MATHPROBLEMS
ISSN: 2217-446X, URL: http://www.mathproblems-ks.org
VOLUME 5, ISSUE 3 (2015), PAGES 442-475

PROBLEMS AND SOLUTIONS

Proposals and solutions must be legible and should appear on separate sheets, each indicating the name of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editors have supplied a solution. The editors encourage undergraduate and pre-college students to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the class and school name. Solutions will be evaluated for publication by a committee of professors according to a combination of criteria. Questions concerning proposals and/or solutions can be sent by e-mail to: *mathproblems-ks@hotmail.com*

Solutions to the problems stated in this issue should arrive before August 15, 2016

Problems

131. *Proposed by Cornel Ioan Vălean, Timiş, Rumania.* Calculate

$$\int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} \frac{\log(1+\cos x) - \log(1+\cos y)}{\cos x - \cos y} dxdy.$$

132. *Proposed by Valmir Krasniqi, University of Prishtina, Republic of Kosova.* Find all functions $f\colon \mathbb{R}^* \to \mathbb{R}^*$ from the non-zero reals to the non-zero reals, such that

$$f(xyz) = f(xy+yz+xz)(f(x)+f(y)+f(z))$$

for all non-zero reals x, y, z such that $xy+yz+xz \neq 0$.

133. *Proposed by Vasile Pop and Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Solve in $\mathcal{M}_2(\mathbb{Z}_5)$ the equation

$$X^5 = \begin{pmatrix} \widehat{4} & \widehat{2} \\ \widehat{4} & \widehat{1} \end{pmatrix}.$$

134. *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia.* Evaluate the following integral

$$\int_0^{\infty} \frac{\sin(a_1 x)}{x} \frac{\sin(a_2 x)}{x} e^{-a_3 x} dx$$

where a_1, a_2, a_3 are positive real numbers.

135. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Calculate

$$\lim_{n\to\infty}\left(\left(\sqrt[n]{n!}\right)^{F_{m+1}}\left(\sqrt[n]{(2n-1)!!}\right)^{F_m}\left(\tan\frac{\pi\sqrt[n+1]{(n+1)!}}{4\sqrt[n]{n!}}-1\right)^{F_{m+2}}\right)$$

where F_m is Fibonacci's sequence.

136. *Proposed by Florin Stanescu, Serban Cioculescu school, city Gaesti, jud. Dambovita, Romania.* We consider $f:\mathbb{R}\to\mathbb{R}$ a function a twice differentiable and f'' continous. Let $(a_n)_{n\geq 1}$ such that $a_i \neq a_{i+1}$ for all $i \geq 1$ which have the following condition

a) $\lim_{n\to\infty}\Big((n+1)^2 f(a_{n+1}) - n^2 f(a_n)\Big) = 0$

b)$\lim_{n\to\infty}\frac{f(a_{n+1})-f(a_n)}{a_{n+1}-a_n} = \theta > 0.$

i) Find an example of such a function and such a string.

ii) Show that $\lim_{n\to\infty} f'(a_n) = \theta$.

137. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria and Anastasios Kotronis, Athens, Greece (Jointly).* Let n be a nonnegative integer, m, p be positive integers and $x \in \mathbb{C}$. Show that for the values of n, p, m, x for which the denominators don't vanish, the following identity holds:

$$\sum_{k=0}^{n}(-1)^k\frac{\binom{n}{k}\binom{x+n-k}{m-k}}{(p+n-k)\binom{x+n-k}{p+n-k}} = \sum_{k=0}^{n}(-1)^k\frac{\binom{n}{k}\binom{x}{m-k}}{(p+n-k)\binom{x+n}{p+n-k}}$$

$$= \begin{cases} (-1)^n\cdot\frac{\binom{x-p}{m-p-n}}{p\binom{m}{p}} & ,m\geq p \\ \\ \frac{\binom{p-1}{m}}{(p-m+n)\binom{x-m+n}{p-m+n}} & ,m<p. \end{cases}$$

Solutions

No problem is ever permanently closed. We will be very pleased considering for publication new solutions or comments on the past problems.

124. *Proposed by Cornel Ioan Vălean, Timiş, Rumania.*
Find an expression $E_n(x)$ whose terms are linearly independent with the other terms in the integrand such that

$$\left|\int_0^1 \left(\frac{\alpha_1}{\log(x)}+\frac{\alpha_2}{\log^2(x)}+\cdots+\frac{\alpha_n}{\log^n(x)}+E_n(x)\right)dx\right|<\infty$$

where $\alpha_i \neq 0$, $\alpha_i \in \mathbb{R}$.
Then, for a specific $E_n(x)$ family that fulfills the requirements above, calculate

$$\int_0^1 \left(\frac{\alpha_1}{\log(x)}+\frac{\alpha_2}{\log^2(x)}+\cdots+\frac{\alpha_n}{\log^n(x)}+E_n(x)\right)dx$$

in closed form.

Solution by the proposer.

We note from the beginning that the text of the problem is meant to avoid the trivial solutions like

$$E_n(x)=-\frac{\alpha_1}{\log(x)}-\frac{\alpha_2}{\log^2(x)}-\cdots-\frac{\alpha_n}{\log^n(x)}.$$

The key observation for getting an expression such that the integral converges is based upon the terms of the form

$$\alpha_i\left(\frac{1-x}{\log(x)}\right)^i,\ i=1,...,n.$$

First observe that

$$\lim_{x\to 0^+}\left(\frac{1-x}{\log(x)}\right)^i=0$$

and

$$\lim_{x\to 1^-}\left(\frac{1-x}{\log(x)}\right)^i=(-1)^i.$$

Also, if we consider $f(x)=\left(\frac{1-x}{\log(x)}\right)^i$, $i=1,...,n$, we see immediately that $|f(x)|\leq M, \forall x\in(0,1)$.
Our aim is to choose $E_n(x)$ such that we obtain terms as above in our integrand, and one of the ways is

$$E_n(x)=\sum_{i=1}^n \alpha_i\left(\frac{1-x}{\log(x)}\right)^i-\sum_{i=1}^n\frac{\alpha_i}{\log^i(x)}.$$

Adding this expression in our integrand, all reduces to testing for convergence

$$\int_0^1\sum_{i=1}^n \alpha_i\left(\frac{1-x}{\log(x)}\right)^i dx$$

Changing the order of summation and integration, we get, in absolute value, that

$$\left|\sum_{i=1}^{n}\alpha_i\int_0^1\left(\frac{1-x}{\log(x)}\right)^i dx\right|$$

$$=\left|\alpha_1\int_0^1\frac{1-x}{\log(x)}dx+\alpha_2\int_0^1\left(\frac{1-x}{\log(x)}\right)^2 dx+\cdots+\alpha_n\int_0^1\left(\frac{1-x}{\log(x)}\right)^n dx\right|<\infty$$

where, as seen above, at the only points where the integrands could have blown up, near 0 and 1, they approach 0 and $(-1)^i$ respectively, and the first part of the problem is finalized.

An important remark to this part is that we may find infinitely many functions that on the numerator near 1 they behave like $\mathcal{O}(1-x)$, and then we get infinitely many solutions for the convergence.

To answer the second part of the question, we first calculate the integral

$$\int_0^1\left(\frac{1-x}{\log(x)}\right)^i dx.$$

If making the variable change $x=e^{-y}$, the integral becomes

$$(-1)^i\int_0^\infty e^{-y}\left(\frac{1-e^{-y}}{y}\right)^i dy=(-1)^i\int_0^\infty\sum_{j=0}^{i}(-1)^j e^{-jy}\binom{i}{j}\frac{e^{-y}}{y^i}\ dy$$

$$=\frac{(-1)^i}{\Gamma(i)}\int_0^\infty\sum_{j=0}^{i}(-1)^j\binom{i}{j}\int_0^\infty x^{i-1}e^{-y(x+j+1)}\ dx\ dy$$

$$=\frac{(-1)^i}{\Gamma(i)}\int_0^\infty x^{i-1}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\int_0^\infty e^{-y(x+j+1)}\ dy\ dx$$

$$=\frac{(-1)^i}{\Gamma(i)}\int_0^\infty x^{i-1}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\frac{1}{x+j+1}\ dx$$

where above I also used that $\frac{1}{y^i}=\frac{1}{\Gamma(i)}\int_0^\infty x^{i-1}e^{-yx}\ dx$.

Using the fact that $\frac{\Gamma(i+1)}{(x+1)(x+2)\cdots(x+i+1)}=\sum_{j=0}^{i}(-1)^j\binom{i}{j}\frac{1}{x+j+1}$, we get that

$$\frac{(-1)^i}{\Gamma(i)}\int_0^\infty x^{i-1}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\frac{1}{x+j+1}\ dx=(-1)^i i\int_0^\infty\frac{x^{i-1}}{(x+1)(x+2)\cdots(x+i+1)}dx.$$

Note that the equality I used above can be proved by writing that

$$\frac{1}{(x+1)(x+2)\cdots(x+m)(x+m+1)(x+m+2)\cdots(x+i+1)}=\sum_{j=0}^{i}\frac{A_j}{x+j+1}.$$

Then, to find the coefficient A_m, multiply both sides of the equality above by $x+m+1$, and we get that

$$\frac{1}{(x+1)(x+2)\cdots(x+m)(x+m+2)\cdots(x+i+1)}=\sum_{j=0}^{i}A_j\frac{x+m+1}{x+j+1}$$

where if we let $x \mapsto -m-1$, we obtain that

$$\frac{(-1)^m}{m!(i-m)!} = \frac{(-1)^m}{\Gamma(i+1)}\binom{i}{m} = A_m$$

whence $A_j = \dfrac{(-1)^j}{\Gamma(i+1)}\dbinom{i}{j}$.
Alternatively, we can use beta function and write that

$$\begin{aligned}\frac{\Gamma(i+1)}{(x+1)(x+2)\cdots(x+i+1)} &= \frac{\Gamma(x+1)\Gamma(i+1)}{\Gamma(x+i+2)}\\ &= B(x+1,i+1)\\ &= \int_0^1 y^{x+1-1}(1-y)^{i+1-1}\ dy\\ &= \int_0^1 y^x\sum_{j=0}^{i}(-1)^j\binom{i}{j}y^j\ dy\\ &= \sum_{j=0}^{i}(-1)^j\int_0^1\binom{i}{j}y^{x+j}\ dy\\ &\sum_{j=0}^{i}(-1)^j\binom{i}{j}\frac{1}{x+j+1}\end{aligned}$$

that shows again that the auxiliary equality I used above is true.
So, we have that

$$(-1)^i\int_0^\infty e^{-y}\left(\frac{1-e^{-y}}{y}\right)^i\ dy = (-1)^i i\int_0^\infty\frac{x^{i-1}}{(x+1)(x+2)\cdots(x+i+1)}dx$$

and for calculating the integral in the right-hand side we use again partial fractions, that is

$$\frac{x^{i-1}}{(x+1)(x+2)\cdots(x+i+1)} = \sum_{k=1}^{i+1}\frac{B_k}{x+k}.$$

To obtain the value of B_m, we multiply both sides of the equality by $x+m$, that is

$$\frac{x^{i-1}}{(x+1)(x+2)\cdots(x+m-1)(x+m+1)\cdots(x+i+1)} = \sum_{k=1}^{i+1}B_k\frac{x+m}{x+k}$$

and then we let $x \mapsto -m$ that leads to

$$\frac{(-1)^{i+m}m^{i-1}}{(m-1)!(i-m+1)!} = \frac{(-1)^{i+m}m^{i-1}}{i!}\cdot\frac{i!}{(m-1)!(i-m+1)!} = \frac{(-1)^{i+m}m^{i-1}}{\Gamma(i+1)}\binom{i}{m-1} = B_m.$$

Upon replacing m by k, we get that $B_k = \dfrac{(-1)^{i+k}k^{i-1}}{\Gamma(i+1)}\binom{i}{k-1}$. Then, we have that

$$\begin{aligned}(-1)^i i\int_0^\infty \frac{x^{i-1}}{(x+1)(x+2)\cdots(x+i+1)}dx &= \frac{(-1)^i}{\Gamma(i)}\lim_{s\to\infty}\int_0^s\sum_{k=1}^{i+1}(-1)^{i+k}k^{i-1}\binom{i}{k-1}\frac{1}{x+k}dx\\ &= \frac{1}{\Gamma(i)}\lim_{s\to\infty}\sum_{k=1}^{i+1}\int_0^s(-1)^k k^{i-1}\binom{i}{k-1}\frac{1}{x+k}dx\\ &= \frac{1}{\Gamma(i)}\lim_{s\to\infty}\sum_{k=1}^{i+1}(-1)^k k^{i-1}\binom{i}{k-1}(\log(k+s)-\log(k)).\end{aligned}$$

Since we have that $1 = \sum_{k=2}^{i+1}(-1)^k k^{i-1}\binom{i}{k-1}$ because

$$0 = \lim_{x\to\infty}\frac{x^i}{(x+1)(x+2)\cdots(x+i+1)} = \lim_{x\to\infty}\sum_{k=1}^{i+1}\frac{B_k x}{x+k} = \sum_{k=1}^{i+1}B_k,$$

then we get

$$\begin{aligned}&\frac{1}{\Gamma(i)}\lim_{s\to\infty}\left(\sum_{k=2}^{i+1}(-1)^k k^{i-1}\binom{i}{k-1}(\log(k+s)-\log(k)) - \sum_{k=2}^{i+1}(-1)^k k^{i-1}\binom{i}{k-1}\log(1+s)\right)\\ &= \frac{1}{\Gamma(i)}\lim_{s\to\infty}\left(\sum_{k=2}^{i+1}(-1)^k k^{i-1}\binom{i}{k-1}(\log(k+s)-\log(k)-\log(1+s))\right)\\ &= \frac{1}{\Gamma(i)}\left(\sum_{k=2}^{i+1}(-1)^k k^{i-1}\binom{i}{k-1}\lim_{s\to\infty}(\log(k+s)-\log(k)-\log(1+s))\right)\\ &= \frac{1}{\Gamma(i)}\sum_{k=2}^{i+1}(-1)^{k+1}k^{i-1}\binom{i}{k-1}\log(k).\end{aligned}$$

Hence

$$\begin{aligned}&\alpha_1\int_0^1\frac{1-x}{\log(x)}dx + \alpha_2\int_0^1\left(\frac{1-x}{\log(x)}\right)^2dx + \cdots + \alpha_n\int_0^1\left(\frac{1-x}{\log(x)}\right)^n dx\\ &= \sum_{i=1}^{n}\sum_{k=2}^{i+1}(-1)^{k+1}\frac{\alpha_i k^{i-1}}{\Gamma(i)}\binom{i}{k-1}\log(k).\end{aligned}$$

Q.E.D.

Editorial comment. The logic behind choosing these E_n's is not strong. One might choose almost anything. The author of the problem should have asked explicitly to evaluate the integrals $\int_0^1\left(\frac{1-x}{\log x}\right)^i dx$ which is the aime of these calculations.

125. *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia.* Let x, y and z be the sides of a triangle and r, R and s be the inradius, circumradius and the semiperimeter of the triangle respectively. Prove that

$$\frac{1}{(x+y)^2}+\frac{1}{(x+z)^2}+\frac{1}{(y+z)^2} \le \frac{r^4+8r^3R+124r^2R^2+2r^2s^2-8rRs^2+s^4}{128r^2R^2s^2}.$$

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria.

The function $f:(0,+\infty)\to\mathbb{R}$ defined by $f(t)=1/t^2$ is convex, so, by Popoviciu's inequality we have

$$\frac{2}{3}\left(f\left(\frac{x+y}{2}\right)+f\left(\frac{y+z}{2}\right)+f\left(\frac{x+z}{2}\right)\right)\le\frac{f(x)+f(y)+f(z)}{3}+f\left(\frac{x+y+z}{3}\right)$$

that is

$$\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(x+z)^2}\le\underbrace{\frac{1}{8}\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)+\frac{27}{8}\frac{1}{(x+y+z)^2}}_{L}\qquad(1)$$

Now, $x+y+z=2s$ and $xyz=4R\cdot\text{area}(ABC)=4Rrs$, thus

$$L=\frac{x^2y^2+y^2z^2+x^2z^2+108R^2r^2}{128R^2r^2s^2}\qquad(2)$$

Moreover, since

$$\begin{aligned}s^2r^2&=(\text{area}(ABC))^2=s(s-x)(s-y)(s-z)\\&=s(s^3-2s^3+(xy+yz+xz)s-xyz)\end{aligned}$$

we get

$$xy+yz+xz=r^2+4Rr+s^2$$

Thus

$$\begin{aligned}x^2y^2+y^2z^2+x^2z^2&=(xy+yz+xz)^2-4xyzs\\&=(r^2+4Rr+s^2)^2-16Rrs^2\end{aligned}$$

Replacing back in (2) and expanding we get

$$L=\frac{r^4+124R^2r^2+s^4+8Rr^3+8r^2s^2-8Rrs^2}{128R^2r^2s^2}$$

and the proposed inequality follows from (1).

Solution 2 by Moti Levy, Rehovot, Israel.

Since $(x+y)^2\ge 4xy$, then

$$\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\le\frac{1}{4}\left(\frac{1}{xy}+\frac{1}{yz}+\frac{1}{zx}\right)=\frac{1}{4}\frac{2s}{xyz}=\frac{1}{4}\frac{1}{2Rr},\qquad(1)$$

and by Euler's inequality $R\ge 2r$, we get

$$\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\le\frac{1}{16r^2}.\qquad(2)$$

Thus, the original inequality is proved once we show that

$$\frac{1}{16r^2}\le\frac{r^4+8r^3R+124r^2R^2+2r^2s^2-8rRs^2+s^4}{128r^2R^2s^2},$$

or that

$$8R^2s^2+8rRs^2\le r^4+8r^3R+124r^2R^2+2r^2s^2+s^4.\qquad(3)$$

To this end, we use two well-known inequalities (Bottema et al., *Geometric Inequalities*, 5.11, page 52, and 5.5 page 49):

$$s^2 \leq 27r^2, \tag{4}$$

$$s^2 \geq 3r\left(4R + r\right). \tag{5}$$

Applying (4) on the left side of (3),

$$8R^2s^2 + 8rRs^2 \leq 216Rr^2\left(R + r\right).$$

Applying (5) on the right side of (3),

$$4r^2\left(67R^2 + 26Rr + 4r^2\right) \leq r^4 + 8r^3R + 124r^2R^2 + 2r^2s^2 + s^4$$

Now it is straightforward to see that

$$216Rr^2\left(R + r\right) \leq 4r^2\left(67R^2 + 26Rr + 4r^2\right),$$

follows from Euler's inequality $R \geq 2r$ and

$$54R\left(R + r\right) \leq 67R^2 + 26Rr + 4r^2$$

$$67R^2 + 26Rr + 4r^2 - 54R\left(R + r\right) = \left(R - 2r\right)\left(13R - 2r\right) \geq 0.$$

Also solved by Ramya Dutta, Chennai Mathematical Institute (student) India; Nicusor Zlota, "Traian Vuia" Technical College, Focsani, Romania and the proposer.

126. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let $n - 2 \geq m \geq 1$ be integers. Calculate

$$\int_0^\infty \frac{x^{m-1} + x^{m-2} + \cdots + x + 1}{x^{n-1} + x^{n-2} + \cdots + x + 1}dx.$$

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

The starting point will be the well-known partial expansion of $z \mapsto \pi\cot(\pi z)$,

$$\pi\cot(\pi z) = \lim_{n\to\infty}\sum_{k=-n}^{n}\frac{1}{z + k}$$

which is valid for $z \notin \mathbb{Z}$. Now, assume that $0 < x < 1$ then

$$\begin{aligned}
\sum_{k=-n}^{n}\frac{1}{x + k} &= \frac{1}{x + n} + \sum_{k=0}^{n-1}\frac{1}{x + k} - \sum_{k=1}^{n}\frac{1}{k - x} \\
&= \frac{1}{x + n} + \sum_{k=0}^{n-1}\int_0^1 t^{x+k-1}dt - \sum_{k=0}^{n-1}\int_0^1 t^{k-x}dt \\
&= \frac{1}{x + n} + \int_0^1 (t^{x-1} - t^{-x})\left(\sum_{k=0}^{n-1}t^k\right)dt \\
&= \frac{1}{x + n} + \int_0^1 \frac{t^{x-1} - t^{-x}}{1 - t}(1 - t^n)dt \\
&= \frac{1}{x + n} + \int_0^1 \frac{t^{x-1} - t^{-x}}{1 - t}t^n dt + \int_0^1 \frac{t^{x-1} - t^{-x}}{1 - t}dt
\end{aligned}$$

letting n tend to infinity we get

$$\pi\cot(\pi x) = \int_0^1 \frac{t^{x-1} - t^{-x}}{1-t}\,dt$$

Now Suppose that $0 < x, y < 1$ then

$$\begin{aligned}
\int_0^\infty \frac{t^{y-1} - t^{x-1}}{1-t}\,dt &= \int_0^1 \frac{t^{y-1} - t^{x-1}}{1-t}\,dt + \underbrace{\int_1^\infty \frac{t^{y-1} - t^{x-1}}{1-t}\,dt}_{t\leftarrow 1/t} \\
&= \int_0^1 \frac{t^{y-1} - t^{x-1}}{1-t}\,dt - \int_0^1 \frac{t^{-y} - t^{-x}}{1-t}\,dt \\
&= \int_0^1 \frac{t^{y-1} - t^{-y}}{1-t}\,dt - \int_0^1 \frac{t^{x-1} - t^{-x}}{1-t}\,dt \\
&= \pi\cot(\pi y) - \pi\cot(\pi x)
\end{aligned}$$

Considering $\beta > 0$ and $\alpha > 1 + \beta$ and taking $x = 1/\alpha$, $y = (\beta+1)/\alpha$ we get

$$\int_0^\infty \frac{t^{(\beta+1)/\alpha-1} - t^{1/\alpha-1}}{1-t}\,dt = \pi\cot\left(\frac{\pi(\beta+1)}{\alpha}\right) - \pi\cot\left(\frac{\pi}{\alpha}\right)$$

Finally, the change of variables $t = u^\alpha$ we obtain

$$\alpha\int_0^\infty \frac{u^\beta - 1}{1-u^\alpha}\,dt = \pi\cot\left(\frac{\pi(\beta+1)}{\alpha}\right) - \pi\cot\left(\frac{\pi}{\alpha}\right)$$

Or, for all $\beta > 0$ and $\alpha > \beta + 1$

$$\int_0^\infty \frac{u^\beta - 1}{u^\alpha - 1}\,dt = \frac{\pi}{\alpha}\left(\cot\frac{\pi}{\alpha} - \cot\frac{\pi(\beta+1)}{\alpha}\right)$$

In particular, taking $\alpha = n$ and $\beta = m$ integers, we get

$$\int_0^\infty \frac{x^{m-1} + x^{m-2} + \cdots + x + 1}{x^{n-1} + x^{n-2} + \cdots + x + 1}\,dx = \frac{\pi}{n}\left(\cot\frac{\pi}{n} - \cot\frac{\pi(m+1)}{n}\right)$$

which is the announced result.

Solution 2 by Moti Levy, Rehovot, Israel.

Following the footsteps of Victor H. Moll, we prove the definite integral **3.246** in Gradshteyn and Rhyzik. Let $I := \int_0^\infty \frac{1-x^m}{1-x^n} x^{p-1} dx$.
Integral representation of the Digamma function is

$$\psi(s) = \int_0^1 \frac{1 - x^{s-1}}{1-x} dx - \gamma. \tag{6}$$

By change of variable $x = t^n$,

$$\psi(s) = n\int_0^1 \frac{t^{n-1} - t^{ns-1}}{1-t^n} dt - \gamma,$$

$$\psi(1-s) = n\int_0^1 \frac{t^{n-1} - t^{n-ns-1}}{1-t^n} dt - \gamma$$

$$\frac{1}{n}\left(\psi\left(1-s\right)-\psi\left(s\right)\right)=\int_0^1\frac{t^{n-1}-t^{n-ns-1}}{1-t^n}dt-\int_0^1\frac{t^{n-1}-t^{ns-1}}{1-t^n}dt \tag{7}$$
$$=\int_0^1\frac{t^{ns-1}-t^{n-ns-1}}{1-t^n}dt.$$

Using (7) we obtain the following expressions:

$$\int_0^1\frac{t^{p-1}-t^{n-p-1}}{1-t^n}dt=\frac{1}{n}\left(\psi\left(1-\frac{p}{n}\right)-\psi\left(\frac{p}{n}\right)\right), \tag{8}$$

$$\int_0^1\frac{t^{m+p-1}-t^{n-p-m-1}}{1-t^n}dt=\frac{1}{n}\left(\psi\left(1-\frac{m+p}{n}\right)-\psi\left(\frac{m+p}{n}\right)\right). \tag{9}$$

Now back to our integral,

$$I=\int_0^1\frac{1-x^m}{1-x^n}x^{p-1}dx+\int_1^\infty\frac{1-x^m}{1-x^n}x^{p-1}dx.$$

By change of variable $x=\frac{1}{t}$,

$$\int_1^\infty\frac{x^m-1}{x^n-1}x^{p-1}dx=\int_0^1\frac{t^{-m}-1}{t^{-n}-1}t^{-p+1}t^{-2}dt$$
$$=\int_0^1\frac{t^{n-m}-t^n}{1-t^n}t^{-p+1}t^{-2}dt=\int_0^1\frac{t^{n-m-p-1}-t^{n-p-1}}{1-t^n}dt$$

$$I=\int_0^1\frac{1-t^m}{1-t^n}t^{p-1}dt+\int_0^1\frac{t^{n-m-p-1}-t^{n-p-1}}{1-t^n}dt$$
$$=\int_0^1\frac{t^{p-1}-t^{n-p-1}}{1-t^n}dt-\int_0^1\frac{t^{m+p-1}-t^{n-m-p-1}}{1-t^n}dt$$
$$=\frac{1}{n}\left(\psi\left(1-\frac{p}{n}\right)-\psi\left(\frac{p}{n}\right)\right)-\frac{1}{n}\left(\psi\left(1-\frac{m+p}{n}\right)-\psi\left(\frac{m+p}{n}\right)\right).$$

The Reflection Formula for the Digamma function is

$$\psi\left(1-s\right)-\psi\left(s\right)=\pi\cot\left(\pi s\right). \tag{10}$$

$$\int_0^\infty\frac{1-x^m}{1-x^n}x^{p-1}dx=\frac{\pi}{n}\left(\cot\left(\frac{p}{n}\pi\right)-\cot\left(\frac{m+p}{n}\pi\right)\right)$$
$$=\frac{\pi}{n}\frac{\sin\frac{m}{n}\pi}{\sin\frac{\pi}{n}p\sin\frac{\pi}{n}\left(m+p\right)}.$$

Setting $p=1$ in (10), we conclude that

$$\int_0^\infty\frac{x^{m-1}+x^{m-2}+\cdots+x+1}{x^{n-1}+x^{n-2}+\cdots+x+1}dx=\frac{\pi}{n}\frac{\sin\left(\frac{m}{n}\pi\right)}{\sin\frac{\pi}{n}\sin\left(\frac{m+1}{n}\pi\right)},\ n-2\geq m\geq 1.$$

Solution 3 by Michel Bataille, Rouen, France.

Let $f:[0,\infty)\to\mathbb{R}$ be the continuous function defined by $f(x)=\frac{x^{m-1}+x^{m-2}+\cdots+x+1}{x^{n-1}+x^{n-2}+\cdots+x+1}$. Then, $f(x)=\frac{x^m-1}{x^n-1}$ if $x\neq 1$ and $f(x)\sim\frac{1}{x^{n-m}}$ as $x\to\infty$ (so that $\int_1^\infty f(x)\,dx$ exists since $n-m\geq 2$). It follows that the required integral is I_1+I_2 with

$$I_1=\int_0^1\frac{1-x^m}{1-x^n}\,dx\quad\text{and}\quad I_2=\int_1^\infty\frac{x^m-1}{x^n-1}\,dx.$$

To evaluate I_1 and I_2, we shall use the following lemma:

Lemma 1.

$$\text{If } a > 0,\ a + b > 0,\ \text{then } \int_0^1 \frac{t^{a-1}(1 - t^b)}{1 - t}\, dt = \psi(a + b) - \psi(a) \qquad (1)$$

where ψ denotes the digamma function defined for $x > 0$ by

$$\psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k + x}\right).$$

[γ is Euler's constant.]

Proof.
For all $t \in (0, 1)$, $\frac{t^{a-1}(1-t^b)}{1-t} = \sum_{n=0}^{\infty}(t^{n+a-1} - t^{n+a+b-1})$ and $t^{n+a-1} - t^{n+a+b-1}$ has the same sign as b. From

$$\sum_{n=0}^{\infty}\int_0^1 |t^{n+a-1} - t^{n+a+b-1}|\, dt = \sum_{n=0}^{\infty}\left|\int_0^1 (t^{n+a-1} - t^{n+a+b-1})\, dt\right| = \sum_{n=0}^{\infty}\frac{|b|}{(n + a)(n + a + b)} < \infty$$

we deduce

$$J = \sum_{n=0}^{\infty}\int_0^1 (t^{n+a-1} - t^{n+a+b-1})\, dt = \sum_{n=0}^{\infty}\left(\frac{1}{n + a} - \frac{1}{n + a + b}\right) = \psi(a + b) - \psi(a).$$

The proof is complet.
The change of variables $x = t^{1/n}$ yields

$$I_1 = \frac{1}{n}\int_0^1 \frac{t^{\frac{1}{n}-1}(1 - t^{m/n})}{1 - t}\, dt = \frac{1}{n}\left(\psi\left(\frac{m + 1}{n}\right) - \psi\left(\frac{1}{n}\right)\right).$$

The change of variables $x = t^{-1/n}$ gives

$$I_2 = \frac{1}{n}\int_0^1 \frac{t^{\frac{n-m-1}{n}-1}(1 - t^{m/n})}{1 - t}\, dt = \frac{1}{n}\left(\psi\left(1 - \frac{1}{n}\right) - \psi\left(1 - \frac{m + 1}{n}\right)\right).$$

Thus,

$$I_1 + I_2 = \frac{1}{n}\left(\psi\left(\frac{m + 1}{n}\right) - \psi\left(1 - \frac{m + 1}{n}\right) - \left(\psi\left(\frac{1}{n}\right) - \psi\left(1 - \frac{1}{n}\right)\right)\right).$$

But we have $\psi(1 - x) = \psi(x) + \pi\cot(\pi x) \quad (0 < x < 1)$ $\Big($by logarithmic differentiation of $\Gamma(x)\Gamma(1 - x) = \frac{\pi}{\sin(\pi x)}\Big)$, hence

$$I_1 + I_2 = \frac{\pi}{n}\left(\cot\left(\frac{\pi}{n}\right) - \cot\left(\frac{(m + 1)\pi}{n}\right)\right) = \frac{\pi}{n}\cdot\frac{\sin\frac{m\pi}{n}}{\left(\sin\frac{\pi}{n}\right)\left(\sin\frac{(m+1)\pi}{n}\right)}.$$

Also solved by Mustafa Samir Khalil (student), Syria; Ramya Dutta, Chennai Mathematical Institute (student) India and the proposer.

127. *Proposed by Serafeim Tsipelis, Ioannina, Greece and Anastasios Kotronis, Athens, Greece (Jointly).* Evaluate $\sum_{k=1}^{+\infty}\frac{\zeta(2k+1)}{(k+1)(2k+1)}$, where ζ is the Riemann's zeta function.

Solution 1 by Ramya Dutta, Chennai Mathematical Institute (student) India.

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{(k+1)(2k+1)} &= 2\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{(2k+1)(2k+2)n^{2k+1}}\\ &= 2\sum_{k=1}^{\infty}\frac{1}{(2k+1)(2k+2)}+2\sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(2k+1)(2k+2)n^{2k+1}}\\ &= 2\left(-\frac{1}{2}+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k}\right)+2\sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(2k+1)(2k+2)n^{2k+1}}\\ &= \log\left(\frac{4}{e}\right)+2\sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(2k+1)(2k+2)n^{2k+1}}\end{aligned}$$

Using,
$\frac{1}{2}\log\left(\frac{1+x}{1-x}\right) = x+\sum_{k=1}^{\infty}\frac{x^{2k+1}}{2k+1}$ and $\frac{1}{2}\log\left(1-x^2\right) = -\frac{x^2}{2}-\sum_{k=1}^{\infty}\frac{x^{2k+2}}{2k+2}$. We can rewrite the summation,

$$2\sum_{n=2}^{\infty}\sum_{k=1}^{\infty}\frac{1}{(2k+1)(2k+2)n^{2k+1}} = \sum_{n=2}^{\infty}\left(\log\left(\frac{1+\frac{1}{n}}{1-\frac{1}{n}}\right)-\frac{1}{n}+n\log\left(1-\frac{1}{n^2}\right)\right)$$

Consider the partial sum,

$$\begin{aligned}&\sum_{n=2}^{N}\left(\log\left(\frac{1+\frac{1}{n}}{1-\frac{1}{n}}\right)-\frac{1}{n}+n\log\left(1-\frac{1}{n^2}\right)\right)\\ &= \sum_{n=2}^{N}\left(\log\left(\frac{n+1}{n-1}\right)-\frac{1}{n}+n\log\left(1-\frac{1}{n^2}\right)\right)\\ &= \sum_{n=2}^{N}\left((n+1)\log(n+1)+(n-1)\log(n-1)-2n\log n-\frac{1}{n}\right)\\ &= (N+1)\log(N+1)-N\log N-2\log 2-\sum_{n=2}^{N}\frac{1}{n}\\ &= 1-2\log 2+(N+1)\log\left(1+\frac{1}{N}\right)-\gamma+O\left(\frac{1}{N}\right)\end{aligned}$$

where, we used the estimate, $\sum_{n=1}^{N}\frac{1}{n} = \log N+\gamma+O\left(\frac{1}{N}\right)$,

Thus,

$$\sum_{k=1}^{\infty}\frac{\zeta(2k+1)}{(k+1)(2k+1)} = \lim_{N\to\infty}(N+1)\log\left(1+\frac{1}{N}\right)-\gamma = 1-\gamma.$$

Solution 2 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

The answer is $1-\gamma$ where γ is the Euler's constant.

For $x \in (-1,1)$, we define

$$f(x) = \log\frac{1+x}{1-x} + \frac{1}{x}\log(1-x^2) - x$$
$$= \left(1+\frac{1}{x}\right)\log(1+x) + \left(\frac{1}{x}-1\right)\log(1-x) - x$$

Clearly

$$f(x) = -x + \sum_{n=0}^{\infty}\frac{2}{2n+1}x^{2n+1} - \sum_{n=0}^{\infty}\frac{1}{n+1}x^{2n+1}$$
$$= \sum_{n=1}^{\infty}\left(\frac{2}{2n+1} - \frac{1}{n+1}\right)x^{2n+1}$$
$$= \sum_{n=1}^{\infty}\frac{1}{(n+1)(2n+1)}x^{2n+1}$$

Thus, setting $x = 1/j$ for $j \geq 2$ and adding we get

$$\sum_{n=1}^{\infty}\frac{\zeta(2n+1)-1}{(n+1)(2n+1)} = \sum_{j=2}^{\infty} f\left(\frac{1}{j}\right) \tag{1}$$

Now,

$$\sum_{j=2}^{n} f\left(\frac{1}{j}\right) = \sum_{j=2}^{n}\left((1+j)\log(1+j) + (j-1)\log(j-1) - 2j\log j\right) - \sum_{j=2}^{n}\frac{1}{j}$$
$$= \sum_{j=3}^{n+1} j\log j + \sum_{j=2}^{n-1} j\log j - 2\sum_{j=2}^{n} j\log j - \sum_{j=2}^{n}\frac{1}{j}$$
$$= -2\log 2 + (n+1)\log(n+1) - n\log n - \sum_{j=2}^{n}\frac{1}{j}$$
$$= 1 - 2\log 2 + n\log\left(1+\frac{1}{n}\right) + \log(n+1) - H_n$$

where $H_n = \sum_{j=1}^{n}\frac{1}{j}$ is the nth harmonic number.
Recalling that $\lim_{n\to\infty}(H_n - \log(n+1)) = \gamma$, we conclude that

$$\sum_{j=2}^{\infty} f\left(\frac{1}{j}\right) = 2 - 2\log 2 - \gamma \tag{2}$$

Finally, note that

$$\sum_{n=0}^{m-1}\frac{1}{(n+1)(2n+1)} = \sum_{n=0}^{m-1}\left(\frac{2}{2n+1} - \frac{1}{n+1}\right)$$
$$= \sum_{n=0}^{m-1}\left(\frac{2}{2n+1} + \frac{2}{2n+2} - \frac{2}{n+1}\right)$$
$$= 2H_{2m} - 2H_m = 2\log 2 + O\left(\frac{1}{m}\right)$$

Hence

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)(2n+1)} = 2\log 2 - 1 \tag{3}$$

Combining (1), (2) and (3) we get

$$\sum_{n=1}^{\infty} \frac{\zeta(2n+1)}{(n+1)(2n+1)} = 1-\gamma$$

which is the announced result.

Solution 3 by Moti Levy, Rehovot, Israel.

The integral representation of the Zeta function is

$$\zeta\left(s\right) = \frac{1}{\Gamma\left(s\right)}\int_0^{\infty} \frac{x^{s-1}}{e^x-1}dx.$$

$$\sum_{k=1}^{\infty} \frac{\zeta\left(2k+1\right)}{\left(2k+1\right)\left(k+1\right)} = \sum_{k=1}^{\infty} \frac{1}{\left(2k+1\right)\left(k+1\right)} \frac{1}{(2k)!}\int_0^{\infty} \frac{x^{2k}}{e^x-1}dx$$

After changing summation with integration,

$$\sum_{k=1}^{\infty} \frac{\zeta\left(2k+1\right)}{\left(2k+1\right)\left(k+1\right)} = \int_0^{\infty} \frac{1}{e^x-1}\sum_{k=1}^{\infty} \frac{x^{2k}}{\left(k+1\right)\left(2k+1\right)!}dx.$$

The Taylor series of hyperbolic sine is $\sinh\left(x\right) = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!}$.

$$\cosh\left(x\right) - 1 = \int_0^x \sinh\left(t\right)dt = \int_0^x \sum_{k=0}^{\infty} \frac{t^{2k+1}}{\left(2k+1\right)!}dt = \sum_{k=0}^{\infty}\int_0^x \frac{t^{2k+1}}{\left(2k+1\right)!}dt$$

$$= \sum_{k=0}^{\infty} \frac{x^{2k+2}}{\left(2k+2\right)\left(2k+1\right)!} = \frac{x^2}{2}\sum_{k=0}^{\infty} \frac{x^{2k}}{\left(k+1\right)\left(2k+1\right)!} = \frac{x^2}{2} + \frac{x^2}{2}\sum_{k=1}^{\infty} \frac{x^{2k}}{\left(k+1\right)\left(2k+1\right)!}.$$

It follows that

$$\sum_{k=1}^{\infty} \frac{x^{2k}}{\left(k+1\right)\left(2k+1\right)!} = \frac{2}{x^2}\left(\cosh\left(x\right)-1\right) - 1.$$

$$\sum_{k=1}^{\infty} \frac{\zeta\left(2k+1\right)}{\left(2k+1\right)\left(k+1\right)} = \int_0^{\infty}\left(\frac{2\left(\cosh\left(t\right)-1\right)}{t^2} - 1\right)\frac{1}{e^t-1}dt$$

$$= \int_0^{\infty}\left(\frac{e^t+e^{-t}-2}{t^2} - 1\right)\frac{1}{e^t-1}dt$$

$$= \int_0^{\infty} \frac{e^t+e^{-t}-2}{t^2\left(e^t-1\right)} - \frac{1}{e^t-1}dt$$

Now we add and subtract $\frac{e^{-t}}{t}$ to the integrand,

$$\sum_{k=1}^{\infty} \frac{\zeta\left(2k+1\right)}{\left(2k+1\right)\left(k+1\right)} = \int_0^{\infty}\left(\frac{e^t+e^{-t}-2}{t^2\left(e^t-1\right)} - \frac{e^{-t}}{t}\right)dt + \int_0^{\infty}\left(\frac{e^{-t}}{t} - \frac{1}{e^t-1}\right)dt$$

The first integral can be simplified

$$\int_0^{\infty}\left(\frac{e^t+e^{-t}-2}{t^2\left(e^t-1\right)} - \frac{e^{-t}}{t}\right)dt = \int_0^{\infty}\left(\frac{1-\left(t+1\right)e^{-t}}{t^2}\right)dt,$$

and by integration by parts

$$\int_0^\infty \left(\frac{1-(t+1)\,e^{-t}}{t^2}\right)dt = \int_0^\infty \left(\frac{(t+1)\,e^{-t}-e^{-t}}{t}\right)dt = \int_0^\infty e^{-t}dt = 1.$$

The second integral $\int_0^\infty \left(\frac{e^{-t}}{t} - \frac{1}{e^t-1}\right)dt$ is equal to $-\gamma$ (the Euler's constant). This follows from the value of the Digamma function at 1

$$\psi(1) = -\gamma,$$

and from the integral representation of the Digamma function,

$$\psi(x) = \int_0^\infty \left(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\right)dt.$$

We conclude that

$$\sum_{k=1}^{\infty} \frac{\zeta(2k+1)}{(k+1)(2k+1)} = 1-\gamma.$$

Also solved by Michel Bataille, Rouen, France and the proposer.

128. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*
Let $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$ be real sequences with $a_n \neq a_{n+1}$ and $b_n \neq b_{n+1}$ such that: $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$, $\lim_{n\to\infty} n(a_{n+1}-a_n) = c$ and $\lim_{n\to\infty} n(b_{n+1} - b_n) = d$, where $a, b, c, d \in \mathbb{R}$. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions with continuous derivatives. Calculate

$$\lim_{n\to\infty} n(f(a_{n+1})g(b_{n+1}) - f(a_n)g(b_n)).$$

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria.

For each n there is α_n between a_n and a_{n+1} such that

$$f(a_{n+1} - f(a_n) = (a_{n+1} - a_n)f'(\alpha_n)$$

From $\lim_{n\to\infty} a_n = a$ we conclude that $\lim_{n\to\infty} \alpha_n = a$ and consequently

$$\lim_{n\to\infty} n(f(a_{n+1} - f(a_n)) = cf'(a) \qquad (1)$$

Similarly, we have

$$\lim_{n\to\infty} n(g(a_{n+1} - g(a_n)) = dg'(b) \qquad (2)$$

Hence, if $\Delta_n = n\,(f(a_{n+1})g(b_{n+1}) - f(a_n)g(b_n))$ then

$$\Delta_n = n\,(g(b_{n+1}) - g(b_n))\,f(a_{n+1}) + n\,(f(a_{n+1}) - f(a_n))\,g(b_n)$$

Therefore,

$$\lim_{n\to\infty} \Delta_n = df(a)g'(b) + cf'(a)g(b)$$

which is the desired conclusion.

Solution 2 by Michel Bataille, Rouen, France.

For every positive integer n, we have

$$n\,(f(a_{n+1})f(b_{n+1}) - f(a_n)f(b_n)) = n\,((f(a_{n+1}) - f(a_n))f(b_{n+1}) + (f(b_{n+1}) - f(b_n))f(a_n))\,.$$

Now, from the Mean Value Theorem, we may write

$$f(a_{n+1}) - f(a_n) = (a_{n+1} - a_n)f'(u_n), \quad f(b_{n+1}) - f(b_n) = (b_{n+1} - b_n)f'(v_n)$$

where u_n (resp. v_n) is a real number between a_n and a_{n+1} (resp. between b_n and b_{n+1}).
We deduce that

$$n\left(f(a_{n+1})f(b_{n+1}) - f(a_n)f(b_n)\right) = n(a_{n+1}-a_n)f'(u_n)f(b_{n+1})+n(b_{n+1}-b_n)f'(v_n)f(a_n) \qquad (1).$$

Since $0 \leq |u_n - a_n| \leq |a_{n+1} - a_n|$ and $\lim_{n\to\infty}(a_{n+1} - a_n) = a - a = 0$, we see that $\lim_{n\to\infty}(u_n - a_n) = 0$, hence $\lim_{n\to\infty} u_n = a$. Similarly, $\lim_{n\to\infty} v_n = b$.
Since f and f' are continuous functions, it follows that

$$\lim_{n\to\infty} f(a_n) = f(a),\ \lim_{n\to\infty} f(b_{n+1}) = f(b),\ \lim_{n\to\infty} f'(u_n) = f'(a),\ \lim_{n\to\infty} f'(v_n) = f'(b)$$

and so

$$\lim_{n\to\infty} n(a_{n+1}-a_n)f'(u_n)f(b_{n+1})+n(b_{n+1}-b_n)f'(v_n)f(a_n) = cf'(a)f(b)+df(a)f'(b)$$

and with (1), we may conclude

$$\lim_{n\to\infty} n\left(f(a_{n+1})f(b_{n+1}) - f(a_n)f(b_n)\right) = cf'(a)f(b) + df'(b)f(a).$$

Editorial comment. This a corrected version of the statement of the problem.

129. *Proposed by Florin Stanescu, Serban Cioculescu school, city Gaesti, jud. Dambovita, Romania.* Let $f : [-1, 1] \to \mathbb{R}$ be a function twice differentiable, with the following properties:
a) $f(-1) = f(1) = 0$.
b) f'' is continuous on $[-1, 1]$.
Prove that

$$\max\{(f(x))^2 : x \in [-1, 1]\} \leq \frac{1}{6}\int_{-1}^{1}(f''(x))^2 dx.$$

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria.

Consider $K(x, t) = (1 - \max(x, t))(1 + \min(x, t))$. Then

$$\begin{aligned}\int_{-1}^{1} K(x,t)f''(t)dt &= (1-x)\int_{-1}^{x}(1+t)f''(t)dt + (1+x)\int_{x}^{1}(1-t)f''(t)dt\\ &= (1-x)\Big[(1+t)f'(t)\Big]_{-1}^{x} - (1-x)\int_{-1}^{x} f'(t)dt\\ &\quad + (1+x)\Big[(1-t)f'(t)\Big]_{x}^{1} + (1+x)\int_{x}^{1} f'(t)dt\\ &= -(1-x)f(x) - (1+x)f(x) = -2f(x)\end{aligned}$$

Thus, by the Cauchy-Schwarz inequality we have

$$4(f(x))^2 \leq \int_{-1}^{1}(K(x,t))^2 dt \int_{-1}^{1}(f''(t))^2 dt$$

But

$$\begin{aligned}\int_{-1}^{1}(K(x,t))^2 dt &= (1-x)^2\int_{-1}^{x}(1+t)^2 + (1+x)^2\int_{x}^{1}(1-t)^2 dt\\ &= (1-x)^2\frac{(1+x)^3}{3} + (1+x)^2\frac{(1-x)^3}{3} = \frac{2}{3}(1-x^2)^2\end{aligned}$$

Thus

$$(f(x))^2 \leq \frac{1}{6}(1-x^2)^2 \int_{-1}^{1} (f''(t))^2 dt$$

and the desired conclusion follows since $(1-x^2)^2 \leq 1$ for $x \in [-1,1]$.

Solution 2 by Moti Levy, Rehovot, Israel.

If $f(x) \equiv 0$ on $[-1,1]$ then the inequality is trivially true. Otherwise, $f^{'}(-1) < 0$ and $f^{'}(1) > 0$.

Now assume that $f(x)$ attains its minimum at $-1 < \xi < 1$. We make the following change of variable:

$$t = \frac{\xi}{1-\xi^2}x^2 + x - \frac{\xi}{1-\xi^2}.$$

Note that $t = 1$ when $x = 1$, $t = -1$when $x = -1$ and $t = 0$ when $x = \xi$.

The function $F(t) := f(x)|_{x=p(t,\xi)}$ attains its minimum at $t = 0$, so that $\frac{dF}{dt}(0) = 0$, and $\max\left\{(f(x))^2;\ x \in [-1,1]\right\} = \max\left\{F^2(t);\ t \in [-1,1]\right\} = F^2(0)$.

In terms of $F(t)$, the original inequality becomes

$$\frac{1}{6}\left(\int_{-1}^{1}\left(\frac{d^2F(t)}{dt^2}\right)^2 \frac{1}{\frac{dp(t,\xi)}{dt}}dt\right) \geq F^2(0). \tag{11}$$

Let $g : [-1,1] \to \mathbb{R}$, be a twice differentiable function defined as follows:

$$g(t) = \begin{cases} \frac{1}{2}(1-t)\frac{dp(t,\xi)}{dt}, & 0 \leq t \leq 1, \\ \frac{1}{2}(1+t)\frac{dp(t,\xi)}{dt}, & -1 \leq t \leq 0. \end{cases} \tag{12}$$

Then $g(t)$ has the following properties:

$$g(-1) = g(1) = 0, \tag{13}$$

$$\left.\frac{d\left(g(t)\frac{1}{\frac{dp(t,\xi)}{dt}}\right)}{dt}\right|_{t=0^+} = -\frac{1}{2}, \tag{14}$$

$$\left.\frac{d\left(g(t)\frac{1}{\frac{dp(t,\xi)}{dt}}\right)}{dt}\right|_{t=0^-} = \frac{1}{2}, \tag{15}$$

$$\int_{-1}^{1}\left(g(t)\frac{1}{\frac{dp(t,\xi)}{dt}}\right)^2 dt = \frac{1}{6}. \tag{16}$$

$$\frac{1}{6}\left(\int_{-1}^{1}\left(\frac{d^2F(t)}{dt^2}\right)^2 \frac{1}{\frac{dp(t,\xi)}{dt}}dt\right) = \left(\int_{-1}^{1}\left(g(t)\frac{1}{\dot{p}(t,\xi)}\right)^2 dt\right)\left(\int_{-1}^{1}\left(\frac{d^2F(t)}{dt^2}\right)^2 \frac{1}{\frac{dp(t,\xi)}{dt}}dt\right)$$

By Cauchy-Schwarz inequality

$$\left(\int_{-1}^{1}\left(g(t)\frac{1}{\dot{p}(t,\xi)}\right)^2 dt\right)\left(\int_{-1}^{1}\left(\frac{d^2F(t)}{dt^2}\right)^2 \frac{1}{\frac{dp(t,\xi)}{dt}}dt\right) \geq \left(\int_{-1}^{1} g(t)\frac{1}{\frac{dp(t,\xi)}{dt}}\frac{d^2F(t)}{dt^2}\right)^2 \tag{17}$$

$$\int_0^1 g(t)\,\frac{1}{\frac{dp(t,\xi)}{dt}}\frac{d^2F(t)}{dt^2}dt = g(t)\,\frac{1}{\frac{dp(t,\xi)}{dt}}\frac{dF(t)}{dt}\Bigg]_0^1 - \int_0^1 \frac{d\left(g(t)\,\frac{1}{\frac{dp(t,\xi)}{dt}}\right)}{dt}\frac{dF(t)}{dt}dt$$
$$= g(1)\,\frac{1}{\frac{dp(1,\xi)}{dt}}\frac{dF(1)}{dt} - \int_0^1 \frac{d\left(g(t)\,\frac{1}{\frac{dp(t,\xi)}{dt}}\right)}{dt}\frac{dF(t)}{dt}dt$$
$$= \left.\frac{d\left(g(t)\,\frac{1}{\frac{dp(t,\xi)}{dt}}\right)}{dt}\right|_{t=0} F(0) + \int_0^1 \frac{d^2\left(g(t)\,\frac{1}{\frac{dp(t,\xi)}{dt}}\right)}{dt^2}F(t)\,dt = -\frac{F(0)}{2}.$$

Similarly,

$$\int_{-1}^0 g(t)\,\frac{1}{\frac{dp(t,\xi)}{dt}}\frac{d^2F(t)}{dt^2}dt$$
$$= -g(-1)\,\frac{1}{\frac{1}{\frac{dp(-1,\xi)}{dt}}}\frac{dF(-1)}{dt} - \left.\frac{d\left(g(t)\,\frac{1}{\frac{dp(t,\xi)}{dt}}\right)}{dt}\right|_{t=0} F(0) + \int_{-1}^0 \frac{d^2\left(g(t)\,\frac{1}{\frac{dp(t,\xi)}{dt}}\right)}{dt^2}F(t)\,dt = -\frac{F(0)}{2}.$$

We have shown that

$$\int_{-1}^1 g(t)\,\frac{1}{\frac{dp(t,\xi)}{dt}}\frac{d^2F(t)}{dt^2}dt = -F(0)\,. \tag{18}$$

The inequality (11) is a consequence of (17) and (18).

Solution 3 by Ramya Dutta, Chennai Mathematical Institute (student) India.

For $x \in [-1, 1]$, integrating by parts,

$$f(x) - f(-1) = \int_{-1}^x f'(t)\,dt = (x+1)f'(x) - \int_{-1}^x (t+1)f''(t)\,dt \tag{1}$$
$$f(1) - f(x) = \int_x^1 f'(t)\,dt = (1-x)f'(x) + \int_x^1 (1-t)f''(t)\,dt \tag{2}$$

Multiplying (1) with $(1-x)$ and (2) with $(1+x)$ and subtracting,

$$2f(x) = -(1-x)\int_{-1}^x (1+t)f''(t)\,dt - (1+x)\int_x^1 (1-t)f''(t)\,dt$$
$$= -\int_{-1}^1 \phi_x(t)f''(t)\,dt$$

where, $\phi_x(t) = \begin{cases} (1-x)(1+t) & \text{when } t \in [-1, x) \\ (1+x)(1-t) & \text{when } t \in [x, 1] \end{cases}$

Now, $\phi_x(t)$ is continuous in $[-1,1]$. Applying Cauchy-Schwarz inequality,

$$4f^2(x) = \left(\int_{-1}^{1} \phi_x(t) f''(t)\,dt\right)^2 \le \int_{-1}^{1} (\phi_x(t))^2\,dt \int_{-1}^{1} (f''(t))^2\,dt$$
$$\le \frac{2}{3}\int_{-1}^{1} (f''(t))^2\,dt$$
$$\implies 6 \max_{x\in[-1,1]} (f(x))^2 \le \int_{-1}^{1} (f''(t))^2\,dt$$

Since, for $x \in [-1,1]$,

$$\int_{-1}^{1} (\phi_x(t))^2\,dt = (1-x)^2\int_{-1}^{x}(1+t)^2\,dt + (1+x)^2\int_{x}^{1}(1-t)^2\,dt$$
$$= \frac{(1-x)^2(1+x)^3}{3} + \frac{(1+x)^2(1-x)^3}{3}$$
$$= \frac{2}{3}\left(1-x^2\right)^2 \le \frac{2}{3}$$

Also solved by the proposer.

130. *Proposed by Mohammed Aassila, Strasbourg, France.* Among the first 2016 positive integers (from 1 to 2016) we underline those which may be represented as the sum of 5 nonnegative integer powers of 2. Is the set of underlined numbers larger than that of the nonunderlined ones ?

Solution 1 by Ramya Dutta, Chennai Mathematical Institute (student) India.

Since, $2016 < 2048 = 2^{11}$, the binary representation of all integers up to 2016 has at most 12 digits. Integers with 6 or more 1's in binary representation cannot be underlined (not representable as sum of 5 non negative integer powers of 2). Integers with five 1's in binary representation,

$\sum_{j=1}^{5} 2^{a_j}$ with $a_1 > a_2 > a_3 > a_4 > a_5$ are underlined, there are $\binom{12}{5}$ such integers less than 2048. Integers with four 1's in binary representation,

$\sum_{j=1}^{4} 2^{a_j} = 2^{a_1-1} + 2^{a_1-1} + \sum_{j=2}^{4} 2^{a_j}$ with $a_1 > a_2 > a_3 > a_4$ are underlined, there are $\binom{12}{4}$ such integers less than 2048. Integers with three 1's in binary representation,

$\sum_{j=1}^{3} 2^{a_j} = 2^{a_1-1} + 2^{a_1-1} + 2^{a_2-1} + 2^{a_2-1} + 2^{a_3}$ with $a_1 > a_2 > a_3$ are underlined, there are $\binom{12}{3}$ such integers less than 2048. Integers with two 1's in binary representation,

$\sum_{j=1}^{2} 2^{a_j} = 2^{a_1-2} + 2^{a_1-2} + 2^{a_1-2} + 2^{a_1-2} + 2^{a_2}$ with $a_1 > a_2$ and $a_1 \ge 2$ are underlined, there are $\binom{12}{2} - 1$ (that is excepting $4 = 11_2$ from $\binom{12}{2}$ such integers) such integers less than 2048.

Thus, at least $\sum_{j=2}^{5}\binom{12}{j} - 1 = 2^{11} - \frac{1}{2}\binom{12}{6} - 14 = 1572$ integers less than 2048 are underlined, i.e., at least $1572 - (2048 - 2016) = 1540$ integers less than 2016 are underlined, which clearly exceeds $1008 = \frac{2016}{2}$. The number of underlined integers exceed the number of non-underlined ones.

Solution 2 by José M. Pacheco and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

Any sum of 5 powers of 2 is an expression of the form $2^{k_1} + 2^{k_2} + 2^{k_3} + 2^{k_4} + 2^{k_5}$. If the sum cannot be larger than 2016, the available values for the exponents range from 0 to 10, since $2^{11} = 2048 > 2016$. There are two different ways of solving the problem, depending on whether repetition of the $k_i's$ is allowed or not.

a) Repeated exponents allowed.

The largest available integers could be obtained as:

$$\begin{aligned} 2^{10} + 2^{10} + 2^{10} + 2^{10} + 2^{10} &= 5120 \text{ (impossible)} \\ 2^{9} + 2^{9} + 2^{9} + 2^{9} + 2^{9} &= 2560 \text{ (impossible)} \\ 2^{8} + 2^{8} + 2^{8} + 2^{8} + 2^{8} &= 1280 < 2016 \text{ (acceptable).} \end{aligned}$$

The number of integers $N \leq 1280$ which are representable as sum of five powers of 2 is

$$CR_5^9 = \binom{9+5-1}{5} = \binom{13}{5} = \frac{13!}{5!\ 8!} = 1287 > 1008 = \frac{2016}{2}.$$

Therefore, the set of underlined integers is **larger** than its complementary.

b) Repeated exponents not allowed.

Now we can allow exponents up to 10: The *worst* case would be $2^{10} + 2^9 + 2^8 + 2^7 + 2^6 = 1984 < 2016$. Therefore, the number of integers $N \leq 1280$ which are also representable as sum of five powers of 2 is

$$C_5^{11} = \binom{11}{5} = \frac{11!}{5!\ 6!} = 462 < 1008 = \frac{2016}{2},$$

and under this assumption, there are **fewer** underlined integers than not underlined ones.

Also solved by the proposer.

MATHCONTEST SECTION

This section of the Journal offers readers an opportunity to solve interesting and elegant mathematical problems mainly appeared in Math Contest around the world and most appropriate for training Math Olympiads. Proposals are always welcomed. The source of the proposals will appear when the solutions be published.

Proposals

90. Let f and g be two continuous, distinct functions from $[0,1] \to (0,+\infty)$ such that $\int_0^1 f(x)dx = \int_0^1 g(x)dx$. Let $y_n = \int_0^1 \frac{f^{n+1}(x)}{g^n(x)}dx$, for $n \geq 0$, natural. Prove that $(y_n)_{n\geq 1}$ is an increasing and divergent sequence.

91. Let $(a_n)_{n\geq 1} \subset \left(\frac{1}{2}, 1\right)$. Define the sequence $x_0 = 0, x_{n+1} = \dfrac{a_{n+1} + x_n}{1 + a_{n+1}x_n}$. Is this sequence convergent? If yes find the limit.

92. For a positive integer n, define $f(n)$ to be the number of sequences $(a_1, a_2, \ldots, a_k)$ such that $a_1 a_2 \cdots a_k = n$ where $a_i \geq 2$ and $k \geq 0$ is arbitrary. Also we define $f(1) = 1$. Now let $\alpha > 1$ be the unique real number satisfying $\sum_{n=1}^{\infty} \frac{1}{n^\alpha} = 2$.
Prove that
(a)

$$\sum_{j=1}^{n} f(j) = \mathcal{O}(n^\alpha)$$

(b) There is no real number $\beta < \alpha$ such that

$$\sum_{j=1}^{n} f(j) = \mathcal{O}(n^\beta).$$

93. Let $c \geq 1$ be a real number. Let G be an Abelian group and let $A \subset G$ be a finite set satisfying $|A + A| \leq c|A|$, where $X + Y := \{x + y | x \in X, y \in Y\}$ and $|Z|$ denotes the cardinality of Z. Prove that

$$|\underbrace{A + A + \cdots + A}_{k}| \leq c^k |A|$$

for every positive integer k.

94. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that
$(f(x + y + z))^2 = (f(x))^2 + (f(y))^2 + (f(z))^2 + 2(f(xy) + f(xz) + f(yz))$, for all x, y, z real numbers.

Solutions

85. Prove that

$$\lim_{n\to\infty} n\left(\frac{\pi}{4} - n\int_0^1 \frac{x^n}{1+x^{2n}}dx\right) = \int_0^1 f(x)dx$$

where $f(x) = \frac{\arctan x}{x}$ if $x \in (0,1]$ and $f(0) = 1$.

(Romania National Olympiad 2005)

Solution by Henry Ricardo, New York Math Circle, New York, USA.
We have, making one substitution and integrating by parts,

$$\begin{aligned}
n\left(\frac{\pi}{4} - n\int_0^1 \frac{x^n}{1+x^{2n}}\,dx\right) &\overset{x^n=y}{=} n\left(\frac{\pi}{4} - \int_0^1 \frac{y^{1/n}}{1+y^2}\,dy\right)\\
&= n\left(\int_0^1 \frac{dx}{1+x^2} - \int_0^1 \frac{x^{1/n}}{1+x^2}\,dx\right) = n\int_0^1 \frac{1-x^{1/n}}{1+x^2}\,dx\\
&= n(1-x^{1/n})\cdot \int_0^x \frac{dt}{1+t^2}\Big|_0^1 + \int_0^1 \frac{\int_0^x \frac{dt}{1+t^2}}{x}\cdot x^{1/n}\,dx\\
&= \int_0^1 \frac{\int_0^x \frac{dt}{1+t^2}}{x}\cdot x^{1/n}\,dx.
\end{aligned}$$

Let $v_n : [0,1] \to \mathbb{R}$ be the sequence defined by

$$v_n(x) = \frac{\int_0^x \frac{dt}{1+t^2}}{x}\cdot x^{1/n} = x^{1/n-1}\int_0^x \frac{dt}{1+t^2}.$$

Then we can calculate the limit function as follows:

$$v(x) = \lim_{n\to\infty} v_n(x) = \begin{cases} \frac{\int_0^x \frac{dt}{1+t^2}}{x} & \text{if } x \in (0,1)\\ 0 & \text{if } x = 0\\ \frac{\pi}{4} & \text{if } x = 1. \end{cases}$$

Finally, noting that $|\int_0^x \frac{dt}{1+t^2}| \le x \le 1$, we use the Bounded Convergence Theorem to conclude that

$$\begin{aligned}
\lim_{n\to\infty} n\left(\frac{\pi}{4} - n\int_0^1 \frac{x^n}{1+x^{2n}}\,dx\right) &= \lim_{n\to\infty}\int_0^1 v_n(x)\,dx = \int_0^1 \left(\lim_{n\to\infty} v_n(x)\right)\,dx\\
&= \int_0^1 \frac{\int_0^x \frac{dt}{1+t^2}}{x}\,dx = \int_0^1 \frac{\arctan x}{x}\,dx,
\end{aligned}$$

where we note that $\lim_{x\to 0}\dfrac{\arctan x}{x} = 1$.

COMMENT: This is a special case of a result (problem 1.64) proved by Ovidiu Furdui in his book *Limits, Series, and Fractional Part Integrals* (Springer, 2013):

Let $f:[0,1]\to\mathbb{R}$ be continuous and $g:[0,1]\to\mathbb{R}$ be a continuously differentiable function. Then (1) $L=\lim_{n\to\infty} n\int_0^1 x^n f(x^n)g(x)dx = g(1)\int_0^1 f(x)dx$ and (2) $\lim_{n\to\infty} n\left(n\int_0^1 x^n f(x^n)g(x)\,dx - L\right) = -(g(1)+g'(1))\int_0^1 \frac{\int_0^x f(t)dt}{x}\,dx$.

Also solved Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; Florin Stanescu, Serban Cioculescu school, city Gaesti, jud. Dambovita, Romania; Arkady Alt, San Jose, California, USA and Michel Bataille, Rouen, France.

86. Define the sequence $a_0, a_1, \ldots,$ inductively by $a_0 = 1$, $a_1 = \frac{1}{2}$, and

$$\forall\, n \geq 1, \qquad a_{n+1} = \frac{na_n^2}{1+(n+1)a_n},$$

Show that the series $\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$ converges and determine its value.

(Romania National Olympiad 2001)

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

It is clear that $a_n > 0$ for $n \geq 0$. Note that for $n \geq 1$ we have

$$\frac{a_{n+1}}{a_n} = na_n - (n+1)a_{n+1}. \qquad (1)$$

This proves that the sequence $\{na_n\}_{n\geq 1}$ is positive decreasing so it must converge to some nonnegative limit ℓ. If $\ell > 0$ we conclude that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)a_{n+1}}{na_n} \times \frac{n}{n+1} = 1$$

Taking the limit as n tends to $+\infty$ in (1) leads to the contradiction $1 = 0$. Thus $\ell = 0$, that is $\lim_{n\to\infty} na_n = 0$. Now, form (1) we get

$$\sum_{n=0}^{m-1} \frac{a_{n+1}}{a_n} = \frac{a_1}{a_0} + a_1 - ma_m.$$

Letting m tend to ∞ we get

$$\sum_{n=0}^{\infty} \frac{a_{n+1}}{a_n} = 1.$$

Solution 2 by Michel Bataille, Rouen, France.

Note that $a_n > 0$ for every positive integer n (by induction). Let k be a positive integer. Then we have

$$\frac{a_{k+1}}{a_k} = \frac{ka_k}{1+(k+1)a_k} = \frac{ka_k(1+(k+1)a_k) - k(k+1)a_k^2}{1+(k+1)a_k} = ka_k - (k+1)a_{k+1} \qquad (1)$$

and for any positive integer K, we may write

$$\sum_{k=0}^{K} \frac{a_{k+1}}{a_k} = \frac{a_1}{a_0} + \sum_{k=1}^{K} \frac{a_{k+1}}{a_k} = \frac{1}{2} + \sum_{k=1}^{K}(ka_k - (k+1)a_{k+1}) = \frac{1}{2} + (a_1 - (K+1)a_{K+1}) = 1 - (K+1)a_{K+1} \qquad (2).$$

Now, from (1), $ka_k - (k+1)a_{k+1} > 0$ for all $k \in \mathbb{N}$, hence the sequence $\{ka_k\}_{k\geq 1}$ is decreasing. As this sequence is also bounded below (by 0), it is convergent. Let

$\ell = \lim_{k\to\infty} ka_k$. Since $a_k = \frac{1}{k}\cdot(ka_k)$, we have $\lim_{k\to\infty} a_k = 0\cdot\ell = 0$; we also have $\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty}(ka_k - (k+1)a_{k+1}) = \ell - \ell = 0$. Thus

$$0 = \lim_{k\to\infty} \frac{a_{k+1}}{a_k} = \lim_{k\to\infty} \frac{ka_k}{1 + a_k + ka_k} = \frac{\ell}{1+0+\ell} = \frac{\ell}{\ell+1}$$

and so $\ell = 0$. From (2), we may conclude that the series $\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$ is convergent and its sum is $1 - \lim_{K\to\infty}(K+1)a_{K+1} = 1$.

Also solved by Arkady Alt, San Jose, California, USA.

87. Edited. Let $f : [0,\infty) \to \mathbb{R}$ be a continuous 1-periodic function. For a strictly increasing and unbounded sequence $(x_n)_{n\geq 0}$ such that $x_0 = 0$, and $\lim_{n\to+\infty}(x_{n+1} - x_n) = 0$, we denote $r(n) = \max\{k | x_k \leq n\}$.

a) Show that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{r(n)} (x_{k+1} - x_k) f(x_k) = \int_0^1 f(t)dt$.

b) Show that $\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\lfloor e^n \rfloor} \frac{f(\ln k)}{k} = \int_0^1 f(t)dt$.

(IMC 2012)

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

a) Consider $\tau_n = (t_0, \ldots, t_{m+1})$ the subdivision of the interval $[0,1]$ defined by

$$t_0 = 0,\ t_{m+1} = 1, \quad t_k = x_{r(n-1)+k} - n + 1 \quad \text{for } 1 \leq k \leq m = r(n) - r(n-1).$$

Clearly, the step h_n of this subdivision satisfies

$$h_n = \max\{t_{k+1} - t_k : 0 \leq k \leq m\} \leq \max\{x_j - x_{j-1} : j \geq r(n-1)\}$$

and the Riemann sum corresponding to this subdivision is

$$R(f,\tau_n) = \sum_{k=0}^{m} (t_{k+1} - t_k) f(t_k)$$

Now, the fact that $\{x_n\}$ is strictly increasing and unbounded implies that $\{r(n)\}$ is also strictly increasing and unbounded. Thus the condition $\lim_{n\to+\infty}(x_{n+1} - x_n) = 0$ implies that $\lim_{n\to\infty} h_n = 0$. Hence

$$\lim_{n\to\infty} R(f,\tau_n) = \int_0^1 f(x)dx.$$

Moreover,

$$\begin{aligned} R(f,\tau_n) = &\sum_{j=r(n-1)+1}^{r(n)} (x_{j+1} - x_j) f(x_j) \\ &+ (x_{r(n-1)+1} - n + 1) f(0) + (n - x_{r(n)+1}) f(x_{r(n)}) \\ = &\ A_n + (x_{r(n-1)+1} - n + 1) f(0) + (n - x_{r(n)+1}) f(x_{r(n)}) \end{aligned}$$

where

$$A_n = \sum_{j=1+r(n-1)}^{r(n)} (x_{j+1} - x_j) f(x_j). \qquad \text{for } n \geq 1.$$

Now, since $0 \leq n - x_{r(n)} \leq x_{r(n)+1} - x_{r(n)}$ we conclude that $\lim_{n\to\infty}(x_{r(n)} - n) = 0$ and the continuity of f at 0 shows that $\lim_{n\to\infty} f(x_{r(n)}) = f(0)$. Therefore

$$\lim_{n\to\infty} A_n = \int_0^1 f(x)dx$$

Finally, by Cesàro's lemma we conclude that

$$\lim_{n\to\infty} \frac{A_1 + \cdots + A_n}{n} = \int_0^1 f(x)dx$$

which is equivalent to a).

Remark. With only the integrability assumption on f, the conclusion of a) is not correct. For example, consider f the characteristic function of the irrational numbers ($f = \mathbb{I}_{\mathbb{R}\setminus\mathbb{Q}}$), which is 1- periodic and integrable on $[0,1]$. Moreover, consider $x_n = \sum_{k=1}^n 1/k$. Clearly $f(x_k) = 0$ for every k, So

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{r(n)} (x_{k+1} - x_k) f(x_k) = 0 \neq 1 = \int_0^1 f(x)dx$$

.

b) Applying a) with $x_n = \ln n$ for $n \geq 1$, we see that $r(n) = \lfloor e^n \rfloor$, so

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\lfloor e^n \rfloor} \ln\left(1 + \frac{1}{k}\right) f(\ln k) = \int_0^1 f(t)dt. \quad (1)$$

But

$$\left| \frac{1}{n} \sum_{k=1}^{\lfloor e^n \rfloor} \left(\frac{1}{k} - \ln\left(1 + \frac{1}{k}\right)\right) f(\ln k) \right| \leq \frac{\sup_{[0,1]} |f|}{n} \sum_{k=1}^{\infty} \left| \frac{1}{k} - \ln\left(1 + \frac{1}{k}\right)\right|$$

Thus

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\lfloor e^n \rfloor} \left(\frac{1}{k} - \ln\left(1 + \frac{1}{k}\right)\right) f(\ln k) = 0. \quad (2)$$

Adding (1) and (2) we get

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{\lfloor e^n \rfloor} \frac{f(\ln k)}{k} = \int_0^1 f(t)dt.$$

Editorial comment: This is Problem 3356 proposed in Crux Mathematicorum (Sept. 2008). A detailed solution appears in Crux Mathematicorum, 34: No 5, September 2009, p. 341-3.

Also solved by Michel Bataille, Rouen, France and Stanescu, Serban Cioculescu school, city Gaesti, jud. Dambovita, Romania.

88. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$(x + y)f(2yf(x) + f(y)) = x^3 f(yf(x))$$

for all $x, y \in \mathbb{R}^+$.

(BMO-shortlist 2015)

Solution by Michel Bataille, Rouen, France.

We show that no such function can exist. Assume that f is a solution. Let us show that f is injective. To this end, consider $x, y > 0$ such that $f(x) = f(y)$. From the equation, we get

$$(x+y)f\left((2y+1)f(y)\right) = x^3 f\left(yf(y)\right) \qquad (1).$$

On the other hand, taking $x = y$ in the equation yields

$$2f\left((2y+1)f(y)\right) = y^2 f\left(yf(y)\right) \qquad (2).$$

Combining (1) and (2), we obtain

$$(x+y)y^2 f\left(yf(y)\right) = 2x^3 f\left(yf(y)\right)$$

or, since $f\left(yf(y)\right) > 0$, $(x+y)y^2 = 2x^3$, that is, $(x-y)((x+y)^2 + x^2) = 0$. Since $(x+y)^2 + x^2 > 0$, we must have $x = y$, as desired. Now, let us take $y = \sqrt{2}$ in (2): $2f\left((2\sqrt{2}+1)f(\sqrt{2})\right) = 2f\left(\sqrt{2}f(\sqrt{2})\right)$. Since f is injective, we have $(2\sqrt{2}+1)f(\sqrt{2}) = \sqrt{2}f(\sqrt{2})$, hence $f(\sqrt{2}) = 0$, a contradiction with $f(x) > 0$ for all $x \in \mathbb{R}^+$. The conclusion follows.

89. Find all real positive solutions (if any) to

$$x^3 + y^3 + z^3 = x + y + z, \text{ and}$$
$$x^2 + y^2 + z^2 = xyz.$$

(Canada National Olympiad 2005)

Solution by Michel Bataille, Rouen, France.

There is no solution. For the purpose of a contradiction, assume that (x, y, z) satisfies the two equations and $x, y, z > 0$. Then, from AM-GM, $xyz = x^2 + y^2 + z^2 \geq 3\sqrt[3]{x^2y^2z^2}$, hence $x^3y^3z^3 \geq 27x^2y^2z^2$ and so $xyz \geq 27$. Since $x + y + z = x^3 + y^3 + z^3 \geq 3xyz$, it follows that $x + y + z \geq 81$.
Now, by the Cauchy-Schwarz inequality

$$3(x^2 + y^2 + z^2) = (1+1+1)(x^2 + y^2 + z^2) \geq (x+y+z)^2,$$

hence we would have

$$x^3 + y^3 + z^3 - 3xyz = x + y + z - 3(x^2 + y^2 + z^2) \leq (x+y+z) - (x+y+z)^2 < 0$$

(the last inequality because $x + y + z \geq 81 > 1$). The obtained inequality $x^3 + y^3 + z^3 - 3xyz < 0$ is the sought contradiction since $x^3 + y^3 + z^3 \geq 3xyz$ by AM-GM.

Also solved by Arkady Alt, San Jose, California, USA and Florin Stanescu, Serban Cioculescu school, city Gaesti, jud. Dambovita, Romania.

MATHNOTES SECTION

Series expansion of a function defined by integral

Anastasios Kotronis

Abstract. In this article we generalize a problem that was proposed by Murray Klamkin and Andy Liu on College Mathematical Journal and a closely related one that was discussed on the Art Of Problem Solving forum. We approach the problem in two ways.

1. Introduction

On January 1992 issue of the College Mathematical Journal (see [1]) Murray Klamkin and Andy Liu proposed the following problem:

$$\text{If}\quad I(n)=\int_1^{+\infty}\frac{1}{1+x^{n+1}}\,dx,\quad n>0,\quad \text{show that}\quad \frac{\ln 2}{n}<I(n)<\frac{\ln 2}{n}+\frac{1}{4n^2}.$$

An asymptotic estimate of the closely related integral $\int_0^1\frac{1}{1+x^n}\,dx$ was also discussed on Art Of Problem Solving forum (see [2]). We generalize the above giving a complete expansion of the mentioned integrals, in terms of the Riemann zeta function, approaching the problem by two ways. On what follows we denote:

$$A_1(t):=\int_0^1\frac{1}{1+x^t}\,dx\quad\text{and}\quad A_2(t):=\int_1^{+\infty}\frac{1}{1+x^t}\,dx;\qquad t>1$$

and we will show that for $t>1$:

$$A_1(t)=1+\sum_{k\geq 1}(-1)^k\frac{(1-2^{1-k})\zeta(k)}{t^k}\quad\text{and}\quad A_2(t)=\sum_{k\geq 1}\frac{(1-2^{1-k})\zeta(k)}{t^k}$$

where for the first summation index, $(1-2^{1-k})\zeta(k)$ is interpreted as $\lim_{k\to 1^+}(1-2^{1-k})\zeta(k)$.

2. Main Results

Theorem. Let

$$A_1(t):=\int_0^1\frac{1}{1+x^t}\,dx\quad\text{and}\quad A_2(t):=\int_1^{+\infty}\frac{1}{1+x^t}\,dx;\qquad t>1.$$

For $t>1$:

$$A_1(t)=1+\sum_{k\geq 1}(-1)^k\frac{(1-2^{1-k})\zeta(k)}{t^k}\quad\text{and}\quad A_2(t)=\sum_{k\geq 1}\frac{(1-2^{1-k})\zeta(k)}{t^k}$$

where for the first summation index, $(1-2^{1-k})\zeta(k)$ is interpreted as $\lim_{k\to 1^+}(1-2^{1-k})\zeta(k)$.

Proof. On this first approach we expand $A_1(t)$ and $A_2(t)$ directly:

$$\begin{aligned}
A_1(t) &= \int_0^1 \sum_{i\geq 0}(-1)^i x^{it}\,dx \overset{(a)}{=} \sum_{i\geq 0}\int_0^1 (-1)^i x^{it}\,dx = \sum_{i\geq 0}\frac{(-1)^i}{it+1} = 1+\sum_{i\geq 1}\frac{(-1)^i}{it}\cdot\frac{1}{1+\frac{1}{it}}\\
&= 1+\sum_{i\geq 1}\frac{(-1)^i}{it}\sum_{k\geq 0}\frac{(-1)^k}{(it)^k} = 1+\sum_{i\geq 1}\sum_{k\geq 1}\frac{(-1)^{i+k-1}}{(it)^k} = 1+\sum_{i\geq 1}\left(\frac{(-1)^i}{it}+\sum_{k\geq 2}\frac{(-1)^{i+k-1}}{(it)^k}\right)\\
&= 1+\sum_{i\geq 1}\frac{(-1)^i}{it}+\sum_{i\geq 1}\sum_{k\geq 2}\frac{(-1)^{i+k-1}}{(it)^k}\\
&\overset{(b)}{=} 1+\sum_{i\geq 1}\frac{(-1)^i}{it}+\sum_{k\geq 2}\sum_{i\geq 1}\frac{(-1)^{i+k-1}}{(it)^k} = 1+\sum_{k\geq 1}\sum_{i\geq 1}\frac{(-1)^{i+k-1}}{(it)^k}\\
&= 1+\sum_{k\geq 1}\frac{(-1)^k}{t^k}\sum_{i\geq 1}\frac{(-1)^{i-1}}{i^k} = 1-\frac{\ln 2}{t}+\sum_{k\geq 2}\frac{(-1)^k}{t^k}\sum_{i\geq 1}\frac{(-1)^{i-1}}{i^k}\\
&= 1-\frac{\ln 2}{t}+\sum_{k\geq 2}\frac{(-1)^k}{t^k}\left(\sum_{i\geq 1}\frac{1}{(2i-1)^k}-\frac{1}{(2i)^k}\right)\\
&= 1-\frac{\ln 2}{t}+\sum_{k\geq 2}\frac{(-1)^k}{t^k}\left(\sum_{i\geq 1}\left(\frac{1}{(2i-1)^k}+\frac{1}{(2i)^k}\right)-2^{1-k}\sum_{i\geq 1}\frac{1}{i^k}\right)\\
&= 1-\frac{\ln 2}{t}+\sum_{k\geq 2}(-1)^k\frac{(1-2^{1-k})\zeta(k)}{t^k}
\end{aligned}$$

where the change of integration and summation order in (a) is justified by the dominated convergence theorem and the change of summation order in (b) is justified by absolute convergence. Now, for $k>1$, since from the monotonicity of x^k on $[1,+\infty)$ we have

$$\frac{1}{k-1}=\int_1^{+\infty}x^{-k}\,dx\leq\sum_{i\geq 1}\frac{1}{i^k}\leq 1+\int_1^{+\infty}x^{-k}\,dt=\frac{k}{k-1},$$

which yields

$$\lim_{k\to 1^+}(1-2^{1-k})\zeta(k)=\lim_{k\to 1^+}\frac{(1-2^{1-k})}{k-1}\cdot(k-1)\zeta(k)=\ln 2,$$

we can write

$$A_1(t)=1+\sum_{k\geq 1}(-1)^k\frac{(1-2^{1-k})\zeta(k)}{t^k},\qquad t>1$$

as desired. For $A_2(t)$ we perform the change of variables $x=1/y$ and the procedure is almost the same. □

Now let us see the second approach of the announced result.

Proof. Here we expand $A_2(t)$ directly, but in a somewhat more complicated way than in the first approach (the same could be done with $A_1(t)$). $A_1(t)$ is expanded indirectly from the expansion of $A_1(t)+A_2(t)$ which is easier to be found.

For $A_1(t)+A_2(t)$, with $t>1$ we have:

$$\int_0^{+\infty}\frac{1}{1+x^t}\,dx \overset{u=x/t}{=\!=\!=} \frac{1}{t}\int_0^{+\infty}\frac{u^{-\frac{1}{t}}}{1+u}\,du=\frac{1}{t}\mathrm{B}\left(1-\frac{1}{t},\frac{1}{t}\right)=\frac{1}{t}\Gamma\left(1-\frac{1}{t}\right)\Gamma\left(\frac{1}{t}\right)=\frac{\pi}{t}\csc\left(\frac{\pi}{t}\right)$$
$$=\sum_{k\geq 0}(-1)^{k+1}\frac{2(2^{2k-1}-1)\mathrm{B}_{2k}\pi^{2k}}{(2k)!t^{2k}}=\sum_{k\geq 0}\frac{2(1-2^{1-2k})\zeta(2k)}{t^{2k}}$$
$$=\sum_{k\geq 0}\frac{\left(1+(-1)^k\right)(1-2^{1-k})\zeta(k)}{t^k}=1+\sum_{k\geq 1}\frac{\left(1+(-1)^k\right)(1-2^{1-k})\zeta(k)}{t^k}$$

where B_k denotes the $k-$th Bernoulli number, $\mathrm{B}(x,y),\Gamma(x)$ denote the Beta and Gamma function respectively and we used that

$$\mathrm{B}(x+y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)},$$

$$\Gamma(1-x)\Gamma(x)=\pi\csc(\pi x),\quad 0<x<1,$$

$$\csc x=\sum_{k\geq 0}(-1)^{k+1}\frac{2(2^{2k-1}-1)\mathrm{B}_{2k}}{(2k)!}x^{2k-1},\quad 0<|x|<\pi,\quad \text{(see [3])}$$

$$\zeta(2k)=(-1)^{k+1}\frac{\mathrm{B}_{2k}(2\pi)^{2k}}{2(2k)!},\quad k\in\mathrm{N}^*,\qquad \zeta(0)=-\frac{1}{2}\quad \text{(see [4])}.$$

We proceed expanding $A_2(t)$ as follows:

$$A_2(t)=\int_1^{+\infty}\frac{x^{-t}}{1+x^{-t}}\,dx=\int_1^{+\infty}\frac{x^{-t}(1-x^{-t})}{1-x^{-2t}}\,dx \overset{x^{-2t}=y}{=\!=\!=} \frac{1}{2t}\int_0^1\frac{y^{-\frac{1}{2}\left(\frac{1}{t}+1\right)}-y^{-\frac{1}{2t}}}{1-y}\,dy$$
$$=\frac{1}{2t}\left(\int_0^1\frac{1-y^{-\frac{1}{2t}}}{1-y}\,dy-\int_0^1\frac{1-y^{-\frac{1}{2}\left(\frac{1}{t}+1\right)}}{1-y}\,dy\right)$$
$$=\frac{1}{2t}\left(\mathrm{H}_{-\frac{1}{2t}}-\mathrm{H}_{-\frac{1}{2}\left(\frac{1}{t}+1\right)}\right)=\frac{1}{2t}\left(\Psi\left(1-\frac{1}{2t}\right)-\Psi\left(1-\frac{1}{2}\left(\frac{1}{t}+1\right)\right)\right),$$

since

$$\mathrm{H}_x=\gamma+\Psi(1+x)\quad\text{for}\quad x>-1,$$

where H_x is the extension of the Harmonic number to real arguments (see [5]) and $\Psi(x)$ is the digamma function.
It is also true that

$$\Psi(1+x)=-\gamma+\sum_{k=2}^{+\infty}(-1)^k\zeta(k)x^{k-1}\quad\text{for}\quad |x|<1\quad\text{(see [6])},$$

so for $t>1$ we have

$$
\begin{aligned}
A_2(t) &= \frac{1}{t}\left(\sum_{k=2}^{+\infty}\frac{\zeta(k)}{2^k}\left(\left(1+\frac{1}{t}\right)^{k-1}-\left(\frac{1}{t}\right)^{k-1}\right)\right) = \sum_{k=2}^{+\infty}\frac{\zeta(k)}{(2t)^k}\left((t+1)^{k-1}-1\right)\\
&= \sum_{k=2}^{+\infty}\frac{\zeta(k)}{(2t)^k}\sum_{m=1}^{k-1}\binom{k-1}{m}t^m = \sum_{k=2}^{+\infty}\sum_{m=1}^{k-1}\frac{\zeta(k)}{2^k}\binom{k-1}{m}\frac{1}{t^{k-m}}\\
&= \sum_{k=1}^{+\infty}t^{-k}\sum_{m=1}^{+\infty}\binom{m+k-1}{m}\frac{\zeta(m+k)}{2^{m+k}} = \sum_{k=1}^{+\infty}t^{-k}\sum_{m=1}^{+\infty}\sum_{\ell=1}^{+\infty}\binom{m+k-1}{m}\frac{1}{(2\ell)^{m+k}}\\
&= \sum_{k=1}^{+\infty}t^{-k}\sum_{\ell=1}^{+\infty}\sum_{m=1}^{+\infty}\binom{m+k-1}{m}\frac{1}{(2\ell)^{m+k}} = \sum_{k=1}^{+\infty}t^{-k}\sum_{\ell=1}^{+\infty}\frac{1}{(2\ell)^k}\sum_{m=1}^{+\infty}\binom{m+k-1}{m}\frac{1}{(2\ell)^m}\\
&= \sum_{k=1}^{+\infty}t^{-k}\sum_{\ell=1}^{+\infty}\frac{1}{(2\ell)^k}\left(\sum_{m=0}^{+\infty}\binom{-k}{m}\left(-\frac{1}{2\ell}\right)^m-1\right) = \sum_{k=1}^{+\infty}t^{-k}\sum_{\ell=1}^{+\infty}\frac{1}{(2\ell)^k}\left(\left(1-\frac{1}{2\ell}\right)^{-k}-1\right)\\
&= \sum_{k=1}^{+\infty}t^{-k}\sum_{\ell=1}^{+\infty}\left(\frac{1}{(2\ell-1)^k}-\frac{1}{(2\ell)^k}\right) = \sum_{k=1}^{+\infty}t^{-k}\left(1-2^{1-k}\zeta(k)\right),
\end{aligned}
$$

with the change of the summation order, wherever takes place, being justified by the constant sign of the summands and the coefficient of t^{-1} being interpreted as in the first approach. Now $A_1(t)$ is immediately evaluated and we are done. □

References

[1] Coll. Math. J., **23** No.1 (Jan., 1992) 69 / Coll. Math. J., **24** No.1 (Jan., 1993) 97-98. https://carma.newcastle.edu.au/jon/Preprints/Books/Klamkin/komplete.pdf#page=228

[2] http://www.artofproblemsolving.com/community/u27772h1128062p5359508

[3] https://mathworld.wolfram.com/Cosecant.html

[4] https://mathworld.wolfram.com/RiemannZetaFunction.html

[5] https://mathworld.wolfram.com/HarmonicNumber.html

[6] P.Sebah, X.Gourdon, *Introduction to the Gamma Function*, p.12, (24), http://www.csie.ntu.edu.tw/~b89089/link/gammaFunction.pdf#page=12

Anastasios Kotronis, Athens, Greece, akotronis@gmail.com, www.asymmetry.gr

JUNIOR PROBLEMS

Solutions to the problems stated in this issue should arrive before August 15, 2016.

Proposals

51. *Proposed by Valmir Krasniqi, University of Prishtina, Republic of Kosova.* Let $0 < x, y, z \leq 2$ such that $xyz = 1$. Find all real numbers x, y, z such that

$$3^{x(4-2y)} + 3^{y(4-2z)} + 3^{z(4-2x)} = 27.$$

52. *Proposed by Florin Stanescu, Serban Cioculescu school, city Gaesti, jud. Dambovita, Romania.* Let a, b, c nonzero real numbers such that $a + b + c = 0$, prove that:

$$\max\left\{\frac{a}{b} + \frac{b}{a}, \frac{b}{c} + \frac{c}{b}, \frac{c}{a} + \frac{a}{c}\right\} \leq 2 + \frac{1}{8}\left(\frac{(a-b)(b-c)(c-a)}{abc}\right)^2.$$

53. *Proposed by Mihály Bencze, Braşov, Romania.* Let ABC be acute triangle. Prove that

$$m_a^a m_b^b m_c^c \leq (R+r)^{2s}$$

where r, R, s, m_a be the inradius, circumradius, semiperimeter and the median of the triangle respectively.

54. *Proposed by Marcel Chiriţă, Bucharest, Romania.* Consider a triangle ABC. Let D be a the midpoint on the median AM and $M \in BC$. Perpendicular on the midpoint of the segment DM passes through the orthocenter of the triangle ABC. Prove that $\sphericalangle(BDC) = 90^\circ$.

55. *Proposed by Dorlir Ahmeti, University of Prishtina, Department of Mathematics, Republic of Kosova.* Let ABC be acute triangle. Let AD be the altitude from A to BC. Let w_1, and w_2 be the circles with diameters BD and CD, respectively. Denote by E the intersection of w_1 with AB, and by F the intersection of w_2 with AC. Let G, H be the intersections of the line EF with w_1, w_2, respectively and their order in this way E, G, H, F. Let I be the intersection of BG and DE and J the intersection of CH and DF. Let O be the intersection of IJ and AD. Prove that O is the circumcenter of the triangle DGH.

Solutions

46. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania.* Solve in real numbers the equation

$$\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}+\frac{4}{x-4}=2x^2-5x-4.$$

Solution by Michel Bataille, Rouen, France.

We show that the equation has six solutions, namely,

$$0\ ;\ \frac{5}{2}\ ;\ \frac{5}{2}+\frac{\sqrt{9+4\sqrt{2}}}{2}\ ;\ \frac{5}{2}-\frac{\sqrt{9+4\sqrt{2}}}{2}\ ;\ \frac{5}{2}+\frac{\sqrt{9-4\sqrt{2}}}{2}\ ;\ \frac{5}{2}-\frac{\sqrt{9-4\sqrt{2}}}{2}.$$

Let $X=x-2$ and $Y=X^2-X-1=x^2-5x+5$. Then, with these notations,

$$\frac{1}{x-1}+\frac{4}{x-4}=\frac{5X+2}{Y-1},\quad \frac{2}{x-2}+\frac{3}{x-3}=\frac{5X-2}{Y+1},\quad 2x^2-5x-4=2Y+5X-4$$

and the equation rewrites as

$$\frac{10XY+4}{Y^2-1}=2Y+5X-4$$

or, after rearranging,

$$(Y^2-2Y-1)(2Y+5X)=0.$$

Thus, the solutions are obtained by grouping the solutions to $2Y+5X=0$ and to $Y^2-2Y-1=0$.

• the equation $2Y+5X=0$ is $2X^2+3X-2=0$ whose solutions for X are $\frac{1}{2}$ and -2. Recalling that $X=x-2$ this provides the solution $\frac{5}{2}$ and 0 for x.

• the equation $Y^2-2Y-1=0$ gives $Y=1+\sqrt{2}$ or $Y=1-\sqrt{2}$, which leads to the equations $X^2-X-(2+\sqrt{2})=0$ and $X^2-X-(2-\sqrt{2})=0$. We obtain $\frac{1\pm\sqrt{9+4\sqrt{2}}}{2}$ and $\frac{1\pm\sqrt{9-4\sqrt{2}}}{2}$ for X. This provides $\frac{5\pm\sqrt{9+4\sqrt{2}}}{2}$ and $\frac{5\pm\sqrt{9-4\sqrt{2}}}{2}$ for x. The announced result follows.

Also solved by Henry Ricardo, New York Math Circle, New York, USA and the proposer.

47. *Proposed by Pham–Thanh Hung, Math. Dept. "Can Tho City" Vietnam.* Let a,b,c be positive real numbers. Prove that

$$\left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2+\frac{10(a+b-c)(b+c-a)(c+a-b)}{(a+b)(b+c)(c+a)}\geq 2.$$

Comment by Michel Bataille, Rouen, France and Ángel Plaza, University of Las Palmas de Gran Canaria, Spain.

Let $L(a,b,c)$ denote the left-hand side of the inequality. We show that neither the inequality $L(a,b,c)\geq 2$ nor the inequality $L(a,b,c)\leq 2$ can hold for all positive real numbers a,b,c.

First, take $a=b=1$ and $c=2$. Then $a+b-c=0$ so that $L(1,1,2)=\frac{1}{9}+\frac{1}{9}+1<2$

and the stated inequality is not correct.
Second, take $a = 1, b = c = \frac{1}{100}$. Then $\left(\frac{a}{b+c}\right)^2 = 2500$ and

$$\frac{10(a+b-c)(b+c-a)(c+a-b)}{(a+b)(b+c)(c+a)} = 10\left(1-\frac{1}{101}\right)^2 \cdot \left(1-\frac{100}{2}\right) > -490$$

and so $L(1, 10^{-2}, 10^{-2}) > 2500-490 > 2$. Thus the reverse inequality is not correct either.

Also solved by Arkady Alt, San Jose, California, USA.

48. *Proposed by Titu Zvonaru, Comăneşti, Romania.* Let P be a point on the hypotenuse BC of the right–angled triangle ABC. If X and Y are the intersections of AP with the external common tangent lines to the circumcircles of the triangles ABP and ACP, prove that $XY = AP\sqrt{2}$ if and only if $BC = AB\sqrt{2}$.

Solution by the proposer.

Let $\Gamma_1(O_1, R_1), \Gamma_2(O_2, R_2)$ be the circumcircles of triangles ABP and ACP, respectively. The tangent line through X intersects these circles at the points T_1, T_2. Using the power of X with respect to the circles Γ_1 and Γ_1, we deduce that X is the midpoint of the segment T_1T_2 ($\mathrm{T}_1X^2 = XA \cdot XP = T_2X^2$).
We denote by M the midpoint of AP (hence the midpoint of XY).
Since, $\sphericalangle(XMO_2) = \sphericalangle(O_2T_2X) = 90^\circ, O_1M = R_1 \cos B$ and $O_2M = R_2 \cos C$, applying pithagorean theorem and the Law of sines, we obtain:
$XM^2 + O_2M^2 = O_2T_2^2 + T_2X^2 \Rightarrow XM^2 = -R_2^2\cos^2 C + R_2^2 + \frac{T_1T_2^2}{4}$
$\Rightarrow 4XM^2 = 4R_2^2 \sin^2 C + O_1O_2^2 - (R_1 - R_2)^2 \Rightarrow XY^2 = 4R_2^2 \sin^2 C + (O_1M +$
$\mathrm{O}_2M)^2 - (R_1 - R_2)^2 \Rightarrow XY^2 = AP^2 R_1^2 \cos^2 C - R_1^2 + R_2^2 \cos^2 C - R_2^2 + 2R_1R_2 \cos B \cos C$
$+2R_1R_2$. Since, $XY^2 = AP^2\left(\frac{1}{2} + \frac{\cos B \cos C}{2\sin B \sin C} + \frac{1}{2\sin B \sin C}\right)$,
we have $\left(\frac{AP}{XY}\right)^2 = \frac{2\sin B \sin C}{1+\sin B \sin C + \cos B \cos C}$. Since, ABC is right-angled triangle, we have $\sin B \sin C = \cos B \cos C = \frac{bc}{a^2}$, then
$XY = AP\sqrt{2} \Leftrightarrow \frac{2bc}{a^2+2bc} = \frac{1}{2} \Leftrightarrow b^2 + c^2 + 2bc = 4bc \Leftrightarrow (b-c)^2 = 0 \Leftrightarrow BC = AB\sqrt{2}$.

49. *Proposed by Proposed by Armend Sh. Shabani, University of Prishtina, Department of Mathematics, Republic of Kosova.* Solve the equation

$$3 \cdot 5^{x+1} + 11 \cdot 3^{x-1} + 5 \cdot 2^x + 2^{x-2} = 2016.$$

Solution by Michel Bataille, Rouen, France.

We show that the unique solution is $x = 3$.
Multiplying its both sides by 12, the equation becomes

$$180 \cdot 5^x + 44 \cdot 3^x + 63 \cdot 2^x = 2^7 \cdot 3^3 \cdot 7.$$

Let $f(x) = 180 \cdot 5^x + 44 \cdot 3^x + 63 \cdot 2^x$. The function f is strictly increasing on $\mathbb{R}$ (as are the functions $x \mapsto 5^x,\ x \mapsto 3^x,\ x \mapsto 2^x$), hence the equation $f(x) = 2^7 \cdot 3^3 \cdot 7$ has at most one solution. However, it is readily checked that $f(3) = 2^7 \cdot 3^3 \cdot 7$. Thus 3 is the unique solution.

Also solved by Ángel Plaza, University of Las Palmas de Gran Canaria, Spain; Henry Ricardo, New York Math Circle, New York, USA and the proposer.

50. *Proposed by Dorlir Ahmeti, University of Prishtina, Department of Mathematics, Republic of Kosova.* Let p be prime number such that $p \equiv 7(\mod 8)$. We define $A = \left\{1, 2, ..., \frac{p-1}{2}\right\}$ and $f(k) = \left|p\left\lfloor\frac{2^k+p-1}{2p}\right\rfloor - 2^{k-1}\right|$ for all $k \in A$ and $\frac{p-1}{2}$ is prime number, where $\lfloor x \rfloor$ is greatest integer not greater than x. Prove that $f(A) = A$.

Solution by proposer.

First we show that $f(A) \in A$. Since $p\left\lfloor\frac{2^k+p-1}{2p}\right\rfloor - 2^{k-1} > p\left(\frac{2^k+p-1}{2p} - 1\right) - 2^{k-1} = -\frac{p+1}{2} \Rightarrow p\left\lfloor\frac{2^k+p-1}{2p}\right\rfloor - 2^{k-1} \geq -\frac{p+1}{2} + 1 = -\frac{p-1}{2}$. Also $p\left\lfloor\frac{2^k+p-1}{2p}\right\rfloor - 2^{k-1} \leq p\left(\frac{2^k+p-1}{2p}\right) - 2^{k-1} = \frac{p-1}{2}$, therefore $\left|p\left\lfloor\frac{2^k+p-1}{2p}\right\rfloor - 2^{k-1}\right| \leq \frac{p-1}{2}$. It is clear that $\left|p\left\lfloor\frac{2^k+p-1}{2p}\right\rfloor - 2^{k-1}\right| \neq 0$ hence $f(A) \in A$. In order to solve the problem it is enough to show that function is injective. Since $p \equiv 7(mod 8)$ we have that $2^{\frac{p-1}{2}} \equiv \left(\frac{2}{p}\right)(modp) \equiv 1(modp)$. We assume that there exist positive integer $t < \frac{p-1}{2}$. Let t be the smallest such number. It is clear that $t \neq 1$; we know that for each $s > t$ such that $2^s \equiv 1(modp)$ we have s is divisible by t and since for $s = \frac{p-1}{2}$ is satisfied the condition: $\frac{p-1}{2}$ is divisible with t, which is not possible since $\frac{p-1}{2}$ is prime number. Therefore the smallest number t such that $2^t \equiv 1(modp)$ is $t = \frac{p-1}{2}$ and thus each two numbers from $2^0, 2^1, ...2^{\frac{p-1}{2}-1}$ have the diffrent reminder when divide p. Let us denote t_k the sequence such that $t_k \in \left[-\frac{p-1}{2}, \frac{p-1}{2}\right]$ and $2^{k-1} \equiv t_k(modp)$ and thus we have that $\left\lfloor\frac{2^k+p-1}{2p}\right\rfloor = \left\lfloor\frac{2^{k-1}+\frac{p-1}{2}}{p}\right\rfloor = \frac{2^{k-1}-t_k}{p}$ therefore $f(k) = \left|p\left\lfloor\frac{2^k+p-1}{2p}\right\rfloor - 2^{k-1}\right| = \left|p\left(\frac{2^{k-1}-t_k}{p}\right) - 2^{k-1}\right| = |t_k|$. Next we show that f is injective. Indeed, let $a, b \in A$ be positive integers such that $f(a) = f(b)$. We show that $a = b$. If $a > b$ then from $f(a) = f(b)$ we have that $|t_a| = |t_b|$ therefore $t_a = t_b$ or $t_a = -t_b$. If $t_a = t_b$ for all different $a, b \in A$ we have that $2^{a-1} \not\equiv 2^{b-1}(modp)$ from where we have $t_a \neq t_b$ which is contradiction. If $t_a = -t_b$ then we have that $2^{a-1} + 2^{b-1} \equiv 0(modp) \Rightarrow 2^{b-1}(2^{a-b} + 1) \equiv 0(modp) \Rightarrow 2^{a-b} \equiv -1(modp)$ Since $1 \leq a - b < \frac{p-1}{2}$ then $2^{2(a-b)} \equiv 1(modp)$ and since $2(a-b) < p-1$ the only possibility is $2(a-b) = \frac{p-1}{2}$ which is not possible since $\frac{p-1}{2}$ is odd. Therefore we have that $t_a \neq -t_b$ which is contradiction. Therefore $f(a) = f(b) \Rightarrow a = b$ and this shows that $f(A) = A$.