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PROBLEMS AND SOLUTIONS

Proposals and solutions must be legible and should appear on separate sheets, each indicating the name of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editors have supplied a solution. The editors encourage undergraduate and pre-college students to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the class and school name. Solutions will be evaluated for publication by a committee of professors according to a combination of criteria. Questions concerning proposals and/or solutions can be sent by e-mail to: *mathproblems-ks@hotmail.com*

Solutions to the problems stated in this issue should arrive before November 5, 2016

Problems

138. *Proposed by Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Rumania.* Let a, b, c, x, y and z be real numbers such that $a+b+c+x+y+z = 3$ and $a^2 + b^2 + c^2 + x^2 + y^2 + z^2 = 9$. Prove that $abcxyz \geq -2$.

139. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let $a, b \in \mathbb{R}$, $a < b$, and let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function. Calculate

$$\lim_{n\to\infty} \int_a^b \frac{f(x)}{1 + \sin^2 x \sin^2(x + 1) \cdots \sin^2(x + n)} \mathrm{d}x.$$

140. *Proposed by Cornel Ioan Vălean, Timiş, Rumania.* Find

$$\sum_{n=1}^{\infty} \frac{H_n^3}{(n+1)2^n},$$

where $H_n = 1 + \frac{1}{2} + ... + \frac{1}{n}$ denotes the nth harmonic number.

141. *Proposed by Valmir Krasniqi, University of Prishtina, Republic of Kosova.* Let $p \in \mathbb{N}$, and let $A_p(x) = \frac{(p+1)!p^{x-1}}{x(x+1)\cdots(x+p)}$. Prove that $(-1)^n \frac{d^n}{dx^n} (\ln \phi(x)) > 0$, for all $n = 1, 2, 3 \ldots$ and $x > 0$, where $\phi(x) = \frac{\sqrt[x]{A_p(x+1)}}{x}$.

142. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Let $(a_n)_{n\geq 1}$ such that $a_1 = 1$ and $a_{n+1} = (n+1)!a_n, \forall n \in \mathbb{N}^*$. Let $(b_n)_{n\geq 1}$ such that $b_n > 0, \forall n \in \mathbb{N}^*$ and $\lim\limits_{n\to\infty} \frac{b_n}{n!} = b > 0$. Compute

$$\lim_{n\to\infty} \frac{\sqrt[2n]{b_n}}{\sqrt[n^2]{a_n}}.$$

143. *Proposed by Florin Stanescu, Serban Cioculescu school, city Gaesti, jud. Dambovita, Romania.* We consider $A, B \in M_2(\mathbb{R})$ two matrices, at least one of which is not invertible. If $A^2 + AB + B^2 = 2BA$, prove that $AB = BA = O_2$.

144. *Proposed by Anastasios Kotronis, Athens, Greece.* Show that

$$\int_1^{+\infty} \frac{y\cosh(yx)}{\sinh(y\pi)}\,dy = \frac{1}{(\pi - x)^2} + \frac{\mathrm{Li}_2\left(e^{-2\pi}\right)}{2\pi^2} - \frac{\ln\left(1 - e^{-2\pi}\right)}{\pi} - \frac{1}{2} + \mathcal{O}(\pi - x)\ \ (x \to \pi^-),$$

where $\mathrm{Li}_2(x) = \sum_{k\geq 1} \frac{x^k}{k^2}$ denotes the Dilogarithm function.

Solutions

No problem is ever permanently closed. We will be very pleased considering for publication new solutions or comments on the past problems.

131. *Proposed by Cornel Ioan Vălean, Timiş, Rumania.* Calculate

$$\int_0^{\pi/2}\int_0^{\pi/2}\frac{\log(1+\cos x)-\log(1+\cos y)}{\cos x-\cos y}dxdy.$$

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria.
Let the proposed integral be denoted by I. We will prove that $I = 2\pi G - \frac{7}{2}\zeta(3)$, where $G = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}$ is the Catalan constant and ζ is the well-known Riemann zeta function. Note that, for $t \in [0,1]$, we have

$$\frac{1}{\cos x-\cos y}\left(\frac{\cos x}{1+t\cos x}-\frac{\cos y}{1+t\cos y}\right)=\frac{1}{(1+t\cos x)(1+t\cos y)}.$$

Thus,

$$\frac{\log(1+\cos x)-\log(1+\cos y)}{\cos x-\cos y}=\int_0^1\frac{dt}{(1+t\cos x)(1+t\cos y)}.$$

It follows that

$$\begin{aligned} I &= \int_0^1\left(\int_0^{\pi/2}\frac{dx}{1+t\cos x}\right)\left(\int_0^{\pi/2}\frac{dy}{1+t\cos y}\right)dt,\\ &= \int_0^1\left(\int_0^{\pi/2}\frac{dx}{1+t\cos x}\right)^2 dt, \qquad t\leftarrow\cos\theta,\\ &= \int_0^{\pi/2}\left(\int_0^{\pi/2}\frac{dx}{1+\cos\theta\,\cos x}\right)^2\sin\theta\,d\theta. \end{aligned}\tag{1}$$

On the other hand, for a fixed $\theta \in (0,\pi/2]$, the change of variables $z = \tan(\theta/2)\tan(x/2)$ shows that

$$\int_0^{\pi/2}\frac{dx}{1+\cos\theta\,\cos x}=\frac{2}{\sin\theta}\int_0^{\tan(\theta/2)}\frac{dz}{1+z^2}=\frac{\theta}{\sin\theta}\tag{2}$$

Replacing (2) in (1) we obtain

$$I=\int_0^{\pi/2}\frac{\theta^2}{\sin\theta}\,d\theta$$

Now, using integration by parts we have

$$I=-2\int_0^{\pi/2}\theta\log\left(\tan\frac{\theta}{2}\right)d=\int_{-\pi/2}^{\pi/2}|\theta|\log\left|\cot\frac{\theta}{2}\right|\,d\theta\tag{3}$$

To evaluate (3) we will use Bessel-Parseval's formula. Note that, for $\theta \in (-\pi, \pi)\backslash\{0\}$ we have

$$\begin{aligned}
\log\left|\cot\frac{\theta}{2}\right| &= \log\left|\frac{1+e^{i\theta}}{1-e^{i\theta}}\right| \\
&= \log\left|1+e^{i\theta}\right| - \log\left|1-e^{i\theta}\right|, \\
&= \Re\left(\mathrm{Log}(1+e^{i\theta}) - \mathrm{Log}(1-e^{i\theta})\right), \\
&= \Re\left(\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}e^{in\theta} + \sum_{n=1}^{\infty}\frac{1}{n}e^{in\theta}\right), \\
&= \sum_{n=0}^{\infty}\frac{2}{2n+1}\cos(2n+1)\theta. \qquad (4)
\end{aligned}$$

Also, if g is the 2π-periodic function that is defined by

$$g(\theta) = \begin{cases} |\theta| & \text{if } \theta \in (-\pi/2, \pi/2), \\ 0 & \text{if } \theta \in (-\pi, -\pi/2] \cup [\pi/2, \pi], \end{cases}$$

then it is straightforward to check that the Fourier series expansion of g is given by

$$g(\theta) = \frac{\pi}{8} + \sum_{n=0}^{\infty}\frac{\cos(4n+2)\theta}{\pi(2n+1)^2} + \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{2n+1} - \frac{2}{\pi(2n+1)^2}\right)\cos(2n+1)\theta$$

So, using Bessel-Parseval's formula we obtain

$$\begin{aligned}
I &= \sum_{n=0}^{\infty}\left(\frac{2\pi(-1)^n}{(2n+1)^2} - \frac{4}{(2n+1)^3}\right) \\
&= 2\pi\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2} - 4\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3} = 2\pi G - \frac{7}{2}\zeta(3).
\end{aligned}$$

which is the announced answer.

Solution 2 by Moti Levy, Rehovot, Israel. Let $I(a)$ be defined as follows:

$$I(a) := \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\log(a+\cos x) - \log(a+\cos y)}{\cos x - \cos y}dxdy, \quad a > 1.$$

Differentiation under the integral sign,

$$\begin{aligned}
\frac{\partial I(a)}{\partial a} &:= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{\frac{1}{a+\cos x} - \frac{1}{a+\cos y}}{\cos x - \cos y}dxdy = -\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{1}{(a+\cos x)(a+\cos y)}dxdy \\
&= -\left(\int_0^{\frac{\pi}{2}}\frac{1}{a+\cos x}dx\right)^2.
\end{aligned}$$

$$\begin{aligned}
\int_0^{\frac{\pi}{2}}\frac{1}{a+\cos x}dx &= \frac{1}{a}\int_0^{\frac{\pi}{2}}\frac{1}{1+a^{-1}\cos x}dx = \int_0^{\frac{\pi}{2}}\sum_{n=0}^{\infty}\frac{(-1)^n}{a^{n+1}}\cos^n x dx \\
&= \sum_{n=0}^{\infty}\frac{(-1)^n}{a^{n+1}}\int_0^{\frac{\pi}{2}}\cos^n x dx.
\end{aligned}$$

Now, (See [1] , entry 3.621 on page 395),

$$\int_0^{\frac{\pi}{2}} \cos^p x dx = 2^{p-1} \frac{\Gamma^2\left(\frac{p+1}{2}\right)}{\Gamma(p+1)}, \quad p > -1,$$

hence

$$\int_0^{\frac{\pi}{2}} \frac{1}{a+\cos x} dx = \sum_{n=0}^{\infty} (-1)^n \frac{2^{n-1}}{a^{n+1}} \frac{\Gamma^2\left(\frac{n+1}{2}\right)}{n!}.$$

One can show that

$$\sum_{n=0}^{\infty} (-1)^n \frac{2^{n-1}}{a^{n+1}} \frac{\Gamma^2\left(\frac{n+1}{2}\right)}{n!} = \frac{\arccos\left(\frac{1}{a}\right)}{\sqrt{a^2-1}},$$

by showing that

$$\sqrt{a^2-1}\left(\sum_{n=0}^{\infty} (-1)^n \frac{2^{n-1}}{a^{n+1}} \frac{\Gamma^2\left(\frac{n+1}{2}\right)}{n!}\right) = \frac{\pi}{2} - \arcsin\left(\frac{1}{a}\right),$$

or

$$\left(\sqrt{1-\left(\frac{1}{a}\right)^2}\right)\left(\sum_{m=0}^{\infty} \frac{2^{2m}(m!)^2}{(2m+1)!}\left(\frac{1}{a}\right)^{2m+2}\right) = \frac{1}{a}\arccos\left(\frac{1}{a}\right)$$

$$= \sum_{m=0}^{\infty} \frac{(2m)!}{2^{2m}(m!)^2(2m+1)}\left(\frac{1}{a}\right)^{2m+2}.$$

It follows that

$$\frac{\partial I(a)}{\partial a} = -\frac{\left(\arccos\left(\frac{1}{a}\right)\right)^2}{a^2-1}.$$

$$\int_1^{\infty} \frac{\partial I(a)}{\partial a} da = \lim_{t\to\infty} I(t) - I(1) = -\int_1^{\infty} \frac{\left(\arccos\left(\frac{1}{a}\right)\right)^2}{a^2-1} da$$

Changing the variable of integration $x = \arccos\left(\frac{1}{a}\right)$ we get, $\int_1^{\infty} \frac{\left(\arccos\left(\frac{1}{a}\right)\right)^2}{a^2-1} da = \int_0^{\frac{\pi}{2}} \frac{x^2}{\sin x} dx$. Since $\lim_{t\to\infty} I(t) = 0$, then

$$I(1) = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\log(1+\cos x) - \log(1+\cos y)}{\cos x - \cos y} dx dy = \int_0^{\frac{\pi}{2}} \frac{x^2}{\sin x} dx.$$

Surprisingly, the original double integral is equal to the second moment of $\frac{1}{\sin x}$ over the interval $\left[0, \frac{\pi}{2}\right]$. The moments of $\frac{1}{\sin x}$ were treated before in several articles. For example in [2], there is a nice recurrence formula for the moments. Let $I_n := \int_0^{\frac{\pi}{2}} \frac{x^n}{\sin x} dx$, be the n-th moment, then

$$(-1)^{n+1}(2n+1)!\left(1-2^{-2n-1}\right)\zeta(2n+1) + \left(n+\frac{1}{2}\right) I_{2n}$$

$$= \sum_{m=0}^{n-1} \binom{2n+1}{2m+1} B_{2(n-m)} \pi^{2(n-m)-1}\left(2^{2(n-m)}-1\right) I_{2m+1},$$

where B_k are Bernoulli's numbers and $\zeta(2n+1)$ is Riemann's zeta function for odd integers. The first moment $I_1 := \int_0^{\frac{\pi}{2}} \frac{x}{\sin x} dx$ is known to be $2G$, where G is the Catalan's constant. Using the recurrence above (by putting $n = 1$ and $B_2 = \frac{1}{6}$),

$$(3)!\left(1-2^{-3}\right)\zeta(3) + \frac{3}{2} I_2 = 3B_2\pi 3I_1,$$

$$I_2 = -\frac{7}{2}\zeta(3) + 2\pi G.$$

We conclude that,

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\log(1+\cos x) - \log(1+\cos y)}{\cos x - \cos y} dxdy = -\frac{7}{2}\zeta(3) + 2\pi G \cong 1.54798.$$

References

[1] Gradshteyn and Rhyzik, "Table of Integrals, Series, and Products", 7th Edition.

[2] Yakubovitch S. "Certain identities, connection and explicit formulas for the Bernoulli, Euler numbers and Riemann zeta values", *Analysis.* Volume 35, Issue 1, Pages 59–71, ISSN (Online) 2196-6753, ISSN (Print) 0174-4747, DOI: 10.1515/anly-2014-1286, February 2015.

Solution 3 by Albert Stadler, Herrliberg, Switzerland.

We claim that $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\log(1+\cos x)-\log(1+\cos y)}{\cos x - \cos y} dxdy = 2\pi G - \frac{7}{2}\zeta(3)$,
where $G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2}$ is Catalan's constant and $\zeta(.)$ is the Riemann zeta function. Put $f(r) = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\log(1+r\cos x)-\log(1+r\cos y)}{\cos x - \cos y} dxdy$, $0 \le r \le 1$.
Then,

$$f'(r) = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{\frac{\cos x}{1+r\cos x} - \frac{\cos y}{1+r\cos y}}{\cos x - \cos y} dxdy = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{1}{(1+r\cos x)(1+r\cos y)} dxdy$$

$$= \left(\int_0^{\frac{\pi}{2}} \frac{1}{1+r\cos x} dx\right)^2.$$

We note that

$$\int \frac{1}{1+r\cos x} dx = \frac{2\arctan\left(\frac{1-r}{\sqrt{1-r^2}}\tan\left(\frac{x}{2}\right)\right)}{\sqrt{1-r^2}} + C,$$

since

$$\frac{d}{dx}\left(\frac{2\arctan\left(\frac{1-r}{\sqrt{1-r^2}}\tan\left(\frac{x}{2}\right)\right)}{\sqrt{1-r^2}}\right) = \frac{2}{\sqrt{1-r^2}} \cdot \frac{1}{\left(\frac{1-r}{\sqrt{1-r^2}}\tan\left(\frac{x}{2}\right)\right)^2 + 1} \cdot \frac{1-r}{\sqrt{1-r^2}} \cdot \frac{1}{2\cos^2\left(\frac{x}{2}\right)}$$

$$= \frac{1-r}{(1-r)^2\sin^2\left(\frac{x}{2}\right) + (1-r^2)\cos^2\left(\frac{x}{2}\right)}$$

$$= \frac{1-r}{\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right) - 2r\sin^2\left(\frac{x}{2}\right) + r^2\left(\sin^2\left(\frac{x}{2}\right) - \cos^2\left(\frac{x}{2}\right)\right)}$$

$$= \frac{1-r}{1 - r(1-\cos x) - r^2\cos x} = \frac{1-r}{(1-r)(1+r\cos x)} = \frac{1}{1+r\cos x}.$$

So

$$f'(r) = \left(\frac{2\arctan\left(\frac{1-r}{\sqrt{1-r^2}}\right)}{\sqrt{1-r^2}}\right)^2 = \frac{4\arctan^2\left(\frac{1-r}{\sqrt{1-r^2}}\right)}{1-r^2}, \text{ and}$$

$$f(1) = \int_0^1 f'(r)dr = 4\int_0^1 \frac{\arctan^2\left(\frac{1-r}{\sqrt{1-r^2}}\right)}{1-r^2} dr$$

$$\overset{r=\cos x}{=} 4\int_0^{\frac{\pi}{2}} \frac{\arctan^2\left(\frac{1-\cos x}{\sin x}\right)}{\sin^2 x} \sin x dx = 4\int_0^{\frac{\pi}{2}} \frac{\arctan^2\left(\tan\left(\frac{x}{2}\right)\right)}{\sin x} dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{x^2}{\sin x} dx \overset{x=2\arctan y}{=} 4\int_0^1 \frac{\arctan^2 y}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)} \cdot 2\cos^2\left(\frac{x}{2}\right) dy$$

$$= 4\int_0^1 \frac{\arctan^2 y}{y} dy = -\int_0^1 \frac{\log^2\left(\frac{1+iy}{1-iy}\right)}{y} dy \overset{z=\frac{1+iy}{1-iy}, y=-i\cdot\frac{z-1}{z+1}, dy=\frac{-2i}{(z+1)^2}dz}{=}$$

$$= 2\int_1^i \frac{\log^2 z}{1-z^2} dz = 2\sum_{k=0}^{\infty} \int_1^i z^{2k} \log^2 z dz$$

$$= 2\sum_{k=0}^{\infty} \left[\frac{z^{2k+1}}{2k+1} \log^2 z - \frac{2z^{2k+1}}{(2k+1)^2} \log z + \frac{2z^{2k+1}}{(2k+1)^3} \right]_1^i$$

$$= 2\sum_{k=0}^{\infty} \frac{i^{2k+1}}{2k+1} \log^2 i - 4\sum_{k=0}^{\infty} \frac{i^{2k+1}}{(2k+1)^2} \log i + 4\sum_{k=0}^{\infty} \frac{i^{2k+1}}{(2k+1)^3} - 4\sum_{k=0}^{\infty} \frac{1}{(2k+1)^3}$$

$$= -\frac{\pi^2 i}{2} \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} + 2\pi \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^2} + 4i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} - 4\left(\sum_{k=1}^{\infty} \frac{1}{k^3} - \sum_{k=1}^{\infty} \frac{1}{(2k)^3} \right)$$

$$= -\frac{\pi^2 i}{2} \arctan(1) + 2\pi G + 4i \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} - 4\zeta(3)\left(1 - \frac{1}{8}\right)$$

$$= 2\pi G - \frac{7}{2}\zeta(3) + 4i\left(-\frac{\pi^3}{32} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} \right) = 2\pi G - \frac{7}{2}\zeta(3),$$

since $f(1)$ is real. Incidentally we have proved that

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^3} = \frac{\pi^3}{32}.$$

Also solved by Ramya Dutta, Chennai Mathematical Institute (student) India and dhe proposer.

132. *Proposed by Valmir Krasniqi, University of Prishtina, Republic of Kosova.* Find all functions $f\colon \mathbb{R}^* \to \mathbb{R}^*$ from the non-zero reals to the non-zero reals, such that

$$f(xyz) = f(xy + yz + xz)(f(x) + f(y) + f(z))$$

for all non-zero reals x, y, z such that $xy + yz + xz \neq 0$.

Solution 1 by Ramya Dutta, Chennai Mathematical Institute (student) India. Let $f : \mathbb{R}^* \to \mathbb{R}^*$ satisfy the functional equation,

$$f(xyz) = f(xy + yz + zx)\,(f(x) + f(y) + f(z)) \qquad \cdots (1)$$

Setting, $(x, y, z) = (1, 1, -1)$ and $(-1, -1, 1)$ in the functional equation (1) respectively gives,

$f(-1) = f(-1)(2f(1) + f(-1)) \implies 2f(1) + f(-1) = 1$ since, $f(-1) \neq 0$.

$f(1) = f(-1)(2f(-1) + f(1)) \implies f(1) = 2f^2(-1) + f(-1)f(1)$.

Solving the system, either **Case 1:** $f(1) = f(-1) = \dfrac{1}{3}$ or, **Case 2:** $f(1) = 1, f(-1) = -1$.

Case 1: Set $(x, y, z) = (x, 1, -1)$ in (1), $f(-x) = f(-1)(f(x) + f(1) + f(-1)) \implies f(-x) = \dfrac{1}{3}f(x) + \dfrac{2}{9}$ and replacing x with $-x$ leads to, $f(x) = \dfrac{1}{3}f(-x) + \dfrac{2}{9}$. Solving the pair of linear equations for $f(x), f(-x)$ gives, $f(x)=f(-x)=\dfrac{1}{3}$, i.e., $f(x)=\dfrac{1}{3}$ (a constant function).

Case 2: Set $(x,y,z)=(x,1,-1)$ in (1), $f(-x)=f(-1)(f(x)+f(1)+f(-1)) \implies f(-x)=-f(x)$ (i.e., f is an odd function).

Again, set $(x,y,z)=(x,-x,z)$ in (1), $f(-x^2z)=f(-x^2)(f(x)+f(-x)+f(z))=f(-x^2)f(z) \implies f(x^2z)=f(x^2)f(z)$, that is $f(xz)=f(x)f(z)$ whenever, $x>0$. Combined with the fact that f is odd, we infer $f(xz)=f(x)f(z)$ for all $x,z\in\mathbb{R}^*$. In particular, $f(x)f\left(\dfrac{1}{x}\right)=f\left(x\cdot\dfrac{1}{x}\right)=f(1)=1$. Since, $f(xy+yz+zx)=f(xyz)f\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)$, equation (1) can be rewritten as: $f\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)(f(x)+f(y)+f(z))=1$. Changing, $(x,y,z)=\left(\dfrac{1}{x},\dfrac{1}{y},\dfrac{1}{z}\right)$, in this equation it becomes,

$$\frac{1}{f(x)}+\frac{1}{f(y)}+\frac{1}{f(z)}=\frac{1}{f(x+y+z)}.$$

Define, $g(x)=\dfrac{1}{f(x)}=f\left(\dfrac{1}{x}\right)$, then g satisfies the functional equation:

$$g(x+y+z)=g(x)+g(y)+g(z) \qquad \cdots (2)$$

where, $g(1)=f(1)=1$ and satisfies $g(x)g(z)=g(xz)$. Now, $g(6)=g(2+2+2)=3g(2)$ and $g(6)=g(4+1+1)=2g(1)+g(4)=2g(1)+g(1+1+2)=4g(1)+g(2)=4+g(2)$. Hence, $g(2)=2$. Setting, $z=x+y$ in (2) we have, $g(2(x+y))=g(x)+g(y)+g(x+y) \implies 2g(x+y)=g(2)g(x+y)=g(2(x+y))=g(x)+g(y)+g(x+y)$. Hence, $g(x+y)=g(x)+g(y)$. Thus, for integers $m,n\in\mathbb{Z}\setminus\{0\}$,

$$g\left(\frac{m}{n}\right)=mg\left(\frac{1}{n}\right)=\frac{m}{n}ng\left(\frac{1}{n}\right)=\frac{m}{n}g(1)=\frac{m}{n}$$

that is $g(x)=x$ for $x\in\mathbb{Q}^*$. Also, since $g(x^2)=(g(x))^2>0$, we may infer $g(x)>0$ whenever $x>0$. That is $g(y+x)=g(y)+g(x)>g(y)$ for $x>0$ (i.e., g is a monotone increasing function). Now, for every real number $x\in\mathbb{R}^*$, there is a monotone increasing sequence in $\mathbb{Q}^*$, $(r_n)_{n\in\mathbb{N}}\to x$ and a monotone decreasing sequence in $\mathbb{Q}^*$, $(r'_n)_{n\in\mathbb{N}}\to x$ such that, $r_n<x<r'_n, \quad (\forall n\in\mathbb{N})$. Since, g is a monotone increasing function, $r_n=g(r_n)<g(x)<g(r'_n)=r'_n, \quad (\forall n\in\mathbb{N}) \implies g(x)=x$ by completeness of the real line. Thus, $f(x)=\dfrac{1}{x}$ is the other solution.

Solution 2 by Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Romania. Replace in the given condition x,y and z by $1,-1$ and -1 respectively and obtain

$$f(1)=f(-1)\left(f(1)+2f(-1)\right).$$

Similarly, replace in the given condition x,y and z by $-1,1$ and 1 respectively and obtain

$$f(-1)=f(-1)\left(f(-1)+2f(1)\right).$$

So that if we denote $f(1)=x$ and $f(-1)=y$, then we get the system

$$\begin{cases} x=y(x+2y) \\ y=y(y+2x)\end{cases} \iff \begin{cases} y=1-2x \\ x=(1-2x)(2-3x)\end{cases} \Rightarrow \begin{cases} f(1)=\frac{1}{3} \\ f(-1)=\frac{1}{3}\end{cases} \vee \begin{cases} f(1)=1 \\ f(-1)=-1\end{cases}.$$

Case1: $f(1)=\frac{1}{3}$ and $f(-1)=\frac{1}{3}$.

For any fixed $x \in \mathbb{R}^*$, we make the substitution $(-x,-1,1)=(x,y,z)$ and get $f(x)=\frac{1}{3}\left(f(-x)+\frac{2}{3}\right)$. Now we make the substitution $(x,1,-1)=(x,y,z)$ and get $f(-x)=\frac{1}{3}\left(f(x)+\frac{2}{3}\right)$. From the system $\left\{\begin{array}{l} f(x)=\frac{1}{3}\left(f(-x)+\frac{2}{3}\right) \\ f(-x)=\frac{1}{3}\left(f(x)+\frac{2}{3}\right)\end{array}\right.$, obtain $f(x)=f(-x)=\frac{1}{3}$. So we've got the function $f:\mathbb{R}^*\to\mathbb{R}^*,\ f(x)=\frac{1}{3}\ \forall\ x\in\mathbb{R}^*$. Replacing by this function in the given relation, observe that it verifies the given equation.

Case 2: $f(1)=1$ and $f(-1)=-1$.

For any fixed $x\in\mathbb{R}^*$, we make the substitution $(x,1,-1)=(x,y,z)$ and get $f(-x)=-f(x)$, hence f is an odd function. Let $x\in\mathbb{R}^*$ and $y>0$ be fixed. Make $\left(-x,-\sqrt{y},\sqrt{y}\right)=(x,y,z)$ and get $f(xy)=-f(y)(-f(x))=f(x)f(y)$.

If $y<0$, then $-y>0 \Longrightarrow f(-xy)=f(x)f(-y) \Longleftrightarrow f(xy)=f(x)f(y)$. In conclusion, $f(xy)=f(x)f(y)\ ,\ \forall\ x,y\in\mathbb{R}^*$.

Consider the function $g:\mathbb{R}^*\to\mathbb{R}^*,\ g(x)=\frac{1}{f(x)}\ \forall\ x\in\mathbb{R}^*$. From above we have $g(1)=1,\ g(-x)=-g(x)\ \forall\ x\in\mathbb{R}^*$ and $g(xy)=g(x)g(y)\ ,\ \forall\ x,y\in\mathbb{R}^*$.

Further, the given condition becomes $g(xy+yz+zx)=g(x)g(y)+g(y)g(z)+g(z)g(x)\Longleftrightarrow g(xy+yz+zx)=g(xy)+\ g(yz)+g(zx)\,,\ \forall\ x,y,z\in\mathbb{R}^*$ such that $xy+yz+zx\neq 0$.

Let $a,b,c\in\mathbb{R}^*$ such that $abc>0$ and $a+b+c\neq 0$. The system $\left\{\begin{array}{l} xy=c \\ yz=a \\ zx=b\end{array}\right.$ admits solutions $\in\mathbb{R}^*$, for example, $x=\frac{\sqrt{abc}}{a}$, $y=\frac{\sqrt{abc}}{b}$ and $z=\frac{\sqrt{abc}}{c}$. In conclusion, we have $g(a+b+c)=g(xy+yz+zx)=g(xy)+g(yz)+g(zx)=g(a)+g(b)+g(c)$.

If $a,b,c\in\mathbb{R}^*$ such that $abc<0$ and $a+b+c\neq 0$ then $(-a)(-b)(-c)>0$ and $-a-b-c\neq 0$, hence from above, $g(-a-b-c)=g(-a)+\ g(-b)+g(-c)\Longleftrightarrow g(a+b+c)=g(a)+\ g(b)+g(c)$.

In conclusion, $g(a+b+c)=g(a)+g(b)+g(c)\ \forall\ a,b,c\in\mathbb{R}^*$ such that $a+b+c\neq 0$.

On the other hand, $1=g(1)=g\left(x\cdot\frac{1}{x}\right)=g(x)\,g\left(\frac{1}{x}\right)\Longrightarrow g\left(\frac{1}{x}\right)=\frac{1}{g(x)}\ \forall\ x\in\mathbb{R}^*$.

We have: $g(3)=g(1+1+1)=g(1)+\ g(1)+g(1)=3g(5)=g(3+1+1)=g(3)+g(1)+g(1)=5$.

But $g(5)=g(2+2+1)=g(2)+g(2)+g(1)\Longrightarrow 5=2g(2)+1\Longrightarrow g(2)=2$.

From here, $g(4)=g(2+1+1)=g(2)+\ g(1)+g(1)=4$.

We can easely show by induction that $g(n)=n\ \forall\ n\in\mathbb{N}^*$. Indeed, the verifying has done. If $n\geq 3$, then

$g(n)=g(n-2+1+1)=g(n-2)+\ g(1)+g(1)=n-2+1+1=n$.

Let $r>0$ be a rational number. Then $\exists\ m,n\in\mathbb{N}^*$ *such that* $r=\frac{m}{n}\Longrightarrow g(r)=g\left(\frac{m}{n}\right)=\frac{g(m)}{g(n)}=\frac{m}{n}=r$. Now let $x,y\in\mathbb{R}^*$ such that $x+y\neq 0$. We have $g(x+y)=g\left(x+\frac{y}{2}+\frac{y}{2}\right)=g(x)+g\left(\frac{y}{2}\right)+g\left(\frac{y}{2}\right)=g(x)+\frac{g(y)}{g(2)}+\frac{g(y)}{g(2)}=g(x)+\frac{g(y)}{2}+\frac{g(y)}{2}$ $=g(x)+g(y)$. If $x>0$, then $g(x)=g\left(\left(\sqrt{x}\right)^2\right)=g^2\left(\sqrt{x}\right)>0$. Now let x and y be non zero real numbers such that $x>y$. Then $0<g(x-y)=g(x)-g(y)\Longrightarrow g$ is strictly increasing on its domain. We assume by absurd that $\exists x\neq 0$ such that $g(x)\neq x$. Let WLOG $g(x)<x$. We consider a rational number r with $g(x)<r<x$. From $r<x$ we get $g(r)<g(x)\Longrightarrow r<g(x)$. But $g(x)<r$. Hence contradiction. Similarly, if $g(x)>x$, then obtain contradiction. In conclusion, $g(x)=x\forall\ x\neq 0$. So that $g(x)=x\ \forall\ x\in\mathbb{R}^*\Longrightarrow f(x)=\frac{1}{x}\ \forall\ x\in\mathbb{R}^*$. The proof is complete.

Also solved by the prposer.

133. *Proposed by Vasile Pop and Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Solve in $\mathcal{M}_2(\mathbb{Z}_5)$ the equation

$$X^5 = \begin{pmatrix} \widehat{4} & \widehat{2} \\ \widehat{4} & \widehat{1} \end{pmatrix}$$

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Let $A = \begin{pmatrix} \widehat{4} & \widehat{2} \\ \widehat{4} & \widehat{1} \end{pmatrix}$. It is straightforward to check that $A^2 = -I$, so that $A^4 = I$ and $A^5 = A$. Therefore, $X = A$ is a solution to the proposed equation.
Now, suppose that $X^5 = A$, and let $d = \det(X)$ and $t = \operatorname{tr}(X)$. From $X^5 = A$ we conclude that $\det(X^5) = \det(A) = \widehat{1}$ so $d^5 = \widehat{1}$, but in $\mathbb{Z}_5$ we have $x^5 = x$ for every x, thus $d = d^5 = \widehat{1}$. Also, from $X^2 = tX - I$ we conclude that

$$-I = A^2 = X^{10} = t^5 X^5 - I = tA - I$$

So $tA = \widehat{0}$ and consequently $t = 0$. It follows that $X^2 = -I$ so $A = X^5 = X$. Thus $X = A$ is the only solution to the equation $X^5 = A$.

Solution 2 by Moti Levy, Rehovot, Israel. Let A be diagonalizable matrix over the field $\mathbb{Z}_5$. By definition, $A = P^{-1}DP$, where P is invertible and D is diagonal.From the theory of finite fields, if $x \in \mathbb{Z}_5$ then $x^5 = x$. It follows that for every diagonal D matrix over $\mathbb{Z}_5$, $D^5 = D$ and

$$A^5 = \left(P^{-1}DP\right)^5 = PD^5P^{-1} = P^{-1}DP = A.$$

We showed that if A is a diagonalizable matrix over the field $\mathbb{Z}_5$, then A is a solution to the equation $X^5 = A$. Suppose B is a solution to the equation $X^5 = A$, i.e., $B^5 = A = A^5$. If v is an eigenvector of B with the corresponding eigenvalue λ , the by definition $Bv = \lambda v$ and $B^5 v = \lambda^5 v$. But $\lambda^5 = \lambda$ in $\mathbb{Z}_5$. It follows that B^5 and B have the same eigenvalues. Consequently, B and A have the same eigenvalues. An $n \times n$ matrix A is diagonalizable over the field $\mathbb{F}$ if it has n distinct eigenvalues in $\mathbb{F}$. If A has two distinct eigenvalues then both A and B are diagonalizable. If B is diagonalizable then $B^5 = B$, and it follows that $B^5 = A = B$.

We have proved the following claim:
If A is a diagonalizable matrix over the field $\mathbb{Z}_5$ with two distinct eigenvalues then the solution to the equation $X^5 = A$ in $\mathbb{Z}_5$ is A.
The eigenvalues of $\begin{bmatrix} 4 & 2 \\ 4 & 1 \end{bmatrix}$ are 2 and 3. Hence the matrix is diagonalizable and the solution to $X^5 = \begin{bmatrix} 4 & 2 \\ 4 & 1 \end{bmatrix}$ is $\begin{bmatrix} 4 & 2 \\ 4 & 1 \end{bmatrix}$.

Solution 3 by Michel Bataille, Rouen, France.
Let $A = \begin{pmatrix} \widehat{4} & \widehat{2} \\ \widehat{4} & \widehat{1} \end{pmatrix}$ and $I = \begin{pmatrix} \widehat{1} & \widehat{0} \\ \widehat{0} & \widehat{1} \end{pmatrix}$. We show that the equation $X^5 = A$ has a unique solution, namely A itself. A simple calculation yields $A^2 = -I$ and so $A^5 = (-I)^2 \cdot A = A$. Thus, A is a solution. Conversely, let X be a solution. Then, recalling that $x^5 = x$ for all $x \in \mathbb{Z}_5$, $\det(X) = (\det(X))^5 = \det(X^5) = \det(A) = \widehat{4}\cdot\widehat{1} - \widehat{4}\cdot\widehat{2} = \widehat{1}$. The characteristic polynomial of X being $\chi(x) = x^2 - \operatorname{tr}(X)\cdot x + \det(X) = x^2 - mx + 1$

(setting $m = \mathrm{tr}(X)$), the Cayley-Hamilton Theorem gives $X^2 = mX - I$. Using this relation repeatedly, we deduce

$$X^5 = (m^4 - 3m^2 + 1)X - (m^3 - 2m)I \qquad (*).$$

It follows that $\mathrm{tr}(X^5) = m(m^4 - 3m^2 + 1) - 2(m^3 - 2m) = m^5 = m$, hence $m = 0$ (since $\mathrm{tr}(X^5) = \mathrm{tr}(A) = \widehat{4} + \widehat{1} = \widehat{0}$). From $(*)$, we obtain $X^5 = X$ and so $X = A$. This completes the proof.

Also solved by Ramya Dutta, Chennai Mathematical Institute (student) India; Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Rumania; Moubinool Omarjee, Paris, France and the proposers.

134. *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia.* Evaluate the following integral

$$\int_0^\infty \frac{\sin(a_1 x)}{x} \frac{\sin(a_2 x)}{x} e^{-a_3 x} dx.$$

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Let the desired integral be denoted by $I(a_1, a_2, a_3)$. We will prove that for $a_3 > 0$ and any real numbers a_1 and a_2 we have

$$I(a_1, a_2, a_3) = \frac{a_1 + a_2}{2} \arctan\left(\frac{a_1 + a_2}{a_3}\right) - \frac{a_1 - a_2}{2} \arctan\left(\frac{a_1 - a_2}{a_3}\right) + \frac{a_3}{4} \log \frac{a_3^2 + (a_1 - a_2)^2}{a_3^2 + (a_1 + a_2)^2}.$$

The Laplace transform of $t \mapsto (1 - \cos t)H(t)$ where H is the Heaviside function is given by

$$\int_0^\infty (1 - \cos t) e^{-pt} dt = \frac{1}{p} - \frac{p}{1 + p^2}, \quad \text{for } p > 0.$$

Integrating with respect to p from s to $+\infty$ we get

$$\int_0^\infty \frac{1 - \cos t}{t} e^{-st} dt = \frac{1}{2} \log\left(1 + \frac{1}{s^2}\right), \quad \text{for } s > 0.$$

Integrating again with respect to s from p to $+\infty$ we get

$$\int_0^\infty \frac{1 - \cos t}{t^2} e^{-pt} dt = \arctan\left(\frac{1}{p}\right) - \frac{p}{2} \log\left(1 + \frac{1}{p^2}\right), \quad \text{for } p > 0.$$

Now, for $a > 0$ the change of variables $t = ax$ and $a_3 = ap$ yield

$$\int_0^\infty \frac{1 - \cos(ax)}{x^2} e^{-a_3 x} dx = a \arctan\left(\frac{a}{a_3}\right) - \frac{a_3}{2} \log\left(1 + \frac{a^2}{a_3^2}\right). \qquad (1)$$

Both sides of this equality are even functions of a, so (1) is valid for $a \in \mathbb{R}$ and $a_3 > 0$. Subtracting the formulas corresponding to $a = a_1 + a_2$ and $a = a_1 - a_2$ we obtain the announced result.

Solution 2 by Michel Bataille, Rouen, France. First, we calculate $F(x) = \int_0^\infty f(x,t)\, dt$ where $f(x,t) = e^{-at} \cdot \frac{\sin(xt)}{t}$ (a is a fixed positive real number and $x \in \mathbb{R}$). For $x \in \mathbb{R}, t > 0$, we have $|f(x,t)| \le |x| e^{-at}$ (since $|\sin(xt)| \le |x|t$) and

$\int_0^\infty e^{-at}\,dt < \infty$. Thus, F is well-defined and continuous on $\mathbb{R}$. In addition, for $|x|, t > 0$ we have

$$\left|\frac{\partial f}{\partial x}(x,t)\right| = |e^{-at}\,\cos(xt)| \le e^{-at}$$

It follows that F is differentiable on $(0,\infty)$ and $(-\infty,0)$ with

$$F'(x) = \int_0^\infty e^{-at}\,\cos(xt)\,dt = \frac{a}{a^2+x^2}.$$

[the second equality is known; for completeness, a quick proof is given at the end]. Therefore $F(x) = \arctan(x/a)$ (since $F(0)=0$).

Now, we consider $G(x,y) = \int_0^\infty g(x,y,t)\,dt$ where $g(x,y,t) = e^{-at}\cdot\frac{\sin(xt)}{t}\cdot\frac{\sin(yt)}{t}$ (with $x, y \ge 0$). As above, we see that G is continuous on $[0,\infty)\times[0,\infty)$ and that for $x, y > 0$:

$$\begin{aligned}
\frac{\partial G}{\partial x}(x,y) &= \int_0^\infty e^{-at}\cdot\frac{\sin(yt)\cos(xt)}{t}\,dt \\
&= \frac{1}{2}\left(\int_0^\infty e^{-at}\frac{\sin((x+y)t)}{t}\,dt - \int_0^\infty e^{-at}\frac{\sin((x-y)t)}{t}\,dt\right) \\
&= \frac{1}{2}\left(\arctan\left(\frac{x+y}{a}\right) - \arctan\left(\frac{x-y}{a}\right)\right).
\end{aligned}$$

In a similar way, $\frac{\partial G}{\partial y}(x,y) = \frac{1}{2}\left(\arctan\left(\frac{x+y}{a}\right) + \arctan\left(\frac{x-y}{a}\right)\right)$. It is readily checked that for $\alpha, \beta \in \mathbb{R}, \alpha \ne 0$, a primitive of $x \mapsto \arctan(\alpha x + \beta)$ is

$$x \mapsto \frac{\alpha x + \beta}{\alpha}\arctan(\alpha x + \beta) - \frac{1}{2\alpha}\ln\left(1 + (\alpha x + \beta)^2\right).$$

It follows that

$$G(x,y) = \frac{1}{2}\left((x+y)\arctan\left(\frac{x+y}{a}\right) - (x-y)\arctan\left(\frac{x-y}{a}\right) - \frac{a}{2}\ln\left(\frac{a^2+(x+y)^2}{a^2+(x-y)^2}\right)\right) + \phi(y)$$

where ϕ is a differentiable function. Calculating $\frac{\partial G}{\partial y}(x,y)$ from this formula and comparing with the previous result show that $\phi'(y) = 0$, hence ϕ is a constant function. Since $G(0,0) = 0$, this constant must be zero. This immediately leads to the value of the required integral $I = \int_0^\infty \frac{\sin(a_1 x)}{x}\frac{\sin(a_2 x)}{x}e^{-a_3 x}\,dx$:

$$I = \frac{1}{2}\left[(a_1+a_2)\arctan\left(\frac{a_1+a_2}{a_3}\right) - (a_1-a_2)\arctan\left(\frac{a_1-a_2}{a_3}\right) - \frac{a_3}{2}\ln\left(\frac{a_3^2+(a_1+a_2)^2}{a_3^2+(a_1-a_2)^2}\right)\right].$$

Proof of $\int_0^\infty e^{-at}\,\cos(xt)\,dt = \frac{a}{a^2+x^2}$.

We remark that $e^{-at}\cos(xt)$ is the real part of the complex number $e^{(-a+ix)t}$. It follows that for $T > 0$, the integral $\int_0^T e^{-at}\,\cos(xt)\,dt$ is the real part of $J(T) = \int_0^T e^{(-a+ix)t}\,dt$. Now, $J(T) = \frac{1}{-a+ix}\left(e^{(-a+ix)T} - 1\right)$ and since

$$\lim_{T\to\infty}\left|e^{(-a+ix)T}\right| = \lim_{T\to\infty} e^{-aT} = 0$$

we see that $\int_0^\infty e^{(-a+ix)t}\,dt = \lim_{T\to\infty} J(T) = \frac{1}{a-ix} = \frac{a+ix}{a^2+x^2}$ whose real part is $\frac{a}{a^2+x^2}$. The result follows.

Solution 3 by Ramya Dutta, Chennai Mathematical Institute (student) India.

Wlog assume $a_1 \geq a_2$, denoting $b = \dfrac{a_1 - a_2}{a_3}$, $c = \dfrac{a_1 + a_2}{a_3}$, then integrating by parts,

$$\begin{aligned}\int_0^\infty \frac{\sin(a_1 x)\sin(a_2 x)}{x^2} e^{-a_3 x}\,dx &= \frac{a_3}{2}\int_0^\infty \frac{\cos(bx)-\cos(cx)}{x^2} e^{-x}\,dx \\ &= \frac{a_3}{2}\int_0^\infty \frac{1}{x}\frac{d}{dx}\left((\cos(bx)-\cos(cx))e^{-x}\right)dx \\ &= \frac{a_3}{2}\Re\int_0^\infty \frac{1}{x}\frac{d}{dx}\left(e^{-(1-ib)x} - e^{-(1-ic)x}\right)dx \\ &= \frac{a_3}{2}\Re\int_0^\infty \frac{-(1-ib)e^{-(1-ib)x} + (1-ic)e^{-(1-ic)x}}{x}\,dx \\ &= \frac{a_3}{2}\Re((1-ib)I(b) - (1-ic)I(c))\end{aligned}$$

where, $I(k) = \displaystyle\int_0^\infty \frac{e^{-x} - e^{-(1-ik)x}}{x}\,dx$, when $k \in \mathbb{R}\setminus\{0\}$ and $I(k) = 0$, when $k = 0$.

Consider, the integral of $f(z) = \dfrac{e^{-z}}{z}$ on the contour $\gamma = \displaystyle\bigcup_{j=1}^4 \gamma_j$, where, $\gamma_1 = [r, R]$, $\gamma_2 = [R, (1-ik)R]$, $\gamma_3 = [(1-ik)R, (1-ik)r]$ and $\gamma_4 = [(1-ik)r, r]$ $(R > r > 0)$. Since, there are no poles of $f(z)$ in the interior of γ,

$$\begin{aligned}0 = \int_\gamma f(z)\,dz &= \sum_{j=1}^4 \int_{\gamma_j} \frac{e^{-z}}{z}\,dz \\ &= \int_r^R \frac{e^{-x}}{x}\,dx + i\int_0^{-k} \frac{e^{-R(1+it)}}{(1+it)}\,dt - \int_r^R \frac{e^{-(1-ik)x}}{x}\,dx + \int_{\gamma_4} \frac{e^{-z}}{z}\,dz\end{aligned}$$

thus,

$$\int_r^R \frac{e^{-x} - e^{-(1-ik)x}}{x}\,dx = -i\int_0^{-k} \frac{e^{-R(1+it)}}{(1+it)}\,dt - \int_{\gamma_4} \frac{dz}{z} - \int_{\gamma_4} \frac{e^{-z}-1}{z}\,dz$$

Now, $\left|\displaystyle\int_0^{-k} \frac{e^{-R(1+it)}}{(1+it)}\,dt\right| < \displaystyle\int_0^{-k} \frac{e^{-R}}{|1+it|}\,dt = O(e^{-R})$, thus it vanishes as $R \to \infty$.

The function $\dfrac{e^{-z}-1}{z}$ is entire and bounded on $|z| < 1$. Letting $r \to 0$, the integral $\left|\displaystyle\int_{\gamma_4} \frac{e^{-z}-1}{z}\,dz\right| < Mr \to 0$.

Thus,

$$\begin{aligned}\int_0^\infty \frac{e^{-x} - e^{-(1-ik)x}}{x}\,dx = -\lim_{r\to 0}\int_{\gamma_4} \frac{dz}{z} &= \lim_{r\to 0} \operatorname{Log}(z)\Big|_{z=r}^{z=r(1-ik)} \\ &= \operatorname{Log}(1-ik) \\ &= \frac{1}{2}\log(1+k^2) - i\arctan k\end{aligned}$$

Now, $\Re((1-ik)I(k)) = \dfrac{1}{2}\log(1+k^2) - k\arctan k$, Hence,

$$\int_0^\infty \frac{\sin(a_1x)\sin(a_2x)}{x^2}e^{-a_3x}\,dx$$
$$= \frac{a_3}{4}\log\left(\frac{a_3^2+(a_1-a_2)^2}{a_3^2+(a_1+a_2)^2}\right)-\frac{1}{2}\left((a_1-a_2)\arctan\left(\frac{a_1-a_2}{a_3}\right)-(a_1+a_2)\arctan\left(\frac{a_1+a_2}{a_3}\right)\right)$$
$$= \frac{a_3}{4}\log\left(\frac{a_3^2+(a_1-a_2)^2}{a_3^2+(a_1+a_2)^2}\right)+\frac{a_1}{2}\arctan\left(\frac{2a_2a_3}{a_3^2+a_1^2-a_2^2}\right)+\frac{a_2}{2}\arctan\left(\frac{2a_1a_3}{a_3^2-a_1^2+a_2^2}\right)$$

Also solved by Moti Levy, Rehovot, Israel and the proposer.

135. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Calculate

$$\lim_{n\to\infty}\left(\left(\sqrt[n]{n!}\right)^{F_{m+1}}\left(\sqrt[n]{(2n-1)!!}\right)^{F_m}\left(\tan\frac{\pi\sqrt[n+1]{(n+1)!}}{4\sqrt[n]{n!}}-1\right)^{F_{m+2}}\right).$$

where $\{F_m\}$ is the Fibonacci sequence, and $n!! = \prod_{0\le k<n/2}(n-2k)$.

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria. The answer is $2^{-F_{m+1}}(\pi/e)^{F_{m+2}}$.
Starting from the well-known asymptotic expansion:

$$\log(n!) = n\log n - n + \frac{1}{2}\log n + \log\sqrt{2\pi} + \mathcal{O}\left(\frac{1}{n}\right)$$

we get

$$\log((n+1)!) = n\log n - n + \frac{3}{2}\log n + \log\sqrt{2\pi} + \mathcal{O}\left(\frac{1}{n}\right)$$

and since $(2n-1)!! = \frac{(2n)!}{2^n n!}$ we have also

$$\log((2n-1)!!) = n\log n + (\log 2 - 1)n + \log\sqrt{2} + \mathcal{O}\left(\frac{1}{n}\right)$$

Thus,

$$\sqrt[n]{n!} = \frac{n}{e}\left(1+\frac{\log n}{2n}+\frac{\log\sqrt{2\pi}}{n}+\mathcal{O}\left(\frac{\log^2 n}{n^2}\right)\right)$$
$$\sqrt[n+1]{(n+1)!} = \frac{n}{e}\left(1+\frac{\log n}{2n}+\frac{1+\log\sqrt{2\pi}}{n}+\mathcal{O}\left(\frac{\log^2 n}{n^2}\right)\right)$$
$$\sqrt[n]{(2n-1)!!} = \frac{2n}{e}\left(1+\frac{\log 2}{2n}+\mathcal{O}\left(\frac{1}{n^2}\right)\right)$$

Consequently,

$$\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}} = 1 + \frac{1}{n} + \mathcal{O}\left(\frac{\log^2 n}{n^2}\right)$$

This implies that

$$\begin{aligned}\tan\frac{\pi\sqrt[n+1]{(n+1)!}}{4\sqrt[n]{n!}} &= \tan\left(\frac{\pi}{4}+\frac{\pi}{4n}+\mathcal{O}\left(\frac{\log^2 n}{n^2}\right)\right)\\ &= 1+\tan'\left(\frac{\pi}{4}\right)\frac{\pi}{4n}+\mathcal{O}\left(\frac{\log^2 n}{n^2}\right)\\ &= 1+\frac{\pi}{2n}+\mathcal{O}\left(\frac{\log^2 n}{n^2}\right)\end{aligned}$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty} n^{-F_{m+1}}\left(\sqrt[n]{n!}\right)^{F_{m+1}} &= \left(\frac{1}{e}\right)^{F_{m+1}},\\ \lim_{n\to\infty} n^{-F_m}\left(\sqrt[n]{(2n-1)!!}\right)^{F_m} &= \left(\frac{2}{e}\right)^{F_m},\\ \lim_{n\to\infty} n^{F_{m+2}}\left(\tan\frac{\pi\sqrt[n+1]{(n+1)!}}{4\sqrt[n]{n!}}-1\right)^{F_{m+2}} &= \left(\frac{\pi}{2}\right)^{F_{m+2}}\end{aligned}$$

Using $F_{m+2}=F_{m+1}+F_m$ and multiplying we obtain the announced result.

Solution 2 by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain. Since $F_{m+2}=F_{m+1}+F_{m+1}$ the proposed limit may be obtained by the product of the following two limits:

$$\lim_{n\to\infty}\left(\sqrt[n]{n!}\left(\tan\frac{\pi\sqrt[n+1]{(n+1)!}}{4\sqrt[n]{n!}}-1\right)\right)^{F_{m+1}},\quad\text{and}$$
$$\lim_{n\to\infty}\left(\sqrt[n]{(2n-1)!!}\left(\tan\frac{\pi\sqrt[n+1]{(n+1)!}}{4\sqrt[n]{n!}}-1\right)\right)^{F_m}.$$

These limits are respectively equal to $\left(\frac{\pi}{2e}\right)^{F_{m+1}}$ and $\left(\frac{\pi}{e}\right)^{F_m}$ from where the result follows. Let us show that

$$\lim_{n\to\infty}\sqrt[n]{n!}\left(\tan\frac{\pi\sqrt[n+1]{(n+1)!}}{4\sqrt[n]{n!}}-1\right)=\frac{\pi}{2e}.$$

By Stirling formula $\lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\frac{1}{e}$, and $\lim_{x\to1}\frac{\tan\frac{\pi x}{4}-1}{x-1}=\frac{\pi}{2}$. Then

$$\begin{aligned}\lim_{n\to\infty}\sqrt[n]{n!}\left(\tan\frac{\pi\sqrt[n+1]{(n+1)!}}{4\sqrt[n]{n!}}-1\right) &= \frac{\pi}{2e}\lim_{n\to\infty} n\left(\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!}}-1\right)\\ &= \frac{\pi}{2}\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!}\right)\\ &= \frac{\pi}{2}\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n+1}\ \text{(by Stolz-Cezaro lemma)}\\ &= \frac{\pi}{2e}.\end{aligned}$$

The second limit may be obtained similarly taking into account that $(2n-1)!! = \frac{(2n)!}{2^n\ n!}$, and by Stirling formula that $\lim_{n\to\infty} \frac{\sqrt[n]{(2n-1)!!}}{n} = \frac{2}{e}$.

Also solved by Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania; Ramya Dutta, Chennai Mathematical Institute (student) India; Michel Bataille, Rouen, France and the proposers.

136. *Proposed by Florin Stanescu, Serban Cioculescu school, city Gaesti, jud. Dambovita, Romania.* We consider a twice continuously differentiable function $f:\mathbb{R}\to\mathbb{R}$. Let $(a_n)_{n\geq 1}$ be a sequence such that $a_{i+1}\neq a_i$ for every i, and suppose that the following two conditions hold:

a) $\lim_{n\to\infty}\left((n+1)^2 f(a_{n+1}) - n^2 f(a_n)\right) = 0$,

b) $\lim_{n\to\infty} \frac{f(a_{n+1})-f(a_n)}{a_{n+1}-a_n} = \theta > 0$.

$i)$ Find an example of such a function f and a sequence $(a_n)_{n\geq 1}$.
$ii)$ Show that $\lim_{n\to\infty} f'(a_n) = \theta$.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria.

$i)$ Take $f(x) = x$ and $a_n = \frac{1}{n\sqrt{n}}$.

$ii)$ Using Cesàro's lemma we conclude from **a)** that $\lim_{n\to\infty} nf(a_n) = 0$. Combining this with **a)** again we conclude that $\lim_{n\to\infty} n^2(f(a_{n+1}) - f(a_n)) = 0$. But,

$$n^2(a_{n+1}-a_n) = n^2(f(a_{n+1})-f(a_n)) \times \frac{a_{n+1}-a_n}{f(a_{n+1})-f(a_n)}$$

so, using **b)** we conclude that $\lim_{n\to\infty} n^2(a_{n+1}-a_n) = 0$. This proves that the series $\sum(a_{n+1}-a_n)$ is convergent, and consequently the sequence $(a_n)_{n\geq 1}$ is convergent. to some limit ℓ.

Now, by the mean value theorem, for each n there is a real number ξ_n strictly between a_n and a_{n+1} such that

$$f'(\xi_n) = \frac{f(a_{n+1})-f(a_n)}{a_{n+1}-a_n}$$

and since $\lim_{n\to\infty} a_n = \ell$ we conclude that $\lim_{n\to\infty} \xi_n = \ell$. Further, using the continuity of f' at $x = \ell$ we conclude also that

$$\lim_{n\to\infty} f'(a_n) = f'(\ell) = \lim_{n\to\infty} f'(\xi_n) = \lim_{n\to\infty} \frac{f(a_{n+1})-f(a_n)}{a_{n+1}-a_n} = \theta.$$

which is the desired conclusion.

Remark. As the proof shows, we only need to assume that f has a continuous derivative.

Also solved by the proposer.

137. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria and Anastasios Kotronis, Athens, Greece (Jointly).* Let n be a nonnegative integer, m, p be positive integers and $x \in \mathbb{C}$. Show that for the values of n, p, m, x for which the denominators don't vanish, the following identity holds:

$$\sum_{k=0}^{n}(-1)^k\frac{\binom{n}{k}\binom{x+n-k}{m-k}}{(p+n-k)\binom{x+n-k}{p+n-k}}=\sum_{k=0}^{n}(-1)^k\frac{\binom{n}{k}\binom{x}{m-k}}{(p+n-k)\binom{x+n}{p+n-k}}$$

$$=\begin{cases}(-1)^n\cdot\frac{\binom{x-p}{m-p-n}}{p\binom{m}{p}} & ,m\geq p\\ \\ \frac{\binom{p-1}{m}}{(p-m+n)\binom{x-m+n}{p-m+n}} & ,m<p.\end{cases}$$

Solution 1 by Ramya Dutta, Chennai Mathematical Institute (student) India and dhe proposer. We denote the Forward Difference operator of spacing 1 with a Δ,

$$\Delta(f(x))=f(x+1)-f(x)$$

and the n-fold composition of the operator with $\Delta^n(f(x))=\Delta(\Delta^{n-1}(f(x)))$. Let, f,g be two real/complex valued functions, then the Liebniz rule for higher order difference of product of f,g states,

$$\Delta^n(f(x)g(x))=\sum_{k=0}^{n}\binom{n}{k}\Delta^k(f(x)).\Delta^{n-k}(g(x+k))\qquad\cdots(1)$$

We consider, $f_m(x)=\binom{x}{m}$ and $g_p(x)=\frac{1}{p\binom{x}{p}}$ (where, $x\in\mathbb{C}$).

Then, $\Delta(f_m(x))=\binom{x+1}{m}-\binom{x}{m}=\binom{x}{m-1}=f_{m-1}(x)$, and inductively we have,

$$\Delta^k(f_m(x))=f_{m-k}(x)=\binom{x}{m-k}\qquad\cdots(2)$$

Also,

$$\Delta(g_p(x))=\frac{1}{p\binom{x+1}{p}}-\frac{1}{p\binom{x}{p}}=\frac{-\binom{x}{p-1}}{p\binom{x+1}{p}\binom{x}{p}}=\frac{-1}{(p+1)\binom{x+1}{p+1}}=-g_{p+1}(x+1)$$

and inductively we have,

$$\Delta^k(g_p(x))=(-1)^kg_{p+k}(x+k)=\frac{(-1)^k}{(p+k)\binom{x+k}{p+k}}\qquad\cdots(3)$$

Thus, using (2) and (3) in the RHS of equation (1):

$$\sum_{k=0}^{n}\binom{n}{k}\Delta^k(f_m(x)).\Delta^{n-k}(g_p(x+k))=\sum_{k=0}^{n}\binom{n}{k}\binom{x}{m-k}\frac{(-1)^{n-k}}{(p+n-k)\binom{x+n}{p+n-k}}$$

$$=(-1)^n\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\binom{x}{m-k}}{(p+n-k)\binom{x+n}{p+n-k}}\qquad\cdots(4)$$

and on the other hand for the LHS of equation (1),
If $m\geq p$, then

$$f_m(x)g_p(x)=\frac{\binom{x}{m}}{p\binom{x}{p}}=\frac{1}{p\binom{m}{p}}\binom{x-p}{m-p}=\frac{1}{p\binom{m}{p}}f_{m-p}(x-p)$$

thus,

$$\Delta^n(f_m(x)g_p(x)) = \frac{1}{p\binom{m}{p}}\Delta^n(f_{m-p}(x-p))$$
$$= \frac{1}{p\binom{m}{p}} f_{m-p-n}(x-p) = \frac{\binom{x-p}{m-p-n}}{p\binom{m}{p}}$$

and, if $m < p$, then

$$f_m(x)g_p(x) = \frac{\binom{x}{m}}{p\binom{x}{p}} = \binom{p-1}{m}\frac{1}{(p-m)\binom{x-m}{p-m}} = \binom{p-1}{m} g_{p-m}(x-m)$$

thus,

$$\Delta^n(f_m(x)g_p(x)) = \binom{p-1}{m}\Delta^n(g_{p-m}(x-m))$$
$$= (-1)^n\binom{p-1}{m} g_{p-m+n}(x-m+n)$$
$$= (-1)^n\frac{\binom{p-1}{m}}{(p-m+n)\binom{x-m+n}{p-m+n}}$$

Thus, combining (1) and (4),

$$S = \sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}\binom{x}{m-k}}{(p+n-k)\binom{x+n}{p+n-k}} = \begin{cases} (-1)^n\dfrac{\binom{x-p}{m-p-n}}{p\binom{m}{p}} & \text{when } m \geq p \\ \dfrac{\binom{p-1}{m}}{(p-m+n)\binom{x-m+n}{p-m+n}} & \text{when } m < p \end{cases}$$

Reversing the role of f_m and g_p in (1) we get the other equality,

$$S = \sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}\binom{x+n-k}{m-k}}{(p+n-k)\binom{x+n-k}{p+n-k}}$$

Solution 2 by the proposers. Let us denote with Δ, E and I the forward difference, the shift and the identity operator respectively, which act on functions as follows:

$$f \overset{\Delta}{\mapsto} f(x+1) - f(x), \qquad f \overset{E}{\mapsto} f(x+1), \qquad f \overset{I}{\mapsto} f(x).$$

The following Lemma holds:

Lemma 1. *With $\Delta^0 = I$, for two functions f, g we have:*

$$\Delta^n(fg) = \sum_{i=0}^{n}\binom{n}{i}\Delta^i f\Delta^{n-i}(E^i g), \qquad n \in \cup\{0\}. \tag{1}$$

Proof. Since

$$f(x+1)g(x+1) - f(x)g(x) = (f(x+1) - f(x))\cdot g(x+1) + f(x)\cdot(g(x+1) - g(x))$$

we conclude that $\Delta(fg) = \Delta f \cdot Eg + f \cdot \Delta g$ which is the case $n = 1$. Now suppose the result is proved for $n-1$. Applying Δ^{n-1} to both sides of the above formula, and using the induction hypothesis we obtain

$$\Delta^n(fg) = \sum_{i=0}^{n-1}\binom{n-1}{i}\Delta^i(\Delta f)\Delta^{n-1-i}(E^i(Eg)) + \sum_{i=0}^{n-1}\binom{n-1}{i}\Delta^i f\Delta^{n-1-i}(E^i(\Delta g)) \quad (E\Delta = \Delta E)$$
$$= \sum_{i=0}^{n-1}\binom{n-1}{i}\Delta^{i+1} f\Delta^{n-1-i}(E^{i+1}g) + \sum_{i=0}^{n-1}\binom{n-1}{i}\Delta^i f\Delta^{n-i}(E^i g)$$
$$= \sum_{i=1}^{n}\binom{n-1}{i-1}\Delta^i f\Delta^{n-i}(E^i g) + \sum_{i=0}^{n-1}\binom{n-1}{i}\Delta^i f\Delta^{n-i}(E^i g)$$
$$= \sum_{i=0}^{n}\left(\binom{n-1}{i-1} + \binom{n-1}{i}\right)\Delta^i f\Delta^{n-i}(E^i g)$$
$$= \sum_{i=0}^{n}\binom{n}{i}\Delta^i f\Delta^{n-i}(E^i g).$$

For $n = 0$ (1) is trivially true and the lemma is proved. □

For a nonnegative integer m, and positive integer p, consider the functions

$$f_m(x) := \binom{x}{m}, \qquad \text{and} \qquad g_p(x) := \frac{1}{p\binom{x}{p}}.$$

We observe that, for $m > 0$ we have $\Delta f_m = f_{m-1}$ and, inductively $\Delta^k f_m = f_{m-k}$, for $0 \le k \le m$. Note that this formula remains valid for $k > m$ since both sides become 0 in this case, using the well-known convention : $\binom{a}{j} = 0$ if j is a negative integer.
Similarly $\Delta g_p = -Eg_{p+1}$ and, inductively $\Delta^k g_p = (-1)^k E^k g_{p+k}$, for $k \ge 0$.
Furthermore, it is clear that

$$f_m(x)g_p(x) = \begin{cases} g_p(m)f_{m-p}(x-p), & m \ge p \\ f_{p-1}(m)g_{p-m}(x-m), & m < p \end{cases}$$

so, from the above and the Lemma we get

$$\Delta^n(f_m g_p) = \sum_{k=0}^{n}\binom{n}{k} f_{m-k}(-1)^{n-k}E^n g_{p+n-k} = \begin{cases} g_p(m)f_{m-p-n}(x-p), & m \ge p \\ (-1)^n f_{p-1}(m)g_{p-m+n}(x+n-m), & m < p \end{cases}$$

Multiplying both sides by $(-1)^n$ we obtain

$$\sum_{k=0}^{n}\frac{(-1)^k\binom{n}{k}\binom{x}{m-k}}{(p+n-k)\binom{x+n}{p+n-k}} = \begin{cases} (-1)^n\frac{\binom{x-p}{m-p-n}}{p\binom{m}{p}}, & m \ge p \\ \frac{\binom{m}{p-1}}{(p-m+n)\binom{x-m+n}{p-m+n}}, & m < p \end{cases}$$

which is the one of the two equalities we wanted.
Reversing the roles of f_m and g_p in (1) and performing the change of variables $n - k \mapsto k$ we get the other equality.

111. Edited. *Proposed by Moti Levy, Rehovot, Israel.* Let m, n be integers. Show that if $n \ge \frac{3m}{2} \ge 3$ then

$$\frac{x^n}{x^m + y^m} + \frac{y^n}{y^m + z^m} + \frac{z^n}{z^m + x^m} \ge \frac{3}{2}\left(\frac{1}{\sqrt{3}}\right)^{n-m}$$

where real $x, y, z > 0$ and $xy + yz + zx = 1$.

Solution by the proposer.

$$\frac{x^n}{x^m+y^m}+\frac{y^n}{y^m+z^m}+\frac{z^n}{z^m+x^m}$$
$$=x^{n-m}-\frac{x^{n-m}y^m}{x^m+y^m}+y^{n-m}-\frac{y^{n-m}z^m}{y^m+z^m}+z^{n-m}-\frac{z^{n-m}x^m}{z^m+x^m}.$$

Since $x^m+y^m \geq 2x^{\frac{m}{2}}y^{\frac{m}{2}}$ then

$$\frac{x^n}{x^m+y^m}+\frac{y^n}{y^m+z^m}+\frac{z^n}{z^m+x^m}$$
$$\geq x^{n-m}-\frac{x^{n-m}y^m}{2x^{\frac{m}{2}}y^{\frac{m}{2}}}+y^{n-m}-\frac{y^{n-m}z^m}{2y^{\frac{m}{2}}z^{\frac{m}{2}}}+z^{n-m}-\frac{z^{n-m}x^m}{2z^{\frac{m}{2}}x^{\frac{m}{2}}}$$
$$=x^{n-m}+y^{n-m}+z^{n-m}-\frac{1}{2}\left(x^{n-\frac{3m}{2}}y^{\frac{m}{2}}+y^{n-\frac{3m}{2}}z^{\frac{m}{2}}+z^{n-\frac{3m}{2}}x^{\frac{m}{2}}\right)$$

Assume, w.l.o.g, that $x \leq y \leq z$; then $x^{\frac{m}{2}} \leq y^{\frac{m}{2}} \leq z^{\frac{m}{2}}$ and $x^{n-\frac{3m}{2}} \leq y^{n-\frac{3m}{2}} \leq z^{n-\frac{3m}{2}}$.

By rearrangement inequality,

$$x^{n-m}+y^{n-m}+z^{n-m} \geq x^{n-\frac{3m}{2}}y^{\frac{m}{2}}+y^{n-\frac{3m}{2}}z^{\frac{m}{2}}+z^{n-\frac{3m}{2}}x^{\frac{m}{2}}$$
$$\frac{x^n}{x^m+y^m}+\frac{y^n}{y^m+z^m}+\frac{z^n}{z^m+x^m} \geq \frac{1}{2}\left(x^{n-m}+y^{n-m}+z^{n-m}\right) \qquad (1)$$

Now we prove by mathematical induction that for positive integer p,

$$x^p+y^p+z^p \geq \left(\sqrt{3}\right)^{2-p}. \qquad (2)$$

Proof:

$$(x+y+z)^2 = x^2+y^2+z^2+2(xy+yz+zx)$$

By rearrangement inequality,

$$x^2+y^2+z^2 \geq xy+yz+zx.$$
$$(x+y+z)^2 \geq 3(xy+yz+zx) = 3$$

Then for $p = 1$,

$$x+y+z \geq \sqrt{3}. \qquad (3)$$

Now suppose that (3) is true.

$$(x+y+z)(x^p+y^p+z^p) = x^{p+1}+y^{p+1}+z^{p+1}+xy^p+xz^p+yx^p+yz^p+zx^p+zy^p$$

By Chebyshev's inequality

$$xy^p+xz^p+yx^p+yz^p+zx^p+zy^p \leq 2\left(x^{p+1}+y^{p+1}+z^{p+1}\right)$$
$$(x+y+z)(x^p+y^p+z^p) \leq 3\left(x^{p+1}+y^{p+1}+z^{p+1}\right)$$

By the induction hypothesis (2) and (3) , we obtain

$$x^{p+1}+y^{p+1}+z^{p+1} \geq \frac{1}{3}\sqrt{3}\left(\sqrt{3}\right)^{2-p} = \left(\sqrt{3}\right)^{2-(p+1)}.$$

Moti Levy's COMMENT. I stumbled accidentally on a discussion in "math-Overflow" forum regarding Problem 111, Volume 5, Issue 1 (2015), Pages 369-406 which I have proposed. Professor Peter Muller from Wurzburg University (and maybe others) stated that the inequality in the problem is false and gave counterexample. I checked his counterexample and found out that he is right! So I checked again the problem statement and my proof and concluded that the condition on n and m in the problem statement must be changed in order to get a correct inequality. The condition $n > m \geq 0$ must be replaced by $n \geq \frac{3m}{2} \geq 3$.

MATHCONTEST SECTION

This section of the Journal offers readers an opportunity to solve interesting and elegant mathematical problems mainly appeared in Math Contest around the world and most appropriate for training Math Olympiads. Proposals are always welcomed. The source of the proposals will appear when the solutions be published.

Proposals

95. Let $n \in N$ and let $O_n = 1 + \frac{1}{3} + ... + \frac{1}{2n-1}$. Calculate $\lim\limits_{n\to\infty} \frac{1}{n}\left(1 + \frac{2O_n}{n}\right)^n$.

96. Let p a positive real number and let $\{a_n\}_{n\geq 1}$ be a sequence defined by $a_1 = 1, a_{n+1} = \frac{a_n}{1+a_n^p}$. Find those real values $q \neq 0$ such that the following series converges $\sum\limits_{n=1}^{\infty} \left|(pn)^{-\frac{1}{p}} - a_n\right|^q$.

97. Let $n \in N$, for k integer, $1 \leq k \leq n$, euclidean division n by k gives $n = qk + n_k$, and denote p_n the probability that $n_k \geq \frac{k}{2}$. Calculate p_n and find $\lim\limits_{n\to\infty} p_n$.

98. Let $(x_n)_{n\geq 0}$ be the sequence defined recurrently by $x_{n+2} = x_{n+1} - \frac{1}{2}x_n$ with initial terms $x_0 = 2$ and $x_1 = 1$. Find $\displaystyle\sum_{n=1}^{\infty} \frac{x_n}{n+2}$.

99. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$mn \cdot (f(m) - nm) \cdot (n - f(n^2))$$

is a square for all $m, n \in \mathbb{N}$.

Solutions

90. Let f and g be two continuous, distinct functions from $[0,1]$ to $(0,+\infty)$ such that $\int_0^1 f(x)dx = \int_0^1 g(x)dx$. Let $y_n = \int_0^1 \frac{f^{n+1}(x)}{g^n(x)}dx$ for $n \geq 0$. Prove that $(y_n)_{n\geq 1}$ is an increasing and divergent sequence.

(Brazil Undergrad MO 2005)

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.
Let $A = \{x \in (0,1) : f(x) > g(x)\}$ and $B = \{x \in (0,1) : f(x) < g(x)\}$. These are open sets because of the continuity of f and g. Moreover, if $A = \emptyset$ then $f(x) \leq g(x)$ for every $x \in [0,1]$ and the assumption $\int_0^1 f(x)dx = \int_0^1 g(x)dx$ would then imply that $f = g$ which is absurd. Thus, $A \neq \emptyset$ and similarly $B \neq \emptyset$.
Now, clearly we have

$$\int_A (f(x) - g(x))dx = \int_B (g(x) - f(x))dx$$

hence

$$\begin{aligned}\int_A \frac{f(x)}{g(x)}(f(x) - g(x))dx &> \int_A (f(x) - g(x))dx \\ &= \int_B (g(x) - f(x))dx > \int_B \frac{f(x)}{g(x)}(g(x) - f(x))dx\end{aligned}$$

or equivalently

$$y_1 = \int_0^1 \frac{f^2(x)}{g(x)}dx > \int_0^1 f(x)dx = y_0$$

On the other hand

$$y_{n+2} - 2y_{n+1} + y_n = \int_0^1 \frac{f^{n+1}(x)}{g^{n+2}(x)}(f(x) - g(x))^2 dx > 0$$

This proves that the sequence $(y_{n+1} - y_n)_{n\geq 0}$ is increasing. In particular,

$$y_{n+1} - y_n \geq y_1 - y_0 > 0, \qquad \text{for every } n.$$

Thus $(y_n)_{n\geq 0}$ is increasing.
Moreover, a simple induction shows that $y_n \geq y_0 + n(y_1 - y_0)$ for every n. Thus $\lim_{n\to\infty} y_n = +\infty$.

Solution 2 by Michel Bataille, Rouen, France.
Note that $y_n > 0$ for all $n \geq 0$ (since $\frac{f^{n+1}}{g^n}$ is a continuous, positive function). First, we have $y_1 \geq y_0$ since by the Cauchy-Schwarz inequality,

$$y_1 \cdot y_0 = \left(\int f\right) \cdot \left(\int \frac{f^2}{g}\right) = \left(\int g\right) \cdot \left(\int \frac{f^2}{g}\right) \geq \left(\int \sqrt{g} \cdot \frac{f}{\sqrt{g}}\right)^2 = \left(\int f\right)^2 = y_0^2 \qquad (1).$$

(here and in what follows, for sake of simplification, we use the notation $\int h$ for $\int_0^1 h(x)\,dx$.)

Using the Cauchy-Schwarz inequality again, we obtain

$$y_n \cdot y_{n+2} = \int \frac{f^{n+1}}{g^n} \cdot \int \frac{f^{n+3}}{g^{n+2}} \geq \left(\int \frac{f^{n+2}}{g^{n+1}} \right)^2 = y_{n+1}^2$$

and so $\frac{y_{n+2}}{y_{n+1}} \geq \frac{y_{n+1}}{y_n}$ for all $n \geq 0$. The sequence $\left(\frac{y_{n+1}}{y_n}\right)_{n\geq 0}$ is increasing, and in particular $\frac{y_{n+1}}{y_n} \geq \frac{y_1}{y_0} \geq 1$ for all $n \geq 0$. Thus, $y_{n+1} \geq y_n$ for all $n \geq 0$ and the sequence (y_n) is increasing.

For the purpose of a contradiction, assume that (y_n) is convergent and let ℓ be its limit. Note that $\ell > 0$ (since $\ell \geq y_0$). Since $\lim\limits_{n\to\infty} \frac{y_{n+1}}{y_n} = \frac{\ell}{\ell} = 1$ and $\left(\frac{y_{n+1}}{y_n}\right)_{n\geq 0}$ is increasing, we have $\frac{y_1}{y_0} \leq 1$ and so $y_1 = y_0$. This means that equality holds in (1) so that $\frac{f}{\sqrt{g}} = \lambda\sqrt{g}$ for some (positive) constant λ. But then $f = \lambda g$ and since $\int f = \int g$, we must have $\lambda = 1$ and so $f = g$, in contradiction with the hypotheses. We conclude that the sequence (y_n) is divergent (more precisely $\lim_{n\to\infty} y_n = \infty$ since (y_n) is increasing).

Also solved by Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Rumania.

91. Let $(a_n)_{n\geq 1} \subset (\frac{1}{2}, 1)$. Define the sequence $x_0 = 0$, $x_{n+1} = \dfrac{a_{n+1} + x_n}{1 + a_{n+1}x_n}$. Is this sequence convergent? if yes find the limit.

(IMC 2011)

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Let $b_n = \tanh^{-1}(a_n) = \frac{1}{2}\log\frac{1+a_n}{1-a_n}$ for every $n \geq 1$, and let $(y_n)_{n\geq 0}$ be defined by $y_n = \sum_{k=1}^n b_k$ with the convention $y_0 = 0$. A simple proof by induction shows that $x_n = \tanh(y_n)$ for every n, using the fact that

$$\tanh(b + y) = \frac{\tanh(b) + \tanh(y)}{1 + \tanh(b)\tanh(y)}$$

Now, since $\frac{1}{2} < a_n$ we conclude that $\frac{1}{2}\log 3 < b_n$ for every n. Thus $\lim_{n\to\infty} y_n = +\infty$ and consequently $\lim_{n\to\infty} x_n = 1$ since $\lim_{x\to\infty}\tanh(x) = 1$.

Solution 2 by Michel Bataille, Rouen, France. We show that the sequence (x_n) is convergent with limit 1. For every positive integer n, since $\frac{1}{2} < a_n < 1$, we have $a_n = \tanh b_n$ for some $b_n > \tanh^{-1}(1/2) = \ln(\sqrt{3})$. Then, we may write $x_0 = \tanh 0$ and so $x_1 = \frac{\tanh b_1 + \tanh 0}{1 + \tanh b_1 \cdot \tanh 0} = \tanh(b_1 + 0) = \tanh b_1$. By an easy induction using the formula $\tanh(x+y) = \frac{\tanh x + \tanh y}{1+\tanh x \cdot \tanh y}$, we obtain

$$x_n = \tanh(b_1 + b_2 + \cdots + b_n)$$

for every positive integer n.

Since $b_1 + b_2 + \cdots + b_n \geq n\ln(\sqrt{3})$ and $\ln(\sqrt{3}) > 0$, we see that $\lim\limits_{n\to\infty}(b_1 + b_2 + \cdots + b_n) = \infty$. Since $\lim\limits_{x\to\infty}\tanh(x) = 1$, it follows that (x_n) is convergent and $\lim\limits_{n\to\infty} x_n = 1$.

92. For a positive integer n, define $f(n)$ to be the number of sequences $(a_1, a_2, \ldots, a_k)$ such that $a_1a_2\cdots a_k = n$ where $a_i \geq 2$ and $k \geq 0$ is arbitrary. Also we define $f(1) = 1$. Now let $\alpha > 1$ be the unique real number satisfying $\sum_{n=1}^{\infty}\frac{1}{n^\alpha} = 2$.

Prove that

(a)

$$\sum_{j=1}^{n} f(j) = \mathcal{O}(n^{\alpha})$$

(b) There is no real number $\beta < \alpha$ such that

$$\sum_{j=1}^{n} f(j) = \mathcal{O}(n^{\beta}).$$

(Miklos Schweitzer 2014)

Solution by Drago Grozev (AOPS), Bulgaria. Let $g(n) = \sum_{j=1}^{n} f(j)$. Clearly:

$$g(n) = |\{(a_1, a_2, \ldots, a_k) \mid a_1 a_2 \ldots a_k \leq n\,,\ a_j \geq 2\}|$$

To avoid unnecessary notations let's set $g(x) = g(\lfloor x \rfloor)\,,\ x > 0$. Since a_1 can take values $2, 3, \ldots, \lfloor n/2 \rfloor$ we obtain:

$$g(n) = \sum_{j=2}^{\lfloor n/2 \rfloor} g\left(\frac{n}{j}\right) \qquad (1)$$

Here ends up the combinatorial and NT trait of the problem, that is, the property (1) of the function g is enough to prove a) and b). **a)** We proceed by induction. If $g(j) \leq C \cdot j^{\alpha}\,,\ j = 1, 2, \ldots, m$ and $n \leq 2m$ we have:

$$g(n) = \sum_{j=2}^{\lfloor n/2 \rfloor} g\left(\frac{n}{j}\right) \leq C \cdot n^{\alpha}\,(\zeta(\alpha) - 1) = C \cdot n^{\alpha}$$

b) Suppose on tha contrary there exists $\beta < \alpha$ and absolute constant C with $g(n) \leq C \cdot n^{\beta}$. Denote $c_j = g(j)/j^{\beta}\,,\ j = 1, 2, \ldots$. Clearly $0 < c_j \leq C$. Let $c = \liminf c_j$. There are two possibilities: either $c > 0$ or $c = 0$. Let's consider both of them. 1) Suppose $c > 0$. We take $\varepsilon > 0$, for which $(\zeta(\beta) - 1)(c - \varepsilon) > c + \varepsilon$. It's possible, since $\zeta(\beta) > 2$. Now, $c_j > c - \varepsilon$ for all but finite numbers j. Thus, for sufficiently big n it holds:

$$g(n) = \sum_{j=2}^{\lfloor n/2 \rfloor} g\left(\frac{n}{j}\right) > (c - \varepsilon) n^{\beta} \sum_{j=2}^{m} 1/j^{\beta}$$

where $m \to \infty$ as $n \to \infty$. It means

$$g(n) \geq (c + \varepsilon) n^{\beta}$$

for sufficiently big n. Therefore $\liminf c_n \geq c + \varepsilon$, a contradiction. 2) Suppose $c = 0$. Then for infinitely many n it holds $c_n = \min\{c_1, c_2, \ldots, c_n\}$. For those n, using (1), we get:

$$c_n > c_n \left(\sum_{j=2}^{\lfloor n \rfloor} 1/j^{\beta} \right)$$

Thus, for sufficiently big of those n, $c_n > c_n$, a contradiction.

93. Let $c \geq 1$ be a real number. Let G be an Abelian group and let $A \subset G$ be a finite set satisfying $|A + A| \leq c|A|$, where $X + Y := \{x + y | x \in X, y \in Y\}$ and $|Z|$

denotes the cardinality of Z. Prove that

$$|\underbrace{A+A+\cdots+A}_{k}| \le c^k|A|$$

for every positive integer k.

(IMC 2012)

Solution. We didn't receive any solution. The official solution is in the following link. http://www.imc-math.org.uk/imc2012/IMC2012-day2-solutions.pdf

94. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$(f(x+y+z))^2 = (f(x))^2 + (f(y))^2 + (f(z))^2 + 2(f(xy) + f(xz) + f(yz)),$$

for all x, y, z real numbers.

(Art of Problem Solving 2016)

Solution by Michel Bataille, Rouen, France. It is readily checked that the constant functions $x \mapsto 0$ and $x \mapsto -3$ are solutions and so are the functions $x \mapsto x$ and $x \mapsto x-3$. We now show that there are no other solutions.
Let $E(x, y, z)$ denote the equality

$$(f(x+y+z))^2 = (f(x))^2 + (f(y))^2 + (f(z))^2 + 2(f(xy) + f(xz) + f(yz))$$

and let $m = f(0)$. From $E(0, 0, 0)$, we obtain $m^2 + 3m = 0$, hence $m = 0$ or $m = -3$.
Let x, y be any real numbers. From $E(x, -x, -x)$ and $E(x, y, 0)$ we get

$$(f(x))^2 + (f(-x))^2 + 2f(x^2) + 4f(-x^2) = 0 \qquad (1)$$

$$(f(x+y))^2 = (f(x))^2 + (f(y))^2 + 2f(xy) + m^2 + 4m \qquad (2),$$

and the latter, with $y = -x$ gives

$$(f(x))^2 + (f(-x))^2 + 2f(-x^2) = -4m \qquad (3).$$

Comparing (1) and (3) yields

$$f(x^2) + f(-x^2) = 2m \qquad (4).$$

We set $g(x) = f(x) - m$ (where $m = 0$ or $m = -3$). Then, from (4), we deduce that the function g is odd, a property that allows one to rewrite (1) as $2(g(x))^2 - 2g(x^2) = -6m - 2m^2 = 0$ so that

$$(g(x))^2 = g(x^2) \qquad (5).$$

Taking $x = 1$ in (5) gives $g(1) = 0$ or $g(1) = 1$. We consider the two cases.
• if $g(1) = 0$, then $f(1) = f(-1) = m$ and from (2) (and (4)) we deduce

$$(f(x+1))^2 = (f(x))^2 + 2f(x) + 2m^2 + 4m \quad \text{and} \quad (f(x-1))^2 = (f(x))^2 - 2f(x) + 2m^2 + 8m$$

so that

$$(f(x+1))^2 + (f(x-1))^2 = 2(f(x))^2 \quad (x \in \mathbb{R}).$$

It follows that the function h defined by $h(x) = (f(x))^2 - (f(x-1))^2$ is 1-periodic. In addition, we have $4f(x) - 4m = (f(x+1))^2 - (f(x-1))^2 = 2h(x)$ so that f itself is 1-periodic and the last equality then implies that h is the zero function and so is $g = \frac{1}{2}h$. Thus, $f(x) = 0$ for all x [if $m = 0$] or $f(x) = -3$ for all x [if $m = -3$].
• if $g(1) = 1$, then $f(1) = m + 1$ and this time (2) and (4) give that for all real x:

$$(f(x+1))^2 = (f(x)+1)^2 \quad \text{and} \quad (f(x-1))^2 = (f(x)-1)^2.$$

Thus, $f(x+1) = f(x)+1$ or $f(x+1) = -f(x)-1$. But in the latter case, the second relation gives $(f(x))^2 = (f(x+1-1))^2 = (f(x+1)-1)^2 = (-f(x)-2)^2 = (f(x))^2+4f(x)+4$, hence $f(x) = -1$ and so $f(x+1) = -(-1)-1 = 0 = f(x)+1$. As a result, $f(x+1) = f(x)+1$ holds for all real numbers x. By induction, it immediately follows that $f(x+n) = f(x)+n$ for all $n \in \mathbb{N}$ and all real x and then $f(x) = f(x-n)+n$ so that $f(x+k) = f(x)+k$ whenever $x \in \mathbb{R}$ and $k \in \mathbb{Z}$. Clearly, we also have $g(x+k) = g(x)+k$ for $x \in \mathbb{R}, k \in \mathbb{Z}$. From (2), $(f(x)+k)^2 = (f(x))^2+(k+m)^2+2f(kx)+m^2+4m$ from which we easily obtain $kf(x) = f(kx)+m^2+2m+km$ and then $g(kx) = kg(x)$ $(x \in \mathbb{R}, k \in \mathbb{Z})$. Classically, we deduce $g(rx) = rg(x)$ and $g(x+r) = g(x)+r$ for $x \in \mathbb{R}, r \in \mathbb{Q}$.

Using these results (which in particular imply $f(-m) = 0$), the relation $E(x,y,-m)$ leads to

$$(g(x+y))^2 = (f(x+y-m))^2 = (f(x))^2+(f(y))^2+2(f(xy)+f(-mx)+f(-my))$$

and in consequence $(g(x+y))^2 = (g(x))^2+(g(y))^2+2g(xy)$.

It follows that $(g(x+y))^2-(g(x-y))^2 = 4g(xy)$. With $x = \frac{u+v}{2}, y = \frac{u-v}{2}$ and observing that $g(t) \geq 0$ when $t \geq 0$ (by (5)), we deduce that for $u > v > 0$, we have $(g(u))^2-(g(v))^2 = 4g\left(\frac{u^2-v^2}{4}\right) \geq 0$ and so $g(u) \geq g(v)$. Thus g is nondecreasing on $(0,\infty)$ and even on $\mathbb{R}$ since g is odd. To conclude, let x be any real number and let (r_n) and (s_n) be two sequences of rational numbers converging to x and such that $r_n \leq x \leq s_n$ for all positive integers n. Then $r_n = g(r_n) \leq g(x) \leq g(s_n) = s_n$ and letting $n \to \infty$, we obtain $g(x) = x$. Thus g is the function $x \mapsto x$ and f is either the function $x \mapsto x$ or the function $x \mapsto x-3$. This completes the proof.

Also solved by Dorlir Ahmeti, University of Prishtina, Republic of Kosova.

MATHNOTES SECTION

Nesbitt-Ionescu type inequalities Some generalizations of problem 37, MathProblems (4) 4 (2014)

D.M. Bătineţu-Giurgiu and Neculai Stanciu

Abstract. In this article we generalize a problem that was proposed by Murray Klamkin and Andy Liu on College Mathematical Journal and a closely related one that was discussed on the Art Of Problem Solving forum. We approach the problem in two ways. Flipping Romanian Mathematical Gazette, Volume XXXII (September 15, 1926 - August 15, 1927), we found with great astonishment, at p. 120 that *Ion Ionescu* - one of the founders and pillars of Mathematical Gazette, published the problem **3478.** *If* x, y, z*are positive, show that*:

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq\frac{3}{2}.$$

In the same volume, at pp. 194-196, are presented two solutions to this problem, as well as a generalization. From [2] yields that *Nesbitt* published the inequality in 1903. It is appropriate to say here that the problem. In any triangle ABC, with usual notations holds the inequality

$$a^2+b^2+c^2\geq 4\sqrt{3}S,$$

was published first by *Ion Ionescu* in 1897 (see [1]), and also published by *Roland Weitzenböck* in 1919. However, this inequality has long been called " *Weitzenböck*'s inequality", and after the appearance of paper [1] is called *Ionescu-Weitzenböck* inequality. Compared to the above we suggest that inequality (*) to be called *Nesbitt-Ionescu* inequality. In the next, we will do a retrospective of our results on *Nesbitt-Ionescu* inequality and we shall give some generalizations of problem 37 from MathProblems (see [3]).

1. Main Results

The inequality of the problem 37 from MathProblems (see [3]) is an Ionescu-Nesbitt type inequality. In the following we shall presents some generalizations of this problem.

Theorem 1. Let $a\in\mathbb{R}_+$ and $b,c,d,x,y,z\in\mathbb{R}_+^*$ such that $cX>d\max\{x,y,z\}$, then

$$\sum_{cyc}\frac{aX+bx}{cX-dx}\geq\frac{3(3a+b)}{3c-d},$$

where $X=x+y+z$.

Proof. Let $V=\sum_{cyc}\frac{aX+bx}{cX-dx}$, then

$$V=\sum_{cyc}\frac{(aX+bx)^2}{(aX+bx)(cX-dx)}=\sum_{cyc}\frac{(aX+bx)^2}{acX^2+(bc-ad)Xx-bdx^2}\overset{Bergstrom}{\geq}$$

$$\stackrel{Bergstrom}{\geq} \frac{\left(\sum_{cyc}(aX+bx)\right)^2}{\sum_{cyc}\left(acX^2+(bc-ad)Xx-bdx^2\right)} = \frac{\left(3aX+b\sum_{cyc}x\right)^2}{3acX^2+(bc-ad)X^2-bd\sum_{cyc}x^2}.$$

But, $x^2+y^2+z^2 \geq \frac{(x+y+z)^2}{3} = \frac{X^2}{3}$. Then by yields that

$$V \geq \frac{(3a+b)^2X^2}{\left(3ac+bc-ad-\frac{bd}{3}\right)X^2} = \frac{3(3a+b)^2}{9ac+3bc-3ad-bd} = \frac{3(3a+b)^2}{(3a+b)(3c-d)} = \frac{3(3a+b)}{3c-d}.$$

We have equality if $x=y=z$.

Corollary 1. If in theorem 1 we take $a=0, b=c=d=1$, then we obtain Nesbitt-Ionescu's inequality, i.e.

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y} \geq \frac{3}{2}.$$

Corollary 2. If in theorem 1 we take $a=b=2, c=4, d=1$, then we obtain

$$\frac{2x+2y+4z}{4x+4y+3z}+\frac{2x+4y+2z}{4x+3y+4z}+\frac{4x+2y+2z}{3x+4y+4z} \geq \frac{24}{11},$$

i.e the problem 37 from Junior MathProblems.

Theorem 2. Let $a \in \mathbb{R}_+, m \in [1,\infty)$ and $b,c,d,x,y,z \in \mathbb{R}_+^*$ such that $cX > d\max\{x,y,z\}$, then

$$\frac{aX+bx}{(cX-dx)^m}+\frac{aX+by}{(cX-dy)^m}+\frac{aX+bz}{(cX-dz)^m} \geq \frac{3^m(3a+b)}{(3c-d)^mX^{m-1}},$$

where $X=x+y+z$.

Proof. Let $W=\frac{aX+bx}{(cX-dx)^m}+\frac{aX+by}{(cX-dy)^m}+\frac{aX+bz}{(cX-dz)^m} \geq \frac{3^m(3a+b)}{(3c-d)^mX^{m-1}}$. By J. Radon's inequality and $x^2+y^2+z^2 \geq \frac{X^2}{3}$, we have

$$W=\sum_{cyc}\frac{aX+bx}{(cX-dx)^m}=\sum_{cyc}\frac{(aX+bx)^{m+1}}{((aX+bx)(cX-dx))^m}=\sum_{cyc}\frac{(aX+bx)^{m+1}}{\left(acX^2+(bc-ad)Xx-bcx^2\right)^m} \geq$$

$$\geq \frac{\left(\sum_{cyc}(aX+bx)\right)^{m+1}}{\left(\sum_{cyc}\left(acX^2+(bc-ad)Xx-bdx^2\right)\right)^m} = \frac{(3a+b)^{m+1}X^{m+1}}{\left(3acX^2+(bc-ad)X^2-\frac{bdX^2}{3}\right)^m}$$

$$= \frac{3^m(3a+b)^{m+1}X^{m+1}}{(9ac+3bc-3ad-bd)^mX^{2m}} = \frac{3^m(3a+b)^{m+1}X^{m+1}}{(3a+b)^m(3c-d)^mX^{2m}} = \frac{3^m(3a+b)}{(3c-d)^mX^{m-1}}.$$

Corollary 3. If we take $m=1$ in theorem 2, then we obtain the the result from theorem 1.

Theorem 3. Let $a,m \in \mathbb{R}_+$ and $b,c,d,x,y,z \in \mathbb{R}_+^*$, such that $cX > d\max\{x,y,z\}$, then:

$$\left(\frac{aX+bx}{cX-dx}\right)^{m+1}+\left(\frac{aX+by}{cX-dy}\right)^{m+1}+\left(\frac{aX+bz}{cX-dz}\right)^{m+1} \geq \frac{3(3a+b)^{m+1}}{(3c-d)^{m+1}},$$

where $X=x+y+z$.

Proof. Let $U=\left(\frac{aX+bx}{cX-dx}\right)^{m+1}+\left(\frac{aX+by}{cX-dy}\right)^{m+1}+\left(\frac{aX+bz}{cX-dz}\right)^{m+1}$.

By J. Radon's inequality we have

$$U=\sum_{cyc}\left(\frac{aX+bx}{cX-dx}\right)^{m+1} \geq \frac{1}{3^m}\left(\sum_{cyc}\frac{aX+bx}{cX-dx}\right)^{m+1}.$$

We shall proved in theorem 1 that

$$V=\sum_{cyc}\frac{aX+bx}{cX-dx}\geq\frac{3(3a+b)}{3c-d}.$$

By above yields the given inequality. The equality occurs if $x=y=z$.

Corollary 4. If $m=0$, then we obtain again the result from theorem 1.

Theorem 4. Let $n\in\mathbb{N}^*-\{1\}$,$a\in\mathbb{R}_+$, and $b,c,d,x_k\in\mathbb{R}_+^*$. Let $k=\overline{1,n}$ and $X_n=\sum_{k=1}^n x_k$ such that $cX_n>d\max\limits_{1\leq k\leq n}x_k$, then

$$\sum_{k=1}^n\frac{aX_n+bx_k}{cX_n-dx_k}\geq\frac{(an+b)n}{cn-d}.$$

Proof. Let $U_n=\sum_{k=1}^n\frac{aX_n+bx_k}{cX_n-dx_k}$, then we have

$$U_n=\sum_{k=1}^n\frac{aX_n+bx_k}{cX_n-dx_k}\Leftrightarrow dU_n=\sum_{k=1}^n\frac{adX_n+bdx_k}{cX_n-dx_k}\Leftrightarrow$$

$$\Leftrightarrow dU_n+nb=\sum_{k=1}^n\left(\frac{adX_n+bdx_k}{cX_n-dx_k}+b\right)=(ad+bc)X_n\sum_{k=1}^n\frac{1}{cX_n-dx_k}\overset{Bergstrom}{\geq}$$

$$\overset{Bergstrom}{\geq}(ad+bc)X_n\cdot\frac{n^2}{\sum_{k=1}^n(cX_n-dx_k)}=\frac{(ad+bc)n^2X_n}{cnX_n-X_n}=\frac{(ad+bc)n^2}{cn-d}$$

$$\Leftrightarrow dU_n=\frac{(ad+bc)n^2}{cn-d}-nb=\frac{adn^2+bdn}{cn-d}\Leftrightarrow U_n\geq\frac{(an+b)n}{cn-d}.$$

Corollary 4. Let $n\in\mathbb{N}^*-\{1\}$,$a\in\mathbb{R}_+$, and $b,c,d,x_k\in\mathbb{R}_+^*$. Let $k=\overline{1,n}$ and $X_n=\sum_{k=1}^n x_k$ such that $aX_n+bx_k>0$, $\forall k=\overline{1,n}$, then

$$\sum_{k=1}^n\frac{cX_n-dx_k}{aX_n+bx_k}\geq\frac{(cn-d)n}{an+b}.$$

Corollary 5. If $n=3,a=0,b=c=d=1$, then by theorem 4 we obtain $\frac{x_1}{x_2+x_3}+\frac{x_2}{x_3+x_1}+\frac{x_3}{x_1+x_2}\geq\frac{3}{2}$, i.e Nesbitt-Ionescu' s inequality.

Corollary 6. If $n=3,a=b=2,c=4,d=1$, then by theorem 4 we deduce that

$$\sum_{k=1}^3\frac{2X_3+2x_k}{4X_3-x_k}\geq\frac{(2\cdot3+2)\cdot3}{4\cdot3-1}=\frac{24}{11},$$

with equality if and only if $x_1=x_2=x_3$, where $X_3=x_1+x_2+x_3$, i.e. we obtain the inequality of the problem 37 from MathProblems..

Theorem 5. If $n\in\mathbb{N}^*-\{1\}$,$a,m\in\mathbb{R}_+$ and $b,c,d,x_k\in R_+^*$. Let $k=\overline{1,n}$ and $X_n=\sum_{k=1}^n x_k$ such that $cX_n>d\max\limits_{1\leq k\leq n}x_k$, then

$$\sum_{k=1}^n\frac{(aX_n+bx_k)^{m+1}}{(cX_n-dx_k)^{2m+1}}\geq\frac{(an+b)^{m+1}n^{m+1}}{(cn-d)^{2m+1}X_n^m}.$$

Proof. Let $W_n=\sum_{k=1}^n\frac{(aX_n+bx_k)^{m+1}}{(cX_n-dx_k)^{2m+1}}$, then we have

$$W_n=\sum_{k=1}^n\frac{(aX_n+bx_k)^{m+1}}{(cX_n-dx_k)^{2m+1}}=\sum_{k=1}^n\left(\frac{aX_n+bx_k}{cX_n-dx_k}\right)^{m+1}\cdot\frac{1}{(cX_n-dx_k)^m}\overset{Radon}{\geq}$$

$$\overset{Radon}{\geq} \frac{\left(\sum_{k=1}^{n} \frac{aX_n+bx_k}{cX_n-dx_k}\right)^{m+1}}{\left(\sum_{k=1}^{n}(cX_n-dx_k)\right)^m} = \frac{1}{(cn-d)^m X_n^m}\left(\sum_{k=1}^{n} \frac{aX_n+bx_k}{cX_n-dx_k}\right)^{m+1}.$$

Finally,

$$W_n \geq \frac{1}{(cn-d)^m X_n^m} \cdot \frac{(an+b)^{m+1} n^{m+1}}{(cn-d)^{m+1}} = \frac{(an+b)^{m+1} n^{m+1}}{(cn-d)^{2m+1} X_n^m}.$$

Corollary 7. If we take $m = 0$, then by theorem 5 we obtain the result from theorem 4.

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JUNIOR PROBLEMS

Solutions to the problems stated in this issue should arrive before November 5, 2016.

Proposals

56. *Proposed by Valmir Krasniqi, University of Prishtina, Republic of Kosova.* Find all functions $f:\mathbb{N}\to\mathbb{N}$ such that $f(m!+n!)|f(m)!+f(n)!$ and $m+n$ divides $f(m)+f(n)$ for all $m,n\in\mathbb{N}$.

57. *Proposed by Angel Plaza, University of Las Palmas de Gran Canaria, Spain.* Let $x_1, x_2, ..., x_n > 0$. Prove that

$$\left(\frac{\sum_{k=1}^{n} x_k}{n}\right)^2 \leq \frac{1}{n}\sum_{k=1 \text{ cyclic}}^{n} \frac{x_k^2 + x_k x_{k+1} + x_{k+1}^2}{3} \leq \frac{\sum_{k=1}^{n} x_k^2}{n}.$$

58. *Proposed by Arkady Alt, San Jose, California, USA.* Let P be arbitrary interior point in a triangle ABC and r be inradius. Prove that

$$\frac{a^2}{d_a(P)} + \frac{b^2}{d_b(P)} + \frac{c^2}{d_c(P)} \geq 36r^2$$

if $d_a(P), d_b(P), d_c(P)$ be distances from the point P to sides BC, CA, AB respectively.

59. *Proposed by Marcel Chiriţă, Bucharest, Romania.* Solve in real numbers the system

$$\left.\begin{aligned} 2^x + 2^y &= 12 \\ 3^x + 4^z &= 11 \\ 3^y - 4^z &= 25 \end{aligned}\right\}.$$

60. *Proposed by Dorlir Ahmeti, University of Prishtina, Department of Mathematics, Republic of Kosova.* Let ABC be an acute triangle. Let D be the foot of the altitude from A. Let E, F be the midpoints of AC, AB, respectively. Let $G \neq B$ and $H \neq C$ be the intersection of circumcircle of the triangle ABC with circumcircles of the triangles BFD and CED, respectively. Suppose that A, G, B, H, C are order in this way on the circle they belong. Show that line EF, HB and CG are concurrent.

Solutions

51. *Proposed by Valmir Krasniqi, University of Prishtina, Republic of Kosova.* Let $0 < x, y, z \leq 2$ such that $xyz = 1$. Find all real numbers x, y, z such that

$$3^{x(4-2y)} + 3^{y(4-2z)} + 3^{z(4-2x)} = 27.$$

Solution by Michel Bataille, Rouen, France. We first prove the following lemma: if $0 < x, y, z \leq 2$ and $xyz = 1$, then

$$2(x+y+z) - (xy+yz+zx) \geq 3 \qquad (1)$$

with equality if and only if $x = y = z = 1$.

Proof: We may suppose that $x \geq y \geq z$. Then, $x \geq 1$ and $z \leq 1$, that is, $\frac{1}{xy} \leq 1$ so that $y \geq \frac{1}{x}$. Substituting $\frac{1}{xy}$ for z, we see that the inequality to be proved is equivalent to $f(y) \geq 3 + \frac{1}{x} - 2x$ where $f(y) = 2y + \frac{2}{xy} - xy - \frac{1}{y}$. Differentiating with respect to y, keeping x fixed, we obtain $f'(y) = \frac{x(2-x)(y^2 - \frac{1}{x})}{xy^2}$. It readily follows that $f(y) \geq f\left(\frac{1}{\sqrt{x}}\right) = \frac{4-2x}{\sqrt{x}}$ when y varies between $\frac{1}{x}$ and x. Thus, it is sufficient to show that

$$\frac{4-2x}{\sqrt{x}} \geq 3 + \frac{1}{x} - 2x \qquad (2)$$

whenever $x \geq 1$. We are done since (2) rewrites as $(\sqrt{x}-1)^2(2x + 2\sqrt{x} - 1) \geq 0$ which clearly holds if $x \geq 1$). Equality clearly holds in (1) if $x = y = z = 1$. Conversely, if equality holds in (1), supposing $x \geq y \geq z$, we must have $f(y) = 3 + \frac{1}{x} - 2x$, which implies equality in (2), hence $x = 1$ and then $y = 1$ (since $\frac{1}{x} \leq y \leq x$) and finally $z = \frac{1}{xy} = 1$.

Now, suppose that x, y, z satisfies $0 < x, y, z \leq 2$, $xyz = 1$ and the equation

$$3^{x(4-2y)} + 3^{y(4-2z)} + 3^{z(4-2x)} = 27.$$

Then, using AM-GM,

$$27 = 9^{x(2-y)} + 9^{y(2-z)} + 9^{z(2-x)} \geq 3 \cdot 9^{\frac{x(2-y)+y(2-z)+z(2-x)}{3}}$$

and so $x(2-y) + y(2-z) + z(2-x) \leq 3$, that is, $2(x+y+z) - (xy+yz+zx) \leq 3$. From the lemma, we must have $2(x+y+z) - (xy+yz+zx) = 3$, hence $x = y = z = 1$.
Conversely, $x = y = z = 1$ is obviously a solution to the problem.
We conclude that $x = y = z = 1$ is the unique solution.

Also solved by Shend Zhjeqi, Prishtina, Republic of Kosova; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania and the proposer.

52. *Proposed by Florin Stanescu, Serban Cioculescu school, city Gaesti, jud. Dambovita, Romania.* Let a, b, c nonzero real numbers such that $a + b + c = 0$, prove that:

$$\max\left\{\frac{a}{b} + \frac{b}{a}, \frac{b}{c} + \frac{c}{b}, \frac{c}{a} + \frac{a}{c}\right\} \leq 2 + \frac{1}{8}\left(\frac{(a-b)(b-c)(c-a)}{abc}\right)^2.$$

Solution by Michel Bataille, Rouen, France. Let $(\mathcal{C})$ be the constraint $a+b+c=0$, L the left-hand side and R the right-hand side of the inequality. We observe that from $(\mathcal{C})$ two of the numbers a, b, c must have the same sign while the third one is of the opposite sign. Further, $(\mathcal{C}), L, R$ remain unchanged if we permute a, b, c in some way or change the sign of the three numbers a, b, c. It follows that, without loss of generality, we may suppose that $a < 0$ and $b, c > 0$. Then $a = -(b+c)$ and the desired inequality $L \leq R$ becomes

$$\frac{b}{c} + \frac{c}{b} \leq 2 + \frac{1}{8}\left(\frac{(2b+c)(b-c)(2c+b)}{bc(b+c)}\right)^2,$$

which is equivalent to

$$\frac{(b-c)^2}{bc} \le \frac{1}{8} \cdot \frac{(b-c)^2(2b+c)^2(2c+b)^2}{(bc)^2(b+c)^2} \qquad (1).$$

If $b = c$, then equality holds in (1). Otherwise, (1) rewrites as

$$8bc(b+c)^2 \le (2b+c)^2(2c+b)^2 \qquad (2).$$

Now, on the one hand $2b + c > b + c > 0$, hence $(2b+c)^2 > (b+c)^2$ and on the other hand, $2c + b \ge 2\sqrt{2c \cdot b}$, hence $(2c+b)^2 \ge 8bc$. Inequality (2) follows and we are done.

Also solved by Shend Zhjeqi, Prishtina, Republic of Kosova; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania and the proposer.

53. *Proposed by Mihály Bencze, Braşov, Romania.* Let ABC be acute triangle. Prove that

$$m_a^a m_b^b m_c^c \le (R+r)^{2s}$$

where r, R, s, m_a be the inradius, circumradius, semiperimeter and the median of the triangle respectively.

Solution 1 by Viorel Codreanu, Satulung, Maramures, Romania. The inequality is equivalent to

$$m_a^{\frac{a}{2s}} m_b^{\frac{b}{2s}} m_c^{\frac{c}{2s}} \le R + r.$$

Using the Weighted AM-GM Inequality, we get

$$m_a^{\frac{a}{2s}} m_b^{\frac{b}{2s}} m_c^{\frac{c}{2s}} \le \frac{a}{2s} m_a + \frac{b}{2s} m_b + \frac{c}{2s} m_c = \frac{1}{2s}\left(am_a + bm_b + cm_c\right).$$

And so it is enough to prove that

$$am_a + bm_b + cm_c \le 2s\left(R + r\right).$$

In triangle ABC let M the midpoint of the side BC and O circumcircle. In triangle AOM we have

$$AM \le AO + OM = R + R\cos A = R\left(1 + \cos A\right),$$

namely $m_a \le R\left(1 + \cos A\right)$ and the similar relations.
We have

$$am_a \le Ra\left(1 + \cos A\right) = aR + 2R^2 \sin A \cos A = aR + R^2 \sin 2A.$$

Of course, we have two other similar inequalities. Then

$$\begin{aligned}&\sum am_a \le \sum \left(aR + R^2 \sin 2A\right) = 2sR + R^2 \sum \sin 2A = 2sR + 4R^2 \prod \sin A = \\ &= 2sR + 2S = 2sR + 2sr = 2s\left(R + r\right),\end{aligned}$$

and the conclusion follows.

Solution 2 by Michel Bataille, Rouen, France. The inequality is equivalent to

$$\frac{a}{2s}\ln(m_a) + \frac{b}{2s}\ln(m_b) + \frac{c}{2s}\ln(m_c) \le \ln(R+r).$$

Since $2s = a + b + c$ and the function ln is concave, Jensen's inequality provides

$$\frac{a}{2s}\ln(m_a) + \frac{b}{2s}\ln(m_b) + \frac{c}{2s}\ln(m_c) \le \ln\left(\frac{am_a + bm_b + cm_c}{a+b+c}\right),$$

hence it is sufficient to prove that

$$\frac{am_a + bm_b + cm_c}{a+b+c} \le R + r \qquad (1).$$

Let A', B', C' denote the midpoints of BC, CA, AB, respectively and let O be the circumcentre of ΔABC. Note that O is interior to the triangle ABC (because ABC is acute).

From the triangle inequality, we deduce that $AA' \le AO+OA'$, that is, $m_a \le R+d(O,BC)$. Similarly, $m_b \le R+d(O,CA)$, $m_c \le R+d(O,AB)$ and so

$$\begin{aligned}\frac{am_a+bm_b+cm_c}{a+b+c} &\le R+\frac{ad(O,BC)+bd(O,CA)+cd(O,AB)}{a+b+c}\\ &= R+\frac{2\text{Area}(BOC)+2\text{Area}(COA)+2\text{Area}(AOB)}{2s}\\ &= R+\frac{2\text{Area}(ABC)}{2s}\\ &= R+\frac{2rs}{2s}=R+r,\end{aligned}$$

and (1) follows at once.

Also solved by Arkady Alt, San Jose, California, USA and the proposer.

54. *Proposed by Marcel Chiriţă, Bucharest, Romania.* Consider a triangle ABC. Let D be a the midpoint on the median AM and $M \in BC$. Perpendicular on the midpoint of the segment DM passes through the orthocenter of the triangle ABC. Prove that $\angle(BDC)=90°$.

Solution 1 by Michel Bataille, Rouen, France. Note that the perpendicular on the midpoint of the segment DM is the perpendicular bisector ℓ of DM. We denote by $(\mathcal{P}_1)$ the property: $\angle(BDC)=90°$ and $(\mathcal{P}_2)$: the orthocenter H of ABC lies on ℓ and we show that $(\mathcal{P}_1)$ and $(\mathcal{P}_2)$ are equivalent properties of ABC. We remark that $(\mathcal{P}_2)$ is equivalent to $DH=MH$ while $(\mathcal{P}_1)$ is equivalent to $MD=MB=MC$, that is, to $AM=BC$. Thus, all boils down to proving that $AM=BC$ is equivalent to $DH=MH$. We shall use complex numbers, supposing without loss of generality that A,B,C are on the circle centered at the circumcentre O of ABC and with radius 1. Taking O as the origin, we denote by a,b,c the affixes of A,B,C, respectively. Note that $a\overline{a}=b\overline{b}=c\overline{c}=1$. The affixes of M,D and H are $\frac{b+c}{2}$, $\frac{2a+b+c}{4}$ and $a+b+c$ (from the well-known vector relation $\overrightarrow{OH}=\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}$) so that the affixes of $\overrightarrow{AM},\overrightarrow{DH},\overrightarrow{MH}$ are $\frac{b+c-2a}{2}$, $\frac{2a+3b+3c}{4}$, $\frac{2a+b+c}{2}$, respectively. With the additional notations $u=\frac{b}{c}+\frac{c}{b}$, $v=\frac{c}{a}+\frac{a}{c}$, $w=\frac{a}{b}+\frac{b}{a}$, simple calculations give

$$BC^2=|b-c|^2=(b-c)(\overline{b}-\overline{c})=(b-c)\left(\frac{1}{b}-\frac{1}{c}\right)=2-u$$

$$AM^2=\frac{1}{4}(b+c-2a)\left(\frac{1}{b}+\frac{1}{c}-\frac{2}{a}\right)=\frac{1}{4}(6+u-2v-2w).$$

Thus, $AM=BC$ is equivalent to $8-4u=6+u-2v-2w$, that is,

$$AM=BC \Longleftrightarrow 5u-2v-2w=2 \qquad (1).$$

In the same way, we readily obtain $DH^2=\frac{1}{16}(22+9u+6v+6w)$ and $MH^2=\frac{1}{4}(6+u+2v+2w)$ so that

$$DH=MH \Longleftrightarrow 22+9u+6v+6w=24+4u+8v+8w \Longleftrightarrow 5u-2v-2w=2 \qquad (2).$$

The result follows from (1) and (2).

Solution 2 by Titu Zvonaru, Comăneşti, Romania and Neculai Stanciu, Buzău Romania.

Lemma. Let ABC be a triangle with the orthocenter H and BB', CC' the altitude. If $B'C'$ intersect the extend of BC in P and M is the midpoint on BC, then $PH\perp AM$.

Proof. Let Tbe the symmetric of $H$with the respect to M; it is well-known that T belongs to circumcircle. Since the symmetric of $H$with the respect to BC belongs to circumcircle we deduce that AT is the diameter. Then $A''=\text{Pr}_{HM}\,A \in$ Circumcircle. BC is radical axis of circles $BCB'C'$ and ABC; and $B'C'$ is radical axis of circles $BCB'C'$ and $AC'HB'$. Hence P is radical center of the circles ABC, $BCB'C'$ and $AC'HB'$. But the radical axis of the circles $AC'HB'$ and ABC is AA'', so the points A, A'' and P are collinear.

Next, we assume that $b > c$ and we denote $m = AM$. By Menelaus theorem we have $\frac{PB}{PC} \cdot \frac{B'C}{B'A} \cdot \frac{C'A}{C'B} = 1 \Rightarrow PB = \frac{ac\cos B}{b\cos C - c\cos B}$, then

$$PM = PB + \frac{a}{2} = \frac{a^2}{2(b\cos C - c\cos B)} = \frac{a^3}{2(b^2 - c^2)}.$$

Since $\cos\angle BMA = \frac{\frac{a^2}{4} + m^2 - c^2}{am} = \frac{b^2 - c^2}{2am}$ and $\cos\angle BMA = \frac{DM}{2PM}$, we obtain

$$\frac{m(b^2 - c^2)}{a^3} = \frac{b^2 - c^2}{am} \Rightarrow a = m.$$

In triangle BDC, the median BM is half of BC, hence the triangle is right-angled with $\angle BDC = \frac{\pi}{2}$ and we are done.

Also solved by the proposer.

55. *Proposed by Dorlir Ahmeti, University of Prishtina, Department of Mathematics, Republic of Kosova.* Let ABC be an acute triangle. Let AD be the altitude from A to BC. Let w_1 and w_2 be the circles with diameters BD and CD, respectively. Denote by E the intersection of w_1 with AB, and by F the intersection of w_2 with AC. Let G, H be the intersections of the line EF with w_1, w_2, respectively and their order in this way E, G, H, F. Let I be the intersection of BG and DE and J the intersection of CH and DF. Let O be the intersection of IJ and AD. Prove that O is the circumcenter of the triangle DGH.

Solution by Michel Bataille, Rouen, France. Since BD is a diameter of w_1, we have $\angle BGD = \angle BED = 90°$. It follows that I is the orthocentre of ΔBKD where K is the point of intersection of DG and AB. Similarly, J is the orthocentre of ΔCLD where L is the point of intersection of DH and AC. As a result, the lines KI, AD, LJ are perpendicular to BC, hence are parallel. Now, let $\mathbf{I}$ denote the inversion with centre D such that $\mathbf{I}(B) = C$. Then $\mathbf{I}(w_1) = w_1'$ and $\mathbf{I}(w_2) = w_2'$ where w_1' and w_2' are the perpendiculars to BC through C and B, respectively, and so $\mathbf{I}(E) = E', \mathbf{I}(G) = G'$ are the points of intersection of w_1' with DE, DG, respectively. In a similar way, DF and DH meet w_2' at $F' = \mathbf{I}(F)$ and $H' = \mathbf{I}(H)$, respectively. The image of the line EF being a circle through D, the points D, E', F', G', H' are all on a circle Γ. Since the points E, F are on the circle with diameter AD, the points E', F' are also on a line perpendicular to AD, hence $E'F'$ is perpendicular to w_1' and w_2'. Thus, E', F', G', H' are concyclic with $\angle H'F'E' = \angle F'E'G' = 90°$, hence $H'F'E'G'$ is a rectangle inscribed in Γ. Therefore $F'G'$ and $H'E'$ are diameters of Γ, and it follows that DG is parallel to AC (both are perpendicular to DF) and similarly DH is parallel to AB. As a result, the quadrilateral $KALD$ is a parallelogram and the midpoint of KL is on AD. Recalling that KI, AD, LJ are parallel, we deduce that O is the midpoint of IJ. Finally, let M be the midpoint of BD (that is, the centre of w_1). Since BI is parallel to DJ (both are perpendicular to GD), MO is parallel to BI (and DJ), hence MO is perpendicular to the chord DG of w_1, hence is the perpendicular bisector of GD (since $MD = MG$) and so $OD = OG$. In a similar way, we obtain $OD = OH$ and we conclude $OD = OG = OH$, as required.

Also solved by Shend Zhjeqi, Prishtina, Republic of Kosova and the proposer.

SOLUTION OF A PROBLEM
ANOTHER PERSPECTIVE!

This section of the Journal presents a solution of a problem, proposed at MathProblems or elsewhere. The presentation of such solution should be done in the style of an article. The articles, should contain ideas and information that readers may find suitable to use in other similar problems.

PROVING A BINOMIAL IDENTITY WITH HYPERGEOMETRIC FUNCTIONS
Solution to Problem 137

Moti Levy

2. Introduction

Problem 137 in Mathproblems Journal, Volume 5, Issue 3, proposed by Omran Kouba and Anastasios Kotronis is solved here using hypergeometric functions.
A very good reference to the application of hypergeometric functions to the evaluation of binomial identities is the wonderful book of Graham, Knut and Patashnik, "Concrete Mathematics". The first step is to express the binomial series as a hypergeometric function and then the second step is trying to use the classical hypergeometric theorems and identities.

3. The problem

Let n be a nonnegative integer, $m,\ p$ be positive integers and $x \in \mathbb{C}$. Show that for the values of n, p, m, x for which the denominators don't vanish, the following identity holds:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\binom{x+n-k}{m-k}}{(p+n-k)\binom{x+n-k}{p+n-k}} = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\binom{x}{m-k}}{(p+n-k)\binom{x+n}{p+n-k}}$$

$$= \begin{cases} (-1)^n \dfrac{\binom{x-p}{m-p-n}}{p\binom{m}{p}}, & m \geq p, \\ \dfrac{\binom{p-1}{m}}{(p-m+n)\binom{x-m+n}{p-m+n}}, & m < p. \end{cases}$$

4. The solution

The following definitions and results are used:
1) The Pochhammer symbol,

$$(c)_n := c(c+1)\cdots(c+n-1) = \frac{\Gamma(c+n)}{\Gamma(c)}, \quad (-c)_n = (-1)^n (c-n+1)_n$$

$$= (-1)^n \frac{\Gamma(c+1)}{\Gamma(c-n+1)}. \tag{4}$$

2) Let $(\alpha_k)_{k\geq 0}$ be a sequence which satisfies the following conditions:

$$\alpha_0 = 1,$$

$$\frac{\alpha_{k+1}}{\alpha_k} = \frac{1}{k+1} \frac{(k+a)(k+b)}{(k+c)} z.$$

Then

$$\sum_{k=0}^{\infty} \alpha_k = 2F1\left[\begin{array}{cc} a & b \\ & c \end{array}\middle| z\right], \tag{5}$$

where $2F1\left[\begin{array}{cc} a & b \\ & c \end{array}\middle| z\right]$ is hypergeometric function.

3) Similarly, let $(\beta_k)_{k\geq 0}$ be a sequence which satisfies the following conditions:

$$\beta_0 = 1,$$
$$\frac{\beta_{k+1}}{\beta_k} = \frac{1}{k+1}\frac{(k+a)(k+b)(k+c)}{(k+d)(k+e)}z.$$

Then

$$\sum_{k=0}^{\infty} \beta_k = 3F2\left[\begin{array}{ccc} a & b & c \\ & d & e \end{array}\middle| z\right], \tag{6}$$

where $3F2\left[\begin{array}{ccc} a & b & c \\ & d & e \end{array}\middle| z\right]$ is hypergeometric function.

4) The Chu-Vandermonde summation formula:

$$2F1\left[\begin{array}{cc} -n & b \\ & c \end{array}\middle| 1\right] = \frac{(c-b)_n}{(c)_n}, \qquad n \geq 0. \tag{7}$$

5) Saalschütz Theorem:

$$3F2\left[\begin{array}{ccc} -n & a & b \\ & c & 1+a+b-c-n \end{array}\middle| 1\right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n} \tag{8}$$

6) Binomial coefficient, where x is complex number and k is integer,

$$\binom{x}{k} = \frac{\Gamma(x+1)}{k!\Gamma(x-k+1)}. \tag{9}$$

a) We first prove that $\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\binom{x}{m-k}}{(p+n-k)\binom{x+n}{p+n-k}} = \begin{cases} (-1)^n \frac{\binom{x-p}{m-p-n}}{p\binom{m}{p}}, & m \geq p, \\ \frac{\binom{p-1}{m}}{(p-m+n)\binom{x-m+n}{p-m+n}}, & m < p. \end{cases}$.

Let

$$\beta_k := (-1)^k \binom{n}{k} \frac{(p+n)\binom{x+n}{p+n}}{\binom{x}{m}} \frac{\binom{x}{m-k}}{(p+n-k)\binom{x+n}{p+n-k}}.$$

Then $\beta_0 = 1$ and

$$\begin{aligned}\frac{\beta_{k+1}}{\beta_k} &= -\frac{(p+n-k)\binom{x+n}{p+n-k}}{\binom{n}{k}\binom{x}{m-k}} \frac{\binom{n}{k+1}\binom{x}{m-k-1}}{(p+n-k-1)\binom{x+n}{p+n-k-1}} \\ &= \frac{1}{k+1}\frac{(-n+k)(-m+k)(x-p+1+k)}{(-p-n+1+k)(x-m+1+k)}.\end{aligned}$$

By (6),

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\binom{x}{m-k}}{(p+n-k)\binom{x+n}{p+n-k}} = \frac{\binom{x}{m}}{(p+n)\binom{x+n}{p+n}} 3F2\left[\begin{array}{ccc} -n & -m & x-p+1 \\ & -p-n+1 & x-m+1 \end{array}\middle| 1\right].$$

By Saalschütz Theorem (8),

$$3F2\left[\begin{array}{ccc} -n & -m & x-p+1 \\ & -p-n+1 & x-m+1 \end{array}\middle| 1\right] = \frac{(m-p-n+1)_n (-n-x)_n}{(-p-n+1)_n (m-n-x)_n},$$

hence:

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\binom{x}{m-k}}{(p+n-k)\binom{x+n}{p+n-k}}$$
$$=\frac{\binom{x}{m}}{(p+n)\binom{x+n}{p+n}}\frac{(m-p-n+1)_n(-n-x)_n}{(-p-n+1)_n(m-n-x)_n}$$
$$=\frac{1}{(p+n)}\frac{\Gamma(x+1)}{m!\Gamma(x-m+1)}\frac{(p+n)!\Gamma(x-p+1)}{\Gamma(x+n+1)}\frac{(m-p-n+1)_n(-1)^n(x+1)_n}{(-1)^n(p)_n(m-n-x)_n}$$
$$=\frac{1}{(p+n)}\frac{\Gamma(x+1)}{m!\Gamma(x-m+1)}\frac{(p+n)!\Gamma(x-p+1)}{\Gamma(x+n+1)}\frac{(m-p-n+1)_n(x+1)_n}{(p)_n(m-n-x)_n}$$
$$=\frac{1}{(p+n)}\frac{\Gamma(x+1)}{m!\Gamma(x-m+1)}\frac{(p+n)!\Gamma(x-p+1)}{\Gamma(x+n+1)}\frac{\Gamma(x+n+1)\Gamma(p)\Gamma(m-n-x)}{\Gamma(x+1)\Gamma(p+n)\Gamma(m-x)}(m-p-n+1)_n$$
$$=\frac{1}{m!\Gamma(x-m+1)}\frac{\Gamma(x-p+1)}{1}\frac{\Gamma(p)\Gamma(m-n-x)}{\Gamma(m-x)}(m-p-n+1)_n$$
$$=\frac{\Gamma(p)\Gamma(x-p+1)}{m!\Gamma(x-m+1)}\frac{\Gamma(m-n-x)}{\Gamma(m-x)}(m-p-n+1)_n.$$

Using this identity,

$$\frac{\Gamma(m-n-x)}{\Gamma(x-m+1)\Gamma(m-x)}=\frac{1}{\Gamma(x-m+1)(m-x-n)_n}=\frac{(-1)^n}{\Gamma(x-m+1)(x-m+1)_n}$$
$$=\frac{(-1)^n}{\Gamma(x-m+n+1)},$$

we obtain

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\binom{x}{m-k}}{(p+n-k)\binom{x+n}{p+n-k}}=(-1)^n\frac{(p-1)!\Gamma(x-p+1)}{m!\Gamma(x-m+n+1)}(m-p-n+1)_n \tag{10}$$

If $m\geq p$, then

$$(-1)^n\frac{(p-1)!\Gamma(x-p+1)}{m!\Gamma(x-m+n+1)}(m-p-n+1)_n=(-1)^n\frac{(p-1)!\Gamma(x-p+1)\Gamma(m-p+1)}{m!\Gamma(x-m+n+1)\Gamma(m-p-n+1)}$$
$$=\frac{(-1)^n}{p}\frac{p!(m-p)!}{m!}\frac{\Gamma(x-p+1)}{(m-p-n)!\Gamma(x-m+n+1)}$$
$$=(-1)^n\frac{1}{p}\frac{\binom{x-p}{m-p-n}}{\binom{m}{p}}.$$

If $m<p$, then

$$(-1)^n\frac{(p-1)!\Gamma(x-p+1)}{m!\Gamma(x-m+n+1)}(m-p-n+1)_n=\frac{(p-1)!\Gamma(x-p+1)}{m!\Gamma(x-m+n+1)}(p-m)_n$$
$$=\frac{(p-1)!\Gamma(x-p+1)}{m!}\frac{\Gamma(p-m+n)}{\Gamma(p-m)}$$
$$=\frac{(p-1)!}{m!(p-m-1)!}\frac{\Gamma(x-p+1)\Gamma(p-m+n+1)}{(p-m+n)\Gamma(x-m+n+1)}$$
$$=\frac{\binom{p-1}{m}}{(p-m+n)\binom{x-m+n}{p-m+n}}.$$

b) Now we prove that $\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\binom{x+n-k}{m-k}}{(p+n-k)\binom{x+n-k}{p+n-k}}=\sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\binom{x}{m-k}}{(p+n-k)\binom{x+n}{p+n-k}}$.

Let

$$\alpha_k := (-1)^k \binom{n}{k} \frac{(p+n)\binom{x+n}{p+n}}{\binom{x+n}{m}} \frac{\binom{x+n-k}{m-k}}{(p+n-k)\binom{x+n-k}{p+n-k}}.$$

Then $\alpha_0 = 1$ and

$$\begin{aligned}\frac{\alpha_{k+1}}{\alpha_k} &= -\frac{\binom{n}{k+1}}{\binom{n}{k}} \frac{(p+n-k)\binom{x+n-k}{p+n-k}}{\binom{x+n-k}{m-k}} \frac{\binom{x+n-k-1}{m-k-1}}{(p+n-k-1)\binom{x+n-k-1}{p+n-k-1}} \\ &= -\frac{\binom{n}{k+1}}{\binom{n}{k}} \frac{(p+n-k)}{(p+n-k-1)} \frac{\binom{x+n-k}{p+n-k}}{\binom{x+n-k-1}{p+n-k-1}} \frac{\binom{x+n-k-1}{m-k-1}}{\binom{x+n-k}{m-k}} \\ &= \frac{1}{k+1} \frac{(k-n)(k-m)}{(k-p-n+1)}.\end{aligned}$$

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{\binom{x+n-k}{m-k}}{(p+n-k)\binom{x+n-k}{p+n-k}} = \frac{\binom{x+n}{m}}{(p+n)\binom{x+n}{p+n}} 2F1\left[\begin{matrix} -n & -m \\ & -p-n+1 \end{matrix} \middle| 1\right].$$

Now, we have to prove that

$$\frac{\binom{x+n}{m}}{(p+n)\binom{x+n}{p+n}} 2F1\left[\begin{matrix} -n & -m \\ & -p-n+1 \end{matrix} \middle| 1\right] = \frac{\binom{x}{m}}{(p+n)\binom{x+n}{p+n}} 3F2\left[\begin{matrix} -n & -m & x-p+1 \\ & -p-n+1 & x-m+1 \end{matrix} \middle| 1\right],$$

or that

$$3F2\left[-n,-m,x-p+1;-p-n+1,x-m+1;1\right] = \frac{\binom{x+n}{m}}{\binom{x}{m}} 2F1\left(-n,-m;-p-n+1;1\right)$$

By Chu-Vandermonde identity,

$$2F1\left(-n,-m;-p-n+1;1\right) = \frac{(m-p-n+1)_n}{(-p-n+1)_n}.$$

By Saalschütz Theorem:

$$3F2\left[-n,-m,x-p+1;-p-n+1,x-m+1;1\right] = \frac{(m-p-n+1)_n\,(-n-x)_n}{(-p-n+1)_n\,(m-n-x)_n}$$

Hence, what is left to prove is that

$$\frac{(m-p-n+1)_n\,(-n-x)_n}{(-p-n+1)_n\,(m-n-x)_n} = \frac{\binom{x+n}{m}}{\binom{x}{m}} \frac{(m-p-n+1)_n}{(-p-n+1)_n},$$

or that

$$\frac{(-n-x)_n}{(m-n-x)_n} = \frac{\binom{x+n}{m}}{\binom{x}{m}}.$$

Using definitions (4) and (9), the required equality is proved,

$$\begin{aligned}\frac{(-n-x)_n}{(m-n-x)_n} &= \frac{(-1)^n (x+1)_n}{(-1)^n (x+1-m)_n} \\ &= \frac{\Gamma(x+n+1)}{\Gamma(x+1)} \frac{\Gamma(x-m+1)}{\Gamma(x+n-m+1)} \\ &= \frac{m!\Gamma(x-m+1)}{\Gamma(x+1)} \frac{\Gamma(x+n+1)}{m!\Gamma(x+n-m+1)} = \frac{\binom{x+n}{m}}{\binom{x}{m}}.\end{aligned}$$

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