{2}$ is equivalent to $2\left\{\frac{n}{k}\right\} \ge 1$. Observing that $0 \le \{a\} < 1$ for all positive real a (so that $0 \le 2\left\{\frac{n}{k}\right\} < 2$), we finally obtain that $n_k \geq \frac{k}{2}$ when $\lfloor 2\left\{\frac{n}{k}\right\}\rfloor = 1$ (and $n_k < \frac{k}{2}$ when $\lfloor 2\left\{\frac{n}{k}\right\}\rfloor = 0$). It follows that the number of $k \in \{1, 2, \ldots, n\}$ such that $n_k \geq \frac{k}{2}$ is equal to $\sum\limits_{k=1}^{n} \lfloor 2\left\{\frac{n}{k}\right\}\rfloor$ and so

$$p_n = \frac{1}{n}\sum_{k=1}^{n}\left\lfloor 2\left\{\frac{n}{k}\right\}\right\rfloor.$$

To evaluate $\lim\limits_{n\to\infty} p_n$, we consider the function f defined on $[0, 1]$ by $f(0) = 0$ and $f(x) = \lfloor 2\left\{\frac{1}{x}\right\}\rfloor$ if $x \in (0, 1]$. The number p_n is a Riemann sum attached to this function f. If m is any positive integer, we have $f(x) = 0$ for $x \in (\frac{1}{m+\frac{1}{2}}, \frac{1}{m}]$ and $f(x) = 1$ for $x \in (\frac{1}{m+1}, \frac{1}{m+\frac{1}{2}}]$. Thus the points of discontinuity of f are all in the set formed by 0 and the numbers $\frac{1}{m}, \frac{1}{m+\frac{1}{2}}$ $(m \in \mathbb{N})$. Thus, f is continuous almost everywhere and bounded, hence Riemann integrable on $[0, 1]$. As a result, we have

$$\lim_{n\to\infty} p_n = \int_0^1 f(x)dx = \sum_{m=1}^{\infty}\int_{1/(m+1)}^{1/(m+1/2)} 1\cdot dx = \sum_{m=1}^{\infty}\left(\frac{1}{m+\frac{1}{2}} - \frac{1}{m+1}\right).$$

Now, if N is a positive integer, we have

$$\sum_{m=1}^{N}\left(\frac{1}{m+\frac{1}{2}} - \frac{1}{m+1}\right) = \sum_{m=1}^{N}\frac{2}{2m+1} - \sum_{m=1}^{N}\frac{1}{m+1}$$

$$= 2(H_{2N+1} - \frac{1}{2}H_N - 1) - (H_{N+1} - 1) = 2H_{2N+1} - 2H_N - 1 - \frac{1}{N+1}$$

where $H_N = \sum\limits_{k=1}^{N}\frac{1}{k}$ is the Nth harmonic number. Since $H_N = \ln(N) + \gamma + o(1)$ as $N \to \infty$ (where γ is the Euler constant), we obtain

$$\lim_{n\to\infty} p_n = \lim_{N\to\infty}\left(2[\ln(2N+1) + \gamma - \ln(N) - \gamma + o(1)] - 1 - \frac{1}{N+1}\right) = 2\ln(2) - 1.$$

Also solved by Albert Stadler, Herrliberg, Switzerland; Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy. and Arkady Alt, San Jose, California, USA.

98. Let $(x_n)_{n\geq 0}$ be the sequence defined inductively by $x_{n+2} = x_{n+1} - \frac{1}{2}x_n$ with initial terms $x_0 = 2$ and $x_1 = 1$. Find $\sum\limits_{n=1}^{\infty}\frac{x_n}{n+2}$.

(Jozsef Wildt IMC 2016)

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. A simple verification, by mathematical induction shows that $x_n = 4 \cdot 2^{-(n+2)/2} \sin \frac{(2+n)\pi}{4}$ for $n \geq 0$. It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{x_n}{n+2} &= 4 \cdot \sum_{n=2}^{\infty} \frac{\sin(n\pi/4)}{n(\sqrt{2})^n} = 4 \cdot \Im\left(\sum_{n=2}^{\infty} \frac{1}{n}\left(\frac{e^{i\pi/4}}{\sqrt{2}}\right)^n\right)\\ &= 4 \cdot \Im\left(-\frac{1+i}{2} + \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1+i}{2}\right)^n\right)\\ &= -2 - 4 \cdot \Im\left(\operatorname{Log}\left(1 - \frac{1+i}{2}\right)\right)\\ &= -2 - 4 \cdot \Im\left(-\frac{i\pi}{4}\right) = \pi - 2,\end{aligned}$$

where Log is the principal branch of the logarithm. Finally, $\displaystyle\sum_{n=1}^{\infty} \frac{x_n}{n+2} = \pi - 3$.

Solution 2 by Julian Spahr (student) and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. Since the roots of the characteristic equation are $a = \frac{1}{2} - \frac{i}{2}$ and $b = \frac{1}{2} + \frac{i}{2}$ and taking into acount the initial values $x_0 = 2$ and $x_1 = 1$, it follows that $x_n = a^n + b^n$.
Since $|a| = |b| = \frac{1}{\sqrt{2}}$, the series expansions $\sum_{n\geq 0}(az)^n = \frac{1}{1-az}$ and $\sum_{n\geq 0}(bz)^n = \frac{1}{1-bz}$ are valid for $|z| < \sqrt{2}$, and for $|z| < \sqrt{2}$ we may add and integrate so

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{a^n + b^n}{n+2} &= \int_0^1 \sum_{n=0}^{\infty} (a^n + b^n)\, z^{n+1} dz = \int_0^1 \left(\frac{z}{1-az} + \frac{z}{1-bz}\right) dz\\ &= \int_0^1 \left(-2 + \frac{4}{z^2 - 2z + 2}\right) dz = [-2z - 4\arctan(1-z)]_0^1\\ &= -2 + \pi.\end{aligned}$$

Therefore, the proposed sum is $\displaystyle\sum_{n=1}^{\infty} \frac{x_n}{n+2} = -1 + \sum_{n=0}^{\infty} \frac{a^n + b^n}{n+2} = \pi - 3$. Let us consider the sum $\displaystyle f(z) = \sum_{n=1}^{\infty} \frac{z^n}{n+2}$ for $z \in$, with $|z| < 1$. Then

$$\begin{aligned}f(z) &= \frac{1}{z^2} \sum_{n=1}^{\infty} \frac{z^{n+2}}{n+2} = \frac{1}{z^2} \sum_{n=1}^{\infty} \int_0^z w^{n+1} dw\\ &= -\frac{\frac{z^2}{2} + z + \ln(1-z)}{z^2}.\end{aligned}$$

Since $\ln(c + di) = \ln\sqrt{c^2 + d^2} + i\arctan(d/c)$, then $\ln\left(\frac{1+i}{2}\right) = \ln\left(\frac{1}{\sqrt{2}}\right) + \frac{\pi}{4}i$, and $\ln\left(\frac{1-i}{2}\right) = \ln\left(\frac{1}{\sqrt{2}}\right) - \frac{\pi}{4}i$. Therefore

$$\sum_{n=1}^{\infty} \frac{x_n}{n+2} = f\left(\frac{1+i}{2}\right) + f\left(\frac{1-i}{2}\right) = \pi - 3.$$

Also solved by Albert Stadler, Herrliberg, Switzerland; Michel Bataille, Rouen, France; Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy.

99. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$mn \cdot (f(m) - nm) \cdot (n - f(n^2))$$

is a square for all $m, n \in \mathbb{N}$.

(Proposed by Valmir Krasniqi, Departament of Mathematics, University of Prishtina, Republic of Kosova.)

Solution by PCO - AOPS. Let $g(m,n) = mn(f(m) - mn)(n - f(n^2))$, then $g(1,1) = -(f(1)-1)^2$ can only be a perfect square when $f(1) = 1$. If $f(n^2) = n$ $\forall n \in \mathbb{N}$, we get a first trivial solution : $f(n^2) = n, \forall n \in \mathbb{N}$ and $f(m)$ is any value when m is not a perfect square.
Suppose now that $\exists a$ such that $f(a^2) \neq a$ $(a-1)^2 g(m,a) = (1-a)m(f(m) - ma)g(1,a)$ and, since both $g(m,a)$ and $g(1,a) \neq 0$ are perfect squares, we get $4(a-1)m(am - f(m))$ is a perfect square. This may be written $(2ma - m - f(m))^2 - (f(m) - m)^2$ is a perfect square $\forall m$. If infinitely many such a exist, this implies $f(n) = n \forall n$ which indeed is a solution. Suppose now that $\exists$ finitely many a such that $f(a^2) \neq a$. So $f(n^2) = n$ $\forall n > M$ for a given M. Let then such $n > M$, then $(a-1)^2 g(n^2, a) = n^3(a-1)(an-1)g(1,a)$. And since both $g(n^2, a)$ and $n^2 g(1,a) \neq 0$ are perfect squares, we get $n(a-1)(an-1)$ perfect square $\forall n > M$. Which is clearly impossible, since $a \neq 1$. So no more solutions.

Also solved by the proposer.

MATHNOTES SECTION

The evaluation of a special fractional part integral with an integrand raised to positive integer powers

Cornel Ioan Vălean

Abstract. The present paper is about calculating in closed-form the following class of special fractional part integrals

$$\int_0^1\int_0^1\int_0^1 \left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^n \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

where $\{x\}$ is the fractional part of x and n is a positive integer. We show the value of the integral reduces to a sum involving specific values of the Riemann zeta function.

Keywords: Fractional part integrals, Leibniz integral rule, Riemann zeta function.

1. Introduction and the main result

Let n be positive integers and I_n denotes the integral with fractional part of power n

$$\mathcal{I}_n = \int_0^1\int_0^1\int_0^1 \left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^n \mathrm{d}x\mathrm{d}y\mathrm{d}z,$$

where $\{x\}$ denotes the fractional part of x.

The aim of the present paper is to calculate in closed-form the proposed triple fractional part integral with integer powers. We will show that its closed-form can be expressed in terms of specific values of the Riemann zeta function.

We state below the theorem we are going to prove.

Let $n \geq 1$ be a positive integer. Then the following equality holds*:*

$$\mathcal{I}_n = \int_0^1\int_0^1\int_0^1 \left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^n \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

$$= 1 - \frac{3}{2(n+1)}\sum_{i=1}^{n}\zeta(i+1) + \frac{1}{(n+1)^2(n+2)}\left(\sum_{i=1}^{n}\zeta(i+1)\right)\left(\sum_{i=1}^{n}(i+1)\zeta(i+2)\right),$$

where $\zeta(k), k \geq 2$ denotes the Riemann zeta function at positive integer values, and it is defined by

$$\zeta(k) = \sum_{n=1}^{\infty}\frac{1}{n^k} = 1 + \frac{1}{2^k} + \frac{1}{3^k} + \cdots + \frac{1}{n^k} + \cdots.$$

Before we prove Theorem 1 we collect some results we need in our analysis.

Next we prove the following lemma which is used in the proof of Theorem 1.

Lemma 1. ***Two special fractional part integrals***

Let $n \geq 1$ be an integer. The following equalities hold:

$$\text{(a) } I_{n,n} = \int_0^1 x^n \left\{\frac{1}{x}\right\}^n dx = 1 - \frac{1}{n+1}\sum_{i=1}^{n}\zeta(i+1);$$

$$\text{(b) } I_{n+1,n} = \int_0^1 x^{n+1} \left\{\frac{1}{x}\right\}^n dx = \frac{1}{2} - \frac{1}{(n+2)(n+1)}\sum_{i=1}^{n}(i+1)\zeta(i+2).$$

Proof. (a) We start with the change of variable $x = 1/y$, and then we have that

$$\begin{aligned}\int_0^1 x^n \left\{\frac{1}{x}\right\}^n \mathrm{d}x &= \int_1^\infty y^{-n-2}\{y\}^n \mathrm{d}y \\ &= \sum_{k=1}^{\infty}\int_k^{k+1} y^{-n-2}\{y\}^n \mathrm{d}y \\ &= \sum_{k=1}^{\infty}\int_k^{k+1} y^{-n-2}(y-\lfloor y\rfloor)^n \mathrm{d}y \\ &= \sum_{k=1}^{\infty}\underbrace{\int_k^{k+1} y^{-n-2}(y-k)^n\ dy}_{J_{n,n,k}}\end{aligned} \tag{27}$$

where we denote the last integral by $J_{n,n,k}$, where $J_{a,b,c} = \int_c^{c+1} x^{-a-2}(x-k)^b \mathrm{d}x$. Integrating by parts, we obtain that

$$\begin{aligned}J_{n,n,k} = -\int_k^{k+1}\left(\frac{y^{-n-1}}{n+1}\right)'(y-k)^n\mathrm{d}y &= -\frac{y^{-n-1}}{n+1}(y-k)^n\Big|_{y=k}^{y=k+1} + \frac{n}{n+1}\int_k^{k+1} y^{-n-1}(y-k)^{n-1}\mathrm{d}y \\ &= -\frac{1}{(n+1)(k+1)^{n+1}} + \frac{n}{n+1}J_{n-1,n-1,k}\end{aligned}$$

whence

$$J_{n,n,k} = -\frac{1}{(n+1)(k+1)^{n+1}} + \frac{n}{n+1}J_{n-1,n-1,k}$$

or rearranging, we get that

$$(n+1)J_{n,n,k} - nJ_{n-1,n-1,k} = -\frac{1}{(k+1)^{n+1}}.$$

If replacing n by i in the relation above

$$(i+1)J_{i,i,k} - iJ_{i-1,i-1,k} = -\frac{1}{(k+1)^{i+1}},$$

and then give values from $i = 1$ to n, we get by the telescoping process that

$$(n+1)J_{n,n,k} = \frac{1}{k(k+1)} - \sum_{i=1}^{n}\frac{1}{(k+1)^{i+1}}$$

or

$$J_{n,n,k} = \frac{1}{k(k+1)(n+1)} - \frac{1}{n+1}\sum_{i=1}^{n}\frac{1}{(k+1)^{i+1}}. \tag{28}$$

Now, we use (28) in (27), and then we get that

$$
\begin{aligned}
I_{n,n} = \int_0^1 x^n \left\{\frac{1}{x}\right\}^n \mathrm{d}x &= \sum_{k=1}^{\infty} \int_k^{k+1} y^{-n-2}(y-k)^n \mathrm{d}y \\
&= \sum_{k=1}^{\infty} J_{n,n,k} \\
&= \sum_{k=1}^{\infty} \left(\frac{1}{k(k+1)(n+1)} - \frac{1}{n+1} \sum_{i=1}^{n} \frac{1}{(k+1)^{i+1}} \right) \\
&= \frac{1}{n+1} \sum_{k=1}^{\infty} \frac{1}{k(k+1)} - \frac{1}{n+1} \sum_{k=1}^{\infty} \sum_{i=1}^{n} \frac{1}{(k+1)^{i+1}} \qquad (29) \\
&= \frac{1}{n+1} - \frac{1}{n+1} \sum_{i=1}^{n} \sum_{k=1}^{\infty} \frac{1}{(k+1)^{i+1}} \\
&= \frac{1}{n+1} - \frac{1}{n+1} \sum_{i=1}^{n} \zeta(i+1) + \frac{1}{n+1} \sum_{i=1}^{n} 1 \\
&= 1 - \frac{1}{n+1} \sum_{i=1}^{n} \zeta(i+1).
\end{aligned}
$$

and the part (a) of the lemma is proved.
(b) Following the same steps as in the part (a) of the lemma, we get that

$$
\begin{aligned}
I_{n+1,n} = \int_0^1 x^{n+1} \left\{\frac{1}{x}\right\}^n \mathrm{d}x &= \int_1^{\infty} y^{-n-3} \{y\}^n \mathrm{d}y \\
&= \sum_{k=1}^{\infty} \int_k^{k+1} y^{-n-3} \{y\}^n \mathrm{d}y \\
&= \sum_{k=1}^{\infty} \int_k^{k+1} y^{-n-3} (y - \lfloor y \rfloor)^n \mathrm{d}y \qquad (30) \\
&= \sum_{k=1}^{\infty} \underbrace{\int_k^{k+1} y^{-n-3}(y-k)^n \, dy}_{J_{n+1,n,k}}
\end{aligned}
$$

where we denote the last integral by $J_{n+1,n,k}$, where $J_{a,b,c} = \int_c^{c+1} x^{-a-2}(x-k)^b \mathrm{d}x$.
Integrating by parts, we obtain that

$$
\begin{aligned}
J_{n+1,n,k} = -\int_k^{k+1} \left(\frac{y^{-n-2}}{n+2}\right)' (y-k)^n \mathrm{d}y &= -\frac{y^{-n-2}}{n+2}(y-k)^n \Big|_{y=k}^{y=k+1} + \frac{n}{n+2} \int_k^{k+1} y^{-n-2}(y-k)^{n-1} \mathrm{d}y \\
&= -\frac{1}{(n+2)(k+1)^{n+2}} + \frac{n}{n+2} J_{n,n-1,k}
\end{aligned}
$$

whence

$$
J_{n+1,n,k} = -\frac{1}{(n+2)(k+1)^{n+2}} + \frac{n}{n+2} J_{n,n-1,k}
$$

or rearranging, we get that

$$
J_{n+1,n,k} - \frac{n}{n+2} J_{n,n-1,k} = -\frac{1}{(n+2)(k+1)^{n+2}}.
$$

Multiplying both sides by $(n+1)(n+2)$, we get that

$$(n+2)(n+1)J_{n+1,n,k}-(n+1)nJ_{n,n-1,k}=-\frac{n+1}{(k+1)^{n+2}}.$$

If replacing n by i in the relation above

$$(i+2)(i+1)J_{i+1,i,k}-(i+1)iJ_{i,i-1,k}=-\frac{i+1}{(k+1)^{i+2}}$$

and then give values from $i=1$ to n, we get by the telescoping process that

$$(n+2)(n+1)J_{n+1,n,k}=\frac{1}{k^2}-\frac{1}{(k+1)^2}-\sum_{i=1}^{n}\frac{i+1}{(k+1)^{i+2}}$$

or

$$J_{n+1,n,k}=\frac{1}{(n+2)(n+1)}\left(\frac{1}{k^2}-\frac{1}{(k+1)^2}\right)-\frac{1}{(n+2)(n+1)}\sum_{i=1}^{n}\frac{i+1}{(k+1)^{i+2}}. \tag{31}$$

Now, we use (31) in (30), and we get

$$\begin{aligned}
I_{n+1,n}&=\int_0^1 x^{n+1}\left\{\frac{1}{x}\right\}^n \mathrm{d}x\\
&=\sum_{k=1}^{\infty}\int_k^{k+1} y^{-n-3}(y-k)^n\mathrm{d}y\\
&=\sum_{k=1}^{\infty}J_{n+1,n,k}\\
&=\sum_{k=1}^{\infty}\left(\frac{1}{(n+2)(n+1)}\left(\frac{1}{k^2}-\frac{1}{(k+1)^2}\right)-\frac{1}{(n+2)(n+1)}\sum_{i=1}^{n}\frac{i+1}{(k+1)^{i+2}}\right)\\
&=\frac{1}{(n+2)(n+1)}\sum_{k=1}^{\infty}\left(\frac{1}{k^2}-\frac{1}{(k+1)^2}\right)-\frac{1}{(n+2)(n+1)}\sum_{k=1}^{\infty}\sum_{i=1}^{n}\frac{i+1}{(k+1)^{i+2}}\\
&=\frac{1}{(n+2)(n+1)}-\frac{1}{(n+2)(n+1)}\sum_{i=1}^{n}\sum_{k=1}^{\infty}\frac{i+1}{(k+1)^{i+2}}\\
&=\frac{1}{(n+2)(n+1)}-\frac{1}{(n+2)(n+1)}\sum_{i=1}^{n}(i+1)(\zeta(i+2)-1)\\
&=\frac{1}{(n+2)(n+1)}-\frac{1}{(n+2)(n+1)}\sum_{i=1}^{n}(i+1)\zeta(i+2)+\frac{1}{(n+2)(n+1)}\sum_{i=1}^{n}(i+1)\\
&=\frac{1}{2}-\frac{1}{(n+2)(n+1)}\sum_{i=1}^{n}(i+1)\zeta(i+2).
\end{aligned}$$

and the second part of the lemma is proved. □

Now we are ready to prove Theorem 1.

Proof. By letting the variable change $x/y=u$, we get that

$$\mathcal{I}_n=\int_0^1\left(\int_0^1\left(\int_0^1\left(\left\{\frac{x}{y}\right\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{x}\right\}\right)^n\mathrm{d}x\right)\mathrm{d}y\right)\mathrm{d}z=\int_0^1\left(\int_0^1\left(y\int_0^{1/y}\left(\{u\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{uy}\right\}\right)^n\mathrm{d}u\right)\mathrm{d}y\right)\mathrm{d}z,$$

and after changing the integration order

$$\mathcal{I}_n = \int_0^1 \left(y \int_0^{1/y} \left(\int_0^1 \left(\{u\}\left\{\frac{y}{z}\right\}\left\{\frac{z}{uy}\right\}\right)^n \mathrm{d}z \right) \mathrm{d}u \right) \mathrm{d}y,$$

we make the change of variable $z/y = t$, and we get that

$$\mathcal{I}_n = \int_0^1 \left(y^2 \int_0^{1/y} \left(\int_0^{1/y} \left(\{u\}\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^n \mathrm{d}t \right) \mathrm{d}u \right) \mathrm{d}y. \tag{32}$$

Now, recall the Leibniz integral rule (see [1])

$$\frac{\partial}{\partial z}\int_{a(z)}^{b(z)} f(x,z)\mathrm{d}x = \int_{a(z)}^{b(z)} \frac{\partial f}{\partial z}\mathrm{d}x + f(b(z),z)\frac{\partial b}{\partial z} - f(a(z),z)\frac{\partial a}{\partial z},$$

where applying the mentioned rule for a function of the form $g(z,x) = \int_0^{b(x)} f(y,z)\, dy$, we obtain that

$$\frac{\partial}{\partial x}\left(\int_0^{a(x)} g(z,x)\mathrm{d}z\right) = \frac{\partial}{\partial x}\left(\int_0^{a(x)}\left(\int_0^{b(x)} f(y,z)\mathrm{d}y\right)\mathrm{d}z\right)$$

$$= \frac{\partial}{\partial x}(b(x))\int_0^{a(x)} f(b(x),z)\mathrm{d}z + \frac{\partial}{\partial x}(a(x))\int_0^{b(x)} f(y,a(x))\mathrm{d}y.$$

Using this result in (32) where we apply the integration by parts, we have that

$$\mathcal{I}_n = \int_0^1 \left(\left(\frac{y^3}{3}\right)' \int_0^{1/y} \left(\int_0^{1/y} \left(\{u\}\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^n \mathrm{d}t \right) \mathrm{d}u \right) \mathrm{d}y$$

$$= \frac{y^3}{3}\int_0^{1/y}\left(\int_0^{1/y}\left(\{u\}\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^n \mathrm{d}t\right)\mathrm{d}u\Big|_{y=0}^{y=1} - \frac{1}{3}\int_0^1 y^3\frac{\partial}{\partial y}\left(\int_0^{1/y}\left(\int_0^{1/y}\left(\{u\}\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^n \mathrm{d}t\right)\mathrm{d}u\right)\mathrm{d}y$$

$$= \frac{1}{3}\int_0^1\left(\int_0^1\left(\{u\}\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^n \mathrm{d}t\right)\mathrm{d}u$$

$$- \lim_{y\to 0}\frac{y^3}{3}\int_0^{1/y}\left(\int_0^{1/y}\left(\{u\}\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^n \mathrm{d}t\right)\mathrm{d}u$$

$$+\frac{1}{3}\int_0^1\left(\int_0^{1/y} y\left(\{u\}\{y\}\left\{\frac{1}{uy}\right\}\right)^n \mathrm{d}u\right)\mathrm{d}y + \frac{1}{3}\int_0^1\left(\int_0^{1/y} y\left(\{ty\}\left\{\frac{1}{y}\right\}\left\{\frac{1}{t}\right\}\right)^n \mathrm{d}t\right)\mathrm{d}y$$

$$= \frac{1}{3}\int_0^1\left(\int_0^1 u^n\left(\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^n \mathrm{d}t\right)\mathrm{d}u$$

$$+\frac{1}{3}\int_0^1\left(\int_0^{1/y} y\left(\{u\}\{y\}\left\{\frac{1}{uy}\right\}\right)^n \mathrm{d}u\right)\mathrm{d}y + \frac{1}{3}\int_0^1\left(\int_0^{1/y} y\left(\{ty\}\left\{\frac{1}{y}\right\}\left\{\frac{1}{t}\right\}\right)^n \mathrm{d}t\right)\mathrm{d}y, (*)$$

where we used that the limit tends to 0 since $0 \le \int_0^{1/y}\left(\int_0^{1/y}\left(\{u\}\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^n \mathrm{d}t\right)\mathrm{d}u \le \frac{1}{y^2}$, and then

$$0 \le \frac{y^3}{3}\int_0^{1/y}\left(\int_0^{1/y}\left(\{u\}\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^n \mathrm{d}t\right)\mathrm{d}u \le \frac{y}{3}.$$

Now we make the changes of variable $uy = w$ in the second integral and $ty = z$ in the third integral, and then we get, for the second integral, that

$$\int_0^1 \left(\int_0^{1/y} y\left(\{u\}\{y\}\left\{\frac{1}{uy}\right\}\right)^n \mathrm{d}u\right)\mathrm{d}y = \int_0^1 \left(\int_0^1 \left(\{y\}\left\{\frac{w}{y}\right\}\left\{\frac{1}{w}\right\}\right)^n \mathrm{d}w\right)\mathrm{d}y$$
$$= \int_0^1 \left(\int_0^1 y^n\left(\left\{\frac{w}{y}\right\}\left\{\frac{1}{w}\right\}\right)^n \mathrm{d}w\right)\mathrm{d}y, \tag{33}$$

and for the third integral we obtain that

$$\int_0^1 \left(\int_0^{1/y} y\left(\{ty\}\left\{\frac{1}{y}\right\}\left\{\frac{1}{t}\right\}\right)^n \mathrm{d}t\right)\mathrm{d}y = \int_0^1 \left(\int_0^1 \left(\{z\}\left\{\frac{1}{y}\right\}\left\{\frac{y}{z}\right\}\right)^n \mathrm{d}z\right)\mathrm{d}y$$
$$= \int_0^1 \left(\int_0^1 z^n\left(\left\{\frac{1}{y}\right\}\left\{\frac{y}{z}\right\}\right)^n \mathrm{d}z\right)\mathrm{d}y. \tag{34}$$

Using (33) and (34) in (*), we get

$$\mathcal{I}_n = = \frac{1}{3}\int_0^1 \left(\int_0^1 u^n\left(\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^n \mathrm{d}t\right)\mathrm{d}u + \frac{1}{3}\int_0^1 \left(\int_0^1 y^n\left(\left\{\frac{w}{y}\right\}\left\{\frac{1}{w}\right\}\right)^n \mathrm{d}w\right)\mathrm{d}y$$
$$+ \frac{1}{3}\int_0^1 \left(\int_0^1 z^n\left(\left\{\frac{1}{y}\right\}\left\{\frac{y}{z}\right\}\right)^n \mathrm{d}z\right)\mathrm{d}y$$
$$= \int_0^1 \left(\int_0^1 u^n\left(\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^n \mathrm{d}u\right)\mathrm{d}t,$$

where above we used the fact that all three integrals are equal, and to see that it's enough to change the integration order in the first and second integrals. □

Next, we split the last integral, and we have that

$$\mathcal{I}_n = \int_0^1 \left(\int_0^1 u^n\left(\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^n \mathrm{d}u\right)\mathrm{d}t$$
$$= \int_0^1 \left(\int_0^t u^n\left(\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^n \mathrm{d}u\right)\mathrm{d}t + \int_0^1 \left(\int_t^1 u^n\left(\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^n \mathrm{d}u\right)\mathrm{d}t \tag{35}$$

If making the change of variable $t/u = v$ in the first integral from (35), we get that

$$\int_0^1 \left(\int_0^t u^n\left(\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^n \mathrm{d}u\right)\mathrm{d}t = \int_0^1 \left(\int_1^\infty \frac{t^{n+1}}{v^{n+2}}\left(\left\{\frac{1}{t}\right\}\{v\}\right)^n \mathrm{d}v\right)\mathrm{d}t$$
$$\overset{v=1/s}{=} \int_0^1 \left(\int_0^1 t^{n+1}s^n\left(\left\{\frac{1}{t}\right\}\left\{\frac{1}{s}\right\}\right)^n \mathrm{d}s\right)\mathrm{d}t$$
$$= \int_0^1 t^{n+1}\left\{\frac{1}{t}\right\}^n \mathrm{d}t \int_0^1 s^n\left\{\frac{1}{s}\right\}^n \mathrm{d}s$$
$$= \left(1 - \frac{1}{n+1}\sum_{i=1}^n \zeta(i+1)\right)\left(\frac{1}{2} - \frac{1}{(n+2)(n+1)}\sum_{i=1}^n (i+1)\zeta(i+2)\right), \tag{36}$$

where we used both results of the Lemma 2.

For the other integral in (35), we have that

$$\int_0^1 \left(\int_t^1 u^n \left(\left\{\frac{1}{t}\right\}\left\{\frac{t}{u}\right\}\right)^n \mathrm{d}u\right) \mathrm{d}t = \int_0^1 \left(\int_t^1 t^n \left\{\frac{1}{t}\right\}^n \mathrm{d}u\right) \mathrm{d}t$$
$$= \int_0^1 (1-t)t^n \left\{\frac{1}{t}\right\}^n \mathrm{d}t$$
$$= \int_0^1 t^n \left\{\frac{1}{t}\right\}^n \mathrm{d}t - \int_0^1 t^{n+1} \left\{\frac{1}{t}\right\}^n \mathrm{d}t$$
$$= \frac{1}{2} + \frac{1}{(n+2)(n+1)} \sum_{i=1}^{n} (i+1)\zeta(i+2) - \frac{1}{n+1} \sum_{i=1}^{n} \zeta(i+1). \tag{37}$$

Hence, plugging (36) and (37) in (35), we obtain that

$$\mathcal{I}_n = 1 - \frac{3}{2(n+1)} \sum_{i=1}^{n} \zeta(i+1) + \frac{1}{(n+1)^2(n+2)} \left(\sum_{i=1}^{n} \zeta(i+1)\right) \left(\sum_{i=1}^{n} (i+1)\zeta(i+2)\right).$$

Editor's comment: Lemma 1, parts a and b are not new. Part (a) is Problem 2.21 on page 103 in [2] and part (b) appears, in a more general form, as part (a) of problem 2.22 on page 103 in [2].

References

[1] Abramowitz, M. and Stegun, I. A. (Eds.). Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables, 9th printing. New York: Dover, p. 11, 1972.

[2] Ovidiu Furdui, Limits, Series and Fractional Part Integrals. Problems in Mathematical Analysis, Springer, New York, 2013.

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JUNIOR PROBLEMS

Solutions to the problems stated in this issue should arrive before June 19, 2017.

Proposals

61. *Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam.* Given a tetrahedron $A_1A_2A_3A_4$ with the volume V, let I and r be incenter and inradius, respectively. Denote by S_i the area of triangle opposite to vertex $A_i (i = 1; 2; 3; 4)$. Prove that

$$\sum_{n=1}^{4} S_i IA_i^2 = \frac{2rS_1S_2S_3S_4}{9V^2} \sum_{1\le i<j\le 4} A_iA_j \sin\angle(A_i, A_j),$$

where $\angle(A_i, A_j)$ is the dihedral angle at edge A_iA_j.

62. *Proposed by Daniel Sitaru, Mathematics Department, Colegiul National Economic Theodor Costescu, Drobeta Turnu - Severin, Mehedinti, Romania.* Let be $A', A'' \in (BC); B', B'' \in (AC); C', C'' \in (AB)$ in ΔABC such that $AA' \cap BB' \cap CC' \neq \emptyset$ and $AA'' \cap BB'' \cap CC'' \neq \emptyset$. Prove that

$$\frac{27[A'B'C']}{[A''B''C'']} \le \left(\frac{BA'}{BA''} + \frac{CB'}{CB''} + \frac{AC'}{AC''}\right)^3,$$

where $[ABC]$ is area of triangle ABC.

63. *Proposed by Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Romania.* Let $a, b, c \in \mathbb{R}$. Prove that

$$9\sqrt{2}(ab(a-b) + bc(b-c) + ca(c-a)) \le \sqrt{3}\left((a-b)^2 + (b-c)^2 + (c-a)^2\right)^{\frac{3}{2}}.$$

64. *Problem proposed by Arkady Alt, San Jose, California, USA.* Let $\Delta(x, y, z) := 2(xy + yz + xz) - (x^2 + y^2 + z^2)$ and let a, b, c be sidelengths of a triangle with area F. Prove that

$$\Delta\left(a^3, b^3, c^3\right) \le \frac{64F^3}{\sqrt{3}}.$$

65. *Proposed by Dorlir Ahmeti, University of Prishtina, Department of Mathematics, Republic of Kosova.* Find all function $f : \mathbb{N} \to \mathbb{N}$ such that $mf(n) + f(m)$ is divisible by $f(m)(f(n) + 1)$ for all $m, n \in \mathbb{N}$.

Solutions

56. *Proposed by Valmir Krasniqi, University of Prishtina, Republic of Kosova.* Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(m!+n!)|f(m)!+f(n)!$ and $m+n$ divides $f(m)+f(n)$ for all $m, n \in \mathbb{N}$.

Solution by Dorlir Ahmeti, University of Prishtina, Department of Mathematics, Republic of Kosova. Taking $m = n$ in second condition we have $n|f(n)$. Let be p very large prime number. Now, from Wilson theorem, we have $(p-1)!+1 \equiv 0(modp)$. Hence from first relation and last relation we have

$$p|f((p-1)!+1)|f(p-1)!+f(1)!$$

Now, since p is very large, then we have that $gcd(f(1), p) = 1$, so we can't have $f(p-1) \geq p$ which means $f(p-1) \leq p-1$.
Now taking $m = p-1$ in second relation and using last relation we have

$$p-1+n|p-1+f(n) \Rightarrow p-1+n|f(n)-n$$

Since p is very large the only possible last raltion to be true is $f(n) - n = 0$, so $f(n) = n$ for every positive integer n. We prove to two relation and we see this is only solution hence done.

Also solved by the proposer.

57. *Proposed by Angel Plaza, University of Las Palmas de Gran Canaria, Spain.* Let $x_1, x_2, \ldots, x_n > 0$, with the assumption $x_{n+1} = x_1$. Prove that

$$\left(\frac{\sum_{k=1}^n x_k}{n}\right)^2 \leq \frac{1}{n}\sum_{k=1}^n \frac{x_k^2 + x_k x_{k+1} + x_{k+1}^2}{3} \leq \frac{\sum_{k=1}^n x_k^2}{n}$$

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.
We consider $\mathbb{C}^n$ equipped with the usual scalar product $\langle \cdot, \cdot \rangle$ and the corresponding norm $\| \cdot \|$ defined by

$$\langle u, v \rangle = \sum_{k=1}^n \overline{u_k}\, v_k, \qquad \|u\|^2 = \sum_{k=1}^n |u_k|^2 .$$

Now, consider the real vectors

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix}, \qquad Y = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ x_1 \end{bmatrix}, \qquad 1 = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix}.$$

and let $Z = X - jY$ with $j = e^{2i\pi/3} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$. Clearly we have $\langle 1, Z \rangle = \left(\sum_{k=1}^n x_k\right)(1-j)$, hence

$$|\langle 1, Z \rangle|^2 = 3\left(\sum_{k=1}^n x_k\right)^2 .$$

Moreover, since $|x_k - jx_{k+1}|^2 = x_k^2 + x_k x_{k+1} + x_{k+1}^2$ we conclude that

$$\|Z\|^2 = \sum_{k=1}^{n}(x_k^2 + x_k x_{k+1} + x_{k+1}^2)$$

Therefore, the Cauchy-Schwarz inequality $|\langle 1, Z\rangle|^2 \le \|1\|^2 \|Z\|^2$ is equivalent to the lower inequality.
The upper inequality is easier, since $\langle X, Y\rangle \le \|X\| \|Y\| = \|X\|^2$, so

$$\sum_{k=1}^{n}(x_k^2 + x_k x_{k+1} + x_{k+1}^2) = \|X\|^2 + \langle X, Y\rangle + \|Y\|^2 \le 3\|X\|^2$$

and the is equivalent to the upper inequality.
Remark. We only need the fact that the x_k's are real. The positivity assumption is unnecessary.

Solution 2 by Arkady Alt, San Jose, California, USA. For cyclic sum $\sum\limits_{k=1\ cyclic}^{n} \dfrac{x_k^2 + x_k x_{k+1} + x_{k+1}^2}{3}$ we will use more compact notation $\sum\limits_{cyc}^{n} \dfrac{x_1^2 + x_1 x_2 + x_2^2}{3}$.
Noting that $\left(\dfrac{x+y}{2}\right)^2 \le \dfrac{x^2+xy+y^2}{3} \iff 0 \le (x-y)^2$ (for any real x, y) we obtain $\dfrac{1}{n}\sum\limits_{cyc}^{n} \dfrac{x_1^2 + x_1 x_2 + x_2^2}{3} \ge \dfrac{1}{n}\sum\limits_{cyc}^{n}\left(\dfrac{x_1+x_2}{2}\right)^2$. By Quadratic Mean-Arithmetic Mean Inequality $\dfrac{1}{n}\sum\limits_{cyc}^{n}\left(\dfrac{x_1+x_2}{2}\right)^2 \ge \left(\dfrac{1}{n}\sum\limits_{cyc}^{n}\left(\dfrac{x_1+x_2}{2}\right)\right)^2 = \left(\dfrac{\sum_{k=1}^{n} x_k}{n}\right)^2$.
(Or, applying Cauchy Inequality to $\left(\dfrac{x_1+x_2}{2}, \dfrac{x_2+x_3}{2}, ..., \dfrac{x_n+x_1}{2}\right)$ and $(1, 1, ..., 1$ we obtain $\sum\limits_{cyc}^{n}\left(\dfrac{x_1+x_2}{2}\right)^2 n \ge \left(\sum\limits_{cyc}^{n}\left(\dfrac{x_1+x_2}{2}\right)\cdot 1\right)^2 \iff \dfrac{1}{n}\sum\limits_{cyc}^{n}\left(\dfrac{x_1+x_2}{2}\right)^2 \ge \left(\dfrac{1}{n}\sum\limits_{cyc}^{n}\dfrac{x_1+x_2}{2}\right)^2 = \left(\dfrac{\sum_{k=1}^{n} x_k}{n}\right)^2$. Since $\dfrac{x^2+xy+y^2}{3} \le \dfrac{x^2+y^2}{2} \iff 0 \le (x-y)^2$ (for any real x, y) we obtain $\dfrac{1}{n}\sum\limits_{cyc}^{n}\dfrac{x_1^2 + x_1 x_2 + x_2^2}{3} \le \dfrac{1}{n}\sum\limits_{cyc}^{n}\dfrac{x_1^2+x_2^2}{2} = \dfrac{\sum_{k=1}^{n} x_k^2}{n}$.

Solution 3 by Michel Bataille, Rouen, France. Since $x_k x_{k+1} \le \frac{x_k^2 + x_{k+1}^2}{2}$, we have $x_k^2 + x_k x_{k+1} + x_{k+1}^2 \le \frac{3}{2}(x_k^2 + x_{k+1}^2)$. It follows that

$$\frac{1}{n}\sum_{k=1\text{ cyclic}}^{n}\frac{x_k^2 + x_k x_{k+1} + x_{k+1}^2}{3} \le \frac{1}{n}\sum_{k=1\text{ cyclic}}^{n}\frac{1}{2}(x_k^2 + x_{k+1}^2) = \frac{1}{n}\sum_{k=1}^{n} x_k^2,$$

hence the right inequality holds.
Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\left(\sum_{k=1}^{n} x_k\right)^2 &= \left(\sum_{k=1 \text{ cyclic}}^{n} 1 \cdot \frac{x_k + x_{k+1}}{2}\right)^2 \\ &\le (1^2 + 1^2 + \cdots + 1^2)\left(\sum_{k=1 \text{ cyclic}}^{n} \frac{(x_k + x_{k+1})^2}{4}\right) \\ &= n\left(\sum_{k=1 \text{ cyclic}}^{n} \frac{(x_k + x_{k+1})^2}{4}\right).\end{aligned}$$

Now, the inequality $\frac{(x_k+x_{k+1})^2}{4} \le \frac{x_k^2+x_kx_{k+1}+x_{k+1}^2}{3}$ holds as being equivalent to $(x_k - x_{k+1})^2 \ge 0$. Thus,

$$\left(\sum_{k=1}^{n} x_k\right)^2 \le n\left(\sum_{k=1 \text{ cyclic}}^{n} \frac{x_k^2 + x_kx_{k+1} + x_{k+1}^2}{3}\right)$$

and this gives the left inequality at once.

Also solved by Adnan Ali (student), Mumbai, India and the proposer.

58. Corrected. *Proposed by Arkady Alt, San Jose, California, USA.* Let P be arbitrary interior point in a triangle ABC and r be inradius. Prove that

$$\frac{a^2}{d_a(P)} + \frac{b^2}{d_b(P)} + \frac{c^2}{d_c(P)} \ge 36r$$

if $d_a(P)$, $d_b(P)$ and $d_c(P)$ are the distances from the point P to the sides BC, CA and AB respectively.

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. (The original statement has r^2 instead of r, and it is clearly not correct, because it is not homogeneous.)
Let $s = (a+b+c)/2$. For $q \ge 1$ we have

$$\frac{2s}{3} = \frac{a+b+c}{3} \le \left(\frac{a^{\frac{q+1}{2}} + b^{\frac{q+1}{2}} + c^{\frac{q+1}{2}}}{3}\right)^{\frac{2}{q+1}}.$$

Equivalently

$$3^{1-q}2^{q+1}s^{q+1} \le \left(a^{\frac{q+1}{2}} + b^{\frac{q+1}{2}} + c^{\frac{q+1}{2}}\right)^2.$$

Now, let F_a, F_b, F_c and F represent the areas of the triangles PBC, PCA, PAB and ABC respectively. Clearly, we have $2F_a = ad_a(P)$, $2F_b = bd_b(P)$ and $2F_c = cd_c(P)$. Consequently, using Cauchy-Schwarz inequality, we have

$$\begin{aligned}3^{1-q}2^{q+1}s^{q+1} &\le \left(a^{\frac{q+1}{2}} + b^{\frac{q+1}{2}} + c^{\frac{q+1}{2}}\right)^2 \\ &\le \left(\frac{a^{q+1}}{2F_a} + \frac{b^{q+1}}{2F_b} + \frac{c^{q+1}}{2F_c}\right)(2(F_a + F_b + F_c)) \\ &\le 2F \cdot \left(\frac{a^q}{d_a(P)} + \frac{b^q}{d_b(P)} + \frac{c^q}{d_c(P)}\right)\end{aligned}$$

Thus, since $F = sr$ we obtain

$$3^{1-q}2^q s^q \le r \cdot \left(\frac{a^q}{d_a(P)} + \frac{b^q}{d_b(P)} + \frac{c^q}{d_c(P)} \right)$$

Finally, using the well-known inequality: $s \ge 3\sqrt{3}r$ we obtain

$$3^{1+q/2}2^q r^{q-1} \le \frac{a^q}{d_a(P)} + \frac{b^q}{d_b(P)} + \frac{c^q}{d_c(P)}.$$

For $q = 1$ we get

$$6\sqrt{3} \le \frac{a}{d_a(P)} + \frac{b}{d_b(P)} + \frac{c}{d_c(P)}$$

and for $q = 2$ we get

$$36r \le \frac{a^2}{d_a(P)} + \frac{b^2}{d_b(P)} + \frac{c^2}{d_c(P)}.$$

which is the announced inequality.

Solution 2 by Michel Bataille, Rouen, France. Let S_a, S_b, S_c denote the areas of $\Delta BPC, \Delta CPA, \Delta APB$, respectively, and let $S = S_a + S_b + S_c$ be the area of ΔABC. Since for $x = a, b, c$ we have $2S_x = x \cdot d_x(P)$, the required inequality (1) rewrites as

$$\frac{a^3}{S_a} + \frac{b^3}{S_b} + \frac{c^3}{S_c} \ge 72r \qquad (2).$$

To prove (2), we apply Holder's inequality as follows

$$\left(\frac{a^3}{S_a} + \frac{b^3}{S_b} + \frac{c^3}{S_c} \right)(S_a+S_b+S_c)(1^3+1^3+1^3) \ge \left(\frac{a}{S_a^{1/3}} \cdot S_a^{1/3} \cdot 1 + \frac{b}{S_b^{1/3}} \cdot S_b^{1/3} \cdot 1 + \frac{c}{S_c^{1/3}} \cdot S_c^{1/3} \cdot 1 \right)^3$$

that is,

$$\left(\frac{a^3}{S_a} + \frac{b^3}{S_b} + \frac{c^3}{S_c} \right) \cdot 3S \ge (a+b+c)^3.$$

With $s = \frac{a+b+c}{2}$, the latter yields

$$\frac{a^3}{S_a} + \frac{b^3}{S_b} + \frac{c^3}{S_c} \ge \frac{8s^2}{3r} \qquad (3).$$

But from Heron's formula ($S = rs = \sqrt{s(s-a)(s-b)(s-c)}$) and AM-GM, we obtain

$$r^2 s = (s-a)(s-b)(s-c) \le \left(\frac{s-a+s-b+s-c}{3} \right)^3 = \frac{s^3}{27}$$

so that $s^2 \ge 27r^2$. Back to (3), we conclude

$$\frac{a^3}{S_a} + \frac{b^3}{S_b} + \frac{c^3}{S_c} \ge \frac{8 \cdot 27r^2}{3r} = 72r.$$

Also solved by Adnan Ali (student), Mumbai, India and the proposer.

59. *Proposed by Marcel Chiriţă, Bucharest, Romania.* Solve in real numbers the system

$$\left.\begin{array}{l} 2^x + 2^y = 12 \\ 3^x + 4^z = 11 \\ 3^y - 4^z = 25 \end{array}\right\}.$$

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and

Technology, Damascus, Syria. Let $\alpha = \ln 2/\ln 3 \in (0,1)$ so that the second equation is equivalent to $2^x = (11-4^z)^\alpha$ and the third one is equivalent to $2^y = (25+4^z)^\alpha$ the first equation is then equivalent to $(11-4^z)^\alpha + (25+4^z)^\alpha = 12$. Now, consider $f(t) = (11-t)^\alpha + (25+t)^\alpha$ for $t \in [0,11)$. Clearly

$$f'(t) = \alpha\left((25+t)^{\alpha-1} - (11-t)^{\alpha-1}\right) < 0$$

because $\alpha - 1 < 0$ and $11 - t \le 11 < 25 + t$ for $t \in [0,11)$. It follows that f is strictly decreasing on $[0,11)$, and since $f(2) = 9^\alpha + 27^\alpha = 2^2 + 2^3 = 12$ we conclude that 2 is the only solution of the equation $f(t) = 12$ that belongs to $[0,11)$. It follows that the equation $f(4^z) = 12$ has $z = 1/2$ as unique solution. But then $2^x = (11-4^z)^\alpha = 2^2$ and $2^y = (25+4^z)^\alpha = 2^3$. Therefore, the proposed system has a unique real solution which is $(x,y,z) = (2,3,\frac{1}{2})$.

Solution 2 by Adnan Ali (student), Mumbai, India.
Eliminating 4^z, we have the system

$$2^x + 2^y = 12$$
$$3^x + 3^y = 36$$

We prove that the only solutions for the above system are $\{x,y\} = \{2,3\}$. The proof is based on the following claim: Fix positive constants a, b and $k > 1$, then the equation in t:

$$t^k + (a-t)^k = b,\ 0 \le t \le a$$

may have at most two solutions. Indeed, we let $f(t) = t^k + (a-t)^k - b$ and observe that since $k - 1 > 0$, the derivative $f'(t) = k(t^{k-1} - (a-t)^{k-1})$ is negative for $t < \frac{a}{2}$, vanishes at $t = \frac{a}{2}$, and is positive for $t > \frac{a}{2}$. Hence $f(t)$ is strictly decreasing from 0 to $\frac{a}{2}$ and strictly increasing from $\frac{a}{2}$ to a, and our claim follows.
Now we let $t = 2^x$, $r = 2^y$ and $k = \frac{\ln 3}{\ln 2} > 1$ so that the system is now

$$t + r = 12$$
$$t^k + r^k = 36$$

Since $t, r > 0$, we have $0 < t < 12$ and by observation $\{t,r\} = \{4,8\}$ are solutions. Since we get two solutions, by our claim, we cannot have any more of them. Thus transforming back to x, y, the only solutions of the system are $\{x,y\} = \{2,3\}$.
Now back to the original system (proposed one), we observe that $3^y > 25 \Rightarrow y = 3 \Rightarrow x = 2$ and $z = \frac{1}{2}$. Clearly they satisfy the system and so they are the only solution, i.e. $(x,y,z) = (2,3,1/2)$.

Also solved by Albert Stadler, Herrliberg, Switzerland and the proposer.

60. *Proposed by Dorlir Ahmeti, University of Prishtina, Department of Mathematics, Republic of Kosova.* Let ABC be an acute triangle. Let D be the foot of the altitude from A. Let E, F be the midpoints of AC, AB, respectively. Let $G \ne B$ and $H \ne C$ be the intersection of circumcircle of the triangle ABC with circumcircles of the triangles BFD and CED, respectively. Suppose that A, G, B, H, C are order in this way on the circle they belong. Show that line EF, HB and CG are concurrent.

Solution 1 by Michel Bataille, Rouen, France. Let Γ be the circumcircle of ΔABC and O its centre. Let γ_b and γ_c be the circumcircles of ΔBFD and ΔCED, respectively.We first note that $AB \ne AC$: otherwise, OD would be perpendicular to BC and γ_b would be the circle with diameter OB (since $OF \perp AB$, OF being the perpendicular bisector of AB). As such, γ_b would be tangent to Γ at B, contradicting $G \neq B$. Since $AB \neq AC$, O is not on AD and so the tangent t to Γ at A is not parallel to EF. Let K be the point of intersection of t and EF, γ_a be the circle with diameter OA (which passes through E and F) and $\mathbf{I}$ the inversion in the circle with centre K and radius KA (so that $\mathbf{I}(A) = A$). Since γ_a is tangent to Γ at A, KA^2 is the power of K with respect to Γ and to γ_a and so $\mathbf{I}(\Gamma) = \Gamma$, $\mathbf{I}(\gamma_a) = \gamma_a$ and $\mathbf{I}(E) = F$ (since $E, F \in \gamma_a$ and $KE \cdot KF = KA^2$). Since $AD \perp DB$, the midpoint F of AB satisfies $FA = FB = FD$ so that the centre I of γ_b is on the perpendicular to BC through F. It follows that $IF \perp FE$ and so γ_b is tangent to EF at F. Similarly, γ_c is tangent to EF at E. Also, the line EF being the perpendicular bisector of AD, we have $KD = KA$, hence $\mathbf{I}(D) = D$. We deduce that $\mathbf{I}(\gamma_c)$ is a circle which is tangent to EF at F and passes through D, that is, $\mathbf{I}(\gamma_c) = \gamma_b$. Now, $\mathbf{I}(C)$ is on $\mathbf{I}(\Gamma) = \Gamma$ and on $\mathbf{I}(\gamma_c) = \gamma_b$, hence $\mathbf{I}(C) = G$ or B. But the latter cannot occur since CB does not pass through K. Thus $\mathbf{I}(C) = G$ and so CG passes through K. In a similar way, $\mathbf{I}(H)$ is on Γ and γ_b and $\mathbf{I}(H) \neq G$ (since $G = \mathbf{I}(C)$ and $C \neq H$), hence $\mathbf{I}(H) = B$ and HB passes through K. In conclusion, EF, HB and GC are concurrent at K.

Solution 2 by Andrea Fanchini, Cantù, Italy. We use barycentric coordinates and the usual Conway's notations with reference to the triangle ABC.
As we know, for the remarkable points D, E, F we have the followings coordinates

$$D(0 : S_C : S_B), \quad E(1 : 0 : 1), \quad F(1 : 1 : 0)$$

• *Circumcircle of the triangle BFD.*
We impose that the circle passes for the three points B, F, D and we find that the equation of the circumcircle is

$$a^2yz + b^2zx + c^2xy - (x + y + z)\left(\frac{c^2}{2}x + S_C z\right) = 0$$

now the center is the point $J(2S^2 - a^2c^2 : 2S^2 + c^2S_C : c^2S_B)$ and the radical axis between this circle and the circumcircle of the triangle ABC has equation $BG : c^2x + 2S_C z = 0$.
• *Coordinates of point G.* We denote with K the intersection point between the radical axis BG and the line that passes for J and that is perpendicular to BG, we find $K\left(2S_C(b^2 - c^2) : 2S_C(3b^2 - 2c^2) + a^2c^2 : -c^2(b^2 - c^2)\right)$.
Then the point G is simmetric of B respect to K, therefore it is

$$G\left(2S_C(b^2 - c^2) : 2b^2S_C : -c^2(b^2 - c^2)\right)$$

• *Coordinates of point H.*
Following the same procedure and using the cyclicity, we find that the point H has coordinates

$$H\left(2S_B(c^2 - b^2) : -b^2(c^2 - b^2) : 2c^2S_B\right)$$

• *Equations of lines EF, BH and CG.*
We have easy that

$$EF : x - y - z = 0, \qquad BH : c^2x + (b^2 - c^2)z = 0, \qquad CG : b^2x + (c^2 - b^2)y = 0$$

Now the three lines EF, BH and CG are concurrent if and only if

$$\begin{vmatrix} 1 & -1 & -1 \\ c^2 & 0 & b^2 - c^2 \\ b^2 & c^2 - b^2 & 0 \end{vmatrix} = 0$$

that it is true as we can easy check.

Also solved by the proposer.