Mathproblems
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PROBLEMS AND SOLUTIONS

Proposals and solutions must be legible and should appear on separate sheets, each indicating the name of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editors have supplied a solution. The editors encourage undergraduate and pre-college students to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the class and school name. Solutions will be evaluated for publication by a committee of professors according to a combination of criteria. Questions concerning proposals and/or solutions can be sent by e-mail to: *mathproblems-ks@hotmail.com*

Solutions to the problems stated in this issue should arrive before June 10, 2016

Problems

124. *Proposed by Cornel Ioan Vălean, Timiş, Rumania.* Find an expression $E_n(x)$ whose terms are linearly independent with the other terms in the integrand such that

$$\left|\int_0^1 \Big(\frac{\alpha_1}{\log(x)}+\frac{\alpha_2}{\log^2(x)}+\cdots+\frac{\alpha_n}{\log^n(x)}+E_n(x)\ \Big)dx\right|<\infty$$

where $\alpha_i \neq 0$, $\alpha_i \in \mathbb{R}$.
Then, for a specific $E_n(x)$ family that fulfills the requirements above, calculate

$$\int_0^1 \Big(\frac{\alpha_1}{\log(x)}+\frac{\alpha_2}{\log^2(x)}+\cdots+\frac{\alpha_n}{\log^n(x)}+E_n(x)\Big)\ dx$$

in closed form.

125. *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia.* Let x, y and z be the sides of a triangle and r, R and s be the inradius, circumradius and the semiperimeter of the triangle respectively. Prove that

$$\frac{1}{(x+y)^2}+\frac{1}{(x+z)^2}+\frac{1}{(y+z)^2}\le\frac{r^4+8r^3R+124r^2R^2+2r^2s^2-8rRs^2+s^4}{128r^2R^2s^2}.$$

126. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let $n-2 \geq m \geq 1$ be integers. Calculate
$$\int_0^{\infty} \frac{x^{m-1}+x^{m-2}+\cdots+x+1}{x^{n-1}+x^{n-2}+\cdots+x+1}\mathrm{d}x.$$

127. *Proposed by Serafeim Tsipelis, Ioannina, Greece and Anastasios Kotronis, Athens, Greece (Jointly).* Evaluate $\sum_{k=1}^{+\infty}\frac{\zeta(2k+1)}{(k+1)(2k+1)}$, where ζ is the Riemann's zeta function.

128. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Let $\{a_n\}_{n\geq 1}$, $\{b_n\}_{n\geq 1}$ be real sequences with $a_n \neq a_{n+1}$ and $b_n \neq b_{n+1}$ such that: $\lim_{n\to\infty} a_n = a$, $\lim_{n\to\infty} b_n = b$,$\lim_{n\to\infty} n(a_{n+1}-a_n) = c$ and $\lim_{n\to\infty} n(b_{n+1}-b_n) = d$, where $a, b, c, d \in \mathbb{R}$. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions with continuous derivatives $\mathbb{R}$. Calculate $\lim_{n\to\infty} n(f(a_{n+1})f(b_{n+1}) - f(a_n)f(b_n))$.

129. *Proposed by Florin Stanescu, Serban Cioculescu school, city Gaesti, jud. Dambovita, Romania.* Let $f : [-1,1] \to \mathbb{R}$, a function a twice differentiable, the following properties:
a) $f(-1) = f(1) = 0$.
b) $f^{''}$ it is integrable and positive on $[-1,1]$.
Prove that:
$$\frac{1}{6}\Big(\int_{-1}^{1}(f^{''}(x))^2dx\Big) \geq \max\{(f(x))^2; x \in [-1,1]\}.$$

130. *Proposed by Mohammed Aassila, Strasbourg, France.* Among the first 2016 positive integers (from 1 to 2016) we underline those which may be represented as the sum of 5 nonnegative integer powers of 2. Is the set of underlined numbers larger than that of the nonunderlined ones ?

Solutions

No problem is ever permanently closed. We will be very pleased considering for publication new solutions or comments on the past problems.

117. *Proposed by Cornel Ioan Vălean, Timiş, Romania.* Calculate.

$$\int_0^{\pi/2} (\mathrm{Chi}(\cot^2 x) + \mathrm{Shi}(\cot^2 x)) \csc^2 x\, e^{-\csc^2 x}\, dx.$$

where $\mathrm{Shi}(x) = \int_0^x \frac{\sinh(t)}{t} dt$ and $\mathrm{Chi}(x) = \gamma + \log x + \int_0^x \frac{\cosh(t)-t}{t} dt$.

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria.

The answer is 0.
First note that $\mathrm{Chi}(z) + \mathrm{Shi}(z) = \mathrm{Ei}(z)$ the "exponential integral" so that, with the change of variables $t = \cot x$, the desired integral is

$$I = \frac{1}{e} \int_0^{\pi/2} \mathrm{Ei}(\cot^2 x) \csc^2 x\, e^{-\cot^2 x}\, dx = \frac{1}{e} \int_0^{\infty} \mathrm{Ei}(t^2)\, e^{-t^2}\, dt \tag{1}$$

Recall that Ei is defined for real non-zero values of x by

$$\mathrm{Ei}(x) = -P\,V \int_{-x}^{+\infty} \frac{e^{-u}}{u} du$$

that is the Cauchy principal value of the integral. So, for $x > 0$ we have

$$\begin{aligned}
\mathrm{Ei}(x) &= -\lim_{\epsilon \to 0^+} \left(\int_{-x}^{-\epsilon} \frac{e^{-u}}{u} du + \int_{\epsilon}^{+\infty} \frac{e^{-u}}{u} du \right) \\
&= \lim_{\epsilon \to 0^+} \left(\int_{-x}^{-\epsilon} \frac{-e^{-u}}{u} du - \int_{\epsilon}^{x} \frac{e^{-u}}{u} du \right) - \int_x^{+\infty} \frac{e^{-u}}{u} du \\
&= \lim_{\epsilon \to 0^+} \int_{\epsilon}^{x} \frac{2\sinh u}{u} du - \int_x^{+\infty} \frac{e^{-u}}{u} du \\
&= \int_0^x \frac{2\sinh u}{u} du - \int_x^{+\infty} \frac{e^{-u}}{u} du \\
&= \int_0^1 \frac{2\sinh(xt)}{t} dt - \int_1^{+\infty} \frac{e^{-xt}}{t} dt
\end{aligned}$$

We conclude that for $x > 0$ we have

$$e^{-x^2} \mathrm{Ei}(x^2) = \int_0^1 \frac{e^{-x^2(1-t)} - e^{-x^2(1+t)}}{t} dt - \int_1^{+\infty} \frac{e^{-x^2(1+t)}}{t} dt$$

Now, using the positivity of the integrands, and Tonelli's theorem we see that

$$\begin{aligned}\int_0^\infty \mathrm{Ei}(x^2)e^{-x^2}dx &= \int_0^1 \left(\int_0^\infty (e^{-x^2(1-t)} - e^{-x^2(1+t)})dx\right)\frac{dt}{t} \\ &\quad - \int_1^\infty \left(\int_0^\infty e^{-x^2(1+t)}dx\right)\frac{dt}{t} \\ &= J\cdot\left(\int_0^1\left(\frac{1}{t\sqrt{1-t}} - \frac{1}{t\sqrt{1+t}}\right)dt - \int_1^\infty \frac{dt}{t\sqrt{1+t}}\right)\end{aligned}$$

where $J = \int_0^\infty e^{-u^2}du$. So,

$$\int_0^\infty \mathrm{Ei}(x^2)e^{-x^2}dx = J\cdot \lim_{\epsilon\to 0^+} K(\epsilon) \tag{2}$$

where $K(\epsilon)$ is defined, for $0<\epsilon<1$, by

$$K(\epsilon) = \int_\epsilon^1 \frac{dt}{t\sqrt{1-t}} - \int_\epsilon^\infty \frac{dt}{t\sqrt{1+t}}$$

The change of variables $t = \dfrac{s}{1+s}$ in the first integral shows that

$$\begin{aligned}K(\epsilon) &= \int_{\epsilon/(1-\epsilon)}^\infty \frac{ds}{s\sqrt{1+s}} - \int_\epsilon^\infty \frac{dt}{t\sqrt{1+t}} \\ &= -\int_\epsilon^{\epsilon/(1-\epsilon)} \frac{dt}{t\sqrt{1+t}} = \int_\epsilon^{\epsilon/(1-\epsilon)}\left(1-\frac{1}{\sqrt{1+t}}\right)\frac{dt}{t} + \log(1-\epsilon) \\ &= \int_\epsilon^{\epsilon/(1-\epsilon)} \frac{dt}{\sqrt{1+t}(\sqrt{1+t}+1)} + \log(1-\epsilon).\end{aligned}$$

Therefore, $\lim_{\epsilon\to 0^+} K(\epsilon) = 0$, and from (2) and (1) the announced conclusion follows.

Solution 2 by Moti Levy, Rehovot, Israel.

By change of variable, $y = \cot^2 x$,

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} \left(Chi\left(\cot^2 x\right) + Shi\left(\cot^2 x\right)\right)\csc^2 x e^{-\csc^2 x}dx \\ &= \frac{1}{2e}\int_0^\infty y^{-\frac{1}{2}}\left(Chi\left(y\right) + Shi\left(y\right)\right)e^{-y}dy.\end{aligned}$$

From the definitions of the Chi and Shi functions, for positive real x,

$$\begin{aligned}Chi\left(x\right) + Shi\left(x\right) &= \gamma + \ln x + \int_0^x \frac{\cosh t - 1}{t}dt + \int_0^x \frac{\sinh t}{t}dt \\ &= \gamma + \ln x + \int_0^x \frac{e^t-1}{t}dt = \gamma + \ln x + \sum_{k=1}^\infty \frac{x^k}{kk!} := \mathrm{Ei}\left(x\right).\end{aligned}$$

$$\int_0^\pi \left(Chi\left(\cot^2 x\right) + Shi\left(\cot^2 x\right)\right)\csc^2 x e^{-\csc^2 x}dx = \frac{1}{2e}\int_0^\infty x^{-\frac{1}{2}}\,\mathrm{Ei}(x)e^{-x}dx$$

For $-1 < p < 0$, and using the fact $\int_0^\infty x^p (\ln x) e^{-x} dx = \Gamma(p+1)\psi(p+1)$,

$$\begin{aligned}\int_0^\infty x^p \operatorname{Ei}(x) e^{-x} dx &= \int_0^\infty x^p \left(\gamma + \ln x + \sum_{k=1}^\infty \frac{x^k}{kk!}\right) e^{-x} dx \\ &= \gamma \int_0^\infty x^p e^{-x} dx + \int_0^\infty x^p (\ln x) e^{-x} dx + \sum_{k=1}^\infty \int_0^\infty \frac{x^{k+p}}{kk!} e^{-x} dx \\ &= \gamma\Gamma(p+1) + \Gamma(p+1)\psi(p+1) + \sum_{k=1}^\infty \frac{\Gamma(k+p+1)}{kk!}.\end{aligned}$$

To evaluate $\sum_{k=1}^\infty \frac{\Gamma(k+p+1)}{kk!}$, we define the generating function

$$f(x) := \sum_{k=1}^\infty \frac{\Gamma(k+p+1)}{kk!} x^k.$$

Clearly $f(1) = \sum_{k=1}^\infty \frac{\Gamma(k+p+1)}{kk!}$.

$$f(x) := \Gamma(p+1) \sum_{k=1}^\infty \frac{\Gamma(k+p+1)}{\Gamma(k+1)\Gamma(p+1)} \frac{x^k}{k} = \Gamma(p+1) \sum_{k=1}^\infty \binom{k+p}{k} \frac{x^k}{k}$$

$$f'(x) = \Gamma(p+1) \sum_{k=1}^\infty \binom{k+p}{k} x^{k-1} = \Gamma(p+1) \left(\frac{1}{x(1-x)^{1+p}} - \frac{1}{x}\right)$$

$$f(x) = \Gamma(p+1) \int_0^x \left(\frac{1}{t(1-t)^{1+p}} - \frac{1}{t}\right) dt$$

$$f(1) = \Gamma(p+1) \int_0^1 \left(\frac{1}{t(1-t)^{1+p}} - \frac{1}{t}\right) dt = -\Gamma(p+1)(\psi(-p) + \gamma)$$

$$\begin{aligned}\int_0^\infty x^p \operatorname{Ei}(x) e^{-x} dx &= \int_0^\infty x^p \left(\gamma + \ln x + \sum_{k=1}^\infty \frac{x^k}{kk!}\right) e^{-x} dx \\ &= \gamma \int_0^\infty x^p e^{-x} dx + \int_0^\infty x^p (\ln x) e^{-x} dx + \sum_{k=1}^\infty \int_0^\infty \frac{x^{k+p}}{kk!} e^{-x} dx \\ &= \gamma\Gamma(p+1) + \Gamma(p+1)\psi(p+1) - \Gamma(p+1)(\psi(-p) + \gamma) \\ &= \Gamma(p+1)(\psi(p+1) - \psi(-p)) = -\Gamma(p+1)\pi\cot(\pi p).\end{aligned}$$

For $p = -\frac{1}{2}$, $\int_0^\infty x^p \operatorname{Ei}(x) e^{-x} dx = -\Gamma\left(\frac{1}{2}\right)\pi\cot\left(\frac{\pi}{2}\right) = 0$.
We conclude that

$$\int_0^{\frac{\pi}{2}} \left(Chi\left(\cot^2 x\right) + Shi\left(\cot^2 x\right)\right) \csc^2 x e^{-\csc^2 x} dx = 0.$$

Also solved by Albert Stadler, Switzerland and the proposer.

118. *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia.*
Compute the following sum

$$\sum_{m=1}^\infty \sum_{n=1}^\infty (-1)^{m+n} \frac{m \log(m+n)}{(m+n)^3}.$$

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

The answer is

$$-\frac{\log^2(2)}{4}+\gamma\frac{\log(2)}{2}-\frac{\pi^2}{24}(\gamma+\log(4\pi)-12\log A)$$

where γ is the Euler-Mascheroni constant, and A is the Glaisher-Kinkelin constant.

Let K_m be defined, for $m \geq 1$, by

$$K_m=\sum_{n=1}^{\infty}(-1)^{m+n}\frac{\log(m+n)}{(m+n)^3}=\sum_{n=m+1}^{\infty}(-1)^n\frac{\log(n)}{n^3}$$

The series defining K_m is absolutely convergent, and since $x \mapsto \dfrac{\log x}{x^3}$ is decreasing on $[2,+\infty)$ we conclude that

$$0<(-1)^{m+1}K_m<\frac{\log(m+1)}{(m+1)^3} \tag{1}$$

Thus the series $\sum_{m=1}^{\infty} mK_m$ is convergent and we are interested in computing the sum $S=\sum_{m=1}^{\infty} mK_m$.
Now, let $S_q=\sum_{m=1}^{q} mK_m$, we have

$$\begin{aligned}
S_q &= \sum_{m=1}^{q}\left(\frac{m(m+1)}{2}-\frac{m(m-1)}{2}\right)K_m\\
&= \sum_{m=2}^{q+1}\frac{m(m-1)}{2}K_{m-1}-\sum_{m=1}^{q}\frac{m(m-1)}{2}K_m\\
&= \frac{q(q+1)}{2}K_q+\sum_{m=2}^{q}\frac{m(m-1)}{2}(K_{m-1}-K_m)\\
&= \frac{q(q+1)}{2}K_q+\sum_{m=2}^{q}\frac{m(m-1)}{2}(-1)^m\frac{\log(m)}{m^3}\\
&= \frac{q(q+1)}{2}K_q+\frac{1}{2}\sum_{m=2}^{q}\frac{(-1)^m(m-1)\log(m)}{m^2}.
\end{aligned}$$

So, using (1) and letting q tend to $+\infty$ we see that

$$S=\sum_{m=2}^{\infty}\frac{(-1)^m(m-1)\log(m)}{2m^2}=\frac{1}{2}\underbrace{\sum_{m=2}^{\infty}\frac{(-1)^m\log(m)}{m}}_{V}-\frac{1}{2}\underbrace{\sum_{m=2}^{\infty}\frac{(-1)^m\log(m)}{m^2}}_{U} \tag{2}$$

Now, for U we have

$$\begin{aligned} U &= \sum_{m=2}^{\infty} \frac{(1+(-1)^m)\log(m)}{m^2} - \sum_{m=2}^{\infty} \frac{\log(m)}{m^2} \\ &= \sum_{m=1}^{\infty} \frac{2\log(2m)}{4m^2} - \sum_{m=2}^{\infty} \frac{\log(m)}{m^2} \\ &= \frac{\log 2}{2}\zeta(2) + \frac{1}{2}\zeta'(2) = \frac{\pi^2 \log 2}{12} + \zeta'(2) \end{aligned}$$

But $\zeta'(2)$ can be calculated in terms of the other known constants by the formula

$$\zeta'(2) = \frac{\pi^2}{6}(\gamma + \log(2\pi) - 12\log A).$$

Thus

$$U = \frac{\pi^2}{12}(\gamma + \log(4\pi) - 12\log A) \tag{3}$$

Let us come to V, let

$$g_n = \frac{\log(n)}{n} - \frac{\log^2(n) - \log^2(n-1)}{2}$$

and define

$$G_m = \sum_{n=2}^{m} g_n = \sum_{n=2}^{m} \frac{\log(n)}{n} - \frac{1}{2}\log^2 m$$

Clearly

$$g_n = \frac{1}{2}\log^2\left(1-\frac{1}{n}\right) + \left(\frac{1}{n} + \log\left(1-\frac{1}{n}\right)\right)\log(n) = \mathcal{O}\left(\frac{\log(n)}{n^2}\right)$$

Thus, the series $\sum g_n$ is convergent. It follows that there is a constant ℓ such that

$$\sum_{k=2}^{n} \frac{\log(k)}{k} = \frac{\log^2(n)}{2} + \ell + o(1) \tag{4}$$

Now, using (4), for $m > 1$ we have

$$\begin{aligned} \sum_{n=2}^{2m} \frac{(-1)^n\log(n)}{n} &= \sum_{n=2}^{2m} \frac{(1+(-1)^n)\log(n)}{n} - \sum_{n=2}^{2m} \frac{\log(n)}{n} \\ &= \sum_{n=1}^{m} \frac{\log(2n)}{n} - \sum_{n=2}^{2m} \frac{\log(n)}{n} \\ &= \log(2)\sum_{n=1}^{m} \frac{1}{n} + \frac{\log^2(m)}{2} - \frac{\log^2(2m)}{2} + o(1) \\ &= -\frac{\log^2(2)}{2} + \log(2)\left(\sum_{n=1}^{m} \frac{1}{n} - \log(m)\right) + o(1) \end{aligned}$$

Taking the limit as m tend to $+\infty$ we get

$$V = -\frac{\log^2(2)}{2} + \gamma\log(2) \tag{5}$$

Replacing (3) and (5) in (2) we get

$$S=-\frac{\log^2(2)}{4}+\gamma\frac{\log(2)}{2}-\frac{\pi^2}{24}(\gamma+\log(4\pi)-12\log A)\approx 0.0292762$$

which is the announced result.

Solution 2 by Moti Levy, Rehovot, Israel.

By D. Borwein trick:

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+n}\frac{m\ln(m+n)}{(m+n)^3}\\
&=\frac{1}{2}\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+n}\frac{m\ln(m+n)}{(m+n)^3}+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+n}\frac{n\ln(m+n)}{(m+n)^3}\right)\\
&=\frac{1}{2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+n}\frac{\ln(m+n)}{(m+n)^2}=\frac{1}{2}\sum_{m=1}^{\infty}\sum_{k=m+1}^{\infty}(-1)^k\frac{\ln k}{k^2}\\
&=\frac{1}{2}\sum_{k=2}^{\infty}\sum_{m=1}^{k-1}(-1)^k\frac{\ln k}{k^2}=\frac{1}{2}\sum_{k=2}^{\infty}(-1)^k\frac{(k-1)\ln k}{k^2}\\
&=\frac{1}{2}\sum_{k=1}^{\infty}(-1)^k\frac{\ln k}{k}-\frac{1}{2}\sum_{k=1}^{\infty}(-1)^k\frac{\ln k}{k^2}.
\end{aligned}$$

Well known facts from analytic number theory are:
1) The Dirichlet eta function is defined as follows,

$$\eta(s)=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^s},\qquad \mathrm{Re}(s)>0.$$

2) The derivative of Dirichlet Eta function is

$$\eta'(s)=\begin{cases}\sum_{k=1}^{\infty}(-1)^k\frac{\ln k}{k^s}=2^{1-s}\ln 2\cdot\zeta(s)+\left(1-2^{1-s}\right)\zeta'(s), & s\neq 1\\ \sum_{k=1}^{\infty}(-1)^k\frac{\ln k}{k}=\gamma\ln 2-\frac{1}{2}\ln^2 2, & s=1.\end{cases}$$

$$\begin{aligned}
\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{m+n}\frac{m\ln(m+n)}{(m+n)^3}&=\frac{1}{2}\left(\eta'(1)-\eta'(2)\right)\\
&=\frac{1}{2}\left(\gamma-\frac{\pi^2}{12}\right)\ln 2-\frac{1}{4}\ln^2 2-\frac{1}{4}\zeta'(2)\cong 0.0292762.
\end{aligned}$$

Solution 3 by Ramya Dutta, Chennai Mathematical Institute (student), India.

Making the change in variable in the sum, $m+n=k$:

$$\begin{aligned}\sum_{m,n=1}^{\infty}(-1)^{m+n}\frac{m\log(m+n)}{(m+n)^3} &= \sum_{k=2}^{\infty}(-1)^k\frac{\log k}{k^3}\left(\sum_{m=1}^{k-1}m\right)\\ &= \frac{1}{2}\sum_{k=2}^{\infty}(-1)^k\frac{\log k}{k^3}(k^2-k)\\ &= \frac{1}{2}\sum_{k=1}^{\infty}(-1)^k\frac{\log k}{k}-\frac{1}{2}\sum_{k=1}^{\infty}(-1)^k\frac{\log k}{k^2}\end{aligned}$$

For $\mathfrak{Re}(s) > 1$, we have $\eta(s) = \displaystyle\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^s} = \left(1-\frac{1}{2^{s-1}}\right)\zeta(s)$
Differentiating both sides with respect to s leads to:

$$\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\log k}{k^s} = -\left(1-\frac{1}{2^{s-1}}\right)\zeta'(s)-\frac{\log 2}{2^{s-1}}\zeta(s)$$

Thus,

$$\sum_{k=1}^{\infty}(-1)^k\frac{\log k}{k^2} = \frac{1}{2}\zeta'(2)+\frac{\log 2}{2}\zeta(2)$$

On the other hand,

$$\begin{aligned}\sum_{k=1}^{2n}(-1)^k\frac{\log k}{k} &= \sum_{k=1}^{n}\frac{\log(2k)}{2k}-\sum_{k=1}^{n}\frac{\log(2k-1)}{2k-1}\\ &= H_n\log 2+\sum_{k=1}^{n}\frac{\log k}{k}-\sum_{k=1}^{2n}\frac{\log k}{k}\end{aligned}$$

For Euler-Maclaurin's formula we have,

$$H_n = \log n+\gamma+\frac{1}{2n}+\mathcal{O}\left(\frac{1}{n^2}\right)$$

and,

$$\sum_{k=1}^{n}\frac{\log k}{k} = \frac{1}{2}\log^2 n+C+\mathcal{O}\left(\frac{\log n}{n}\right)$$

where, C is some constant term.
Thus,

$$\begin{aligned}\sum_{k=1}^{2n}(-1)^k\frac{\log k}{k} &= \log 2\log n+\gamma\log 2+\frac{1}{2}\log^2 n-\frac{1}{2}\log^2(2n)+\mathcal{O}\left(\frac{\log n}{n}\right)\\ &= \gamma\log 2-\frac{1}{2}\log^2 2+\mathcal{O}\left(\frac{\log n}{n}\right)\end{aligned}$$

Combining the results we have:

$$\sum_{m,n=1}^{\infty}(-1)^{m+n}\frac{m\log(m+n)}{(m+n)^3} = \frac{\gamma\log 2}{2}-\frac{\log^2 2}{4}-\frac{1}{4}\zeta'(2)-\frac{\zeta(2)\log 2}{4}$$

Also solved by Albert Stadler, Switzerland; Haroun Meghaichi (student), Algeria; Anastasios Kotronis, Athens, Greece (Jointly) and the proposer.

119. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let $n > 1$ be an integer. Calculate

$$\int_0^\infty \ln^n \left|\frac{1-x}{1+x}\right| dx.$$

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

The answer is $4(-1)^n(n!)(1-2^{-n})\zeta(n)$ where ζ is the Riemann zeta function. Let the considered integral be denoted by I_n. Clearly we have

$$\begin{aligned}
I_n &= \int_0^1 \ln^n\left(\frac{1-x}{1+x}\right)dx + \underbrace{\int_1^\infty \ln^n\left(\frac{x-1}{x+1}\right)dx}_{x\leftarrow 1/x} \\
&= \int_0^1 \ln^n\left(\frac{1-x}{1+x}\right)dx + \int_0^1 \frac{1}{x^2}\ln^n\left(\frac{1-x}{1+x}\right)dx \\
&= \int_0^1 \left(1+\frac{1}{x^2}\right)\ln^n\left(\frac{1-x}{1+x}\right)dx, \qquad x \leftarrow \frac{e^t-1}{e^t+1} \\
&= \int_0^\infty \left(1+\frac{(e^t+1)^2}{(e^t-1)^2}\right)(-t)^n\frac{2e^t}{(e^t+1)^2}dt \\
&= 2(-1)^n\int_0^\infty \left(\frac{1}{(1+e^{-t})}+\frac{1}{(1-e^{-t})}\right)e^{-t}t^n dt
\end{aligned}$$

But, since $\sum_{m=1}^\infty mx^{m-1} = \frac{1}{(1-x)^2}$ for $|x| < 1$, we conclude that

$$\frac{1}{(1+e^{-t})^2}+\frac{1}{(1-e^{-t})^2} = 2\sum_{m=0}^\infty (2m+1)e^{-2mt}, \qquad t > 0,$$

Thus

$$I_n = 4(-1)^n\int_0^\infty \left(\sum_{m=0}^\infty (2m+1)e^{-(2m+1)t}t^n\right)dt$$

Since the summands are positive functions we can interchange the signs of sum and integral, so

$$\begin{aligned}
I_n &= 4(-1)^n\sum_{m=0}^\infty \frac{1}{(2m+1)^n}\int_0^\infty e^{-u}u^n du \\
&= 4(-1)^n(n!)\sum_{m=0}^\infty \frac{1}{(2m+1)^n} \\
&= 4(-1)^n(n!)\left(\sum_{m=1}^\infty \frac{1}{m^n} - \sum_{m=1}^\infty \frac{1}{(2m)^n}\right) \\
&= 4(-1)^n(n!)(1-2^{-n})\zeta(n)
\end{aligned}$$

Which is the desired conclusion.

Solution 2 by Michel Bataille, Rouen, France.

Let I denote the integral. We show that $I = 4(-1)^n(1-2^{-n})n!\zeta(n)$ where ζ is the Riemann function defined by $\zeta(s) = \sum\limits_{k=1}^{\infty} \frac{1}{k^s} \quad (s>1)$.

First, $I = I_1 + I_2$ with

$$I_1 = \int_0^1 \left(\ln\left(\frac{1-x}{1+x}\right)\right)^n dx \quad \text{and} \quad I_2 = \int_1^{\infty} \left(\ln\left(\frac{x-1}{x+1}\right)\right)^n dx.$$

The substitutions $x = \frac{1-y}{1+y}$ in I_1 and $x = \frac{1+y}{1-y}$ in I_2 give

$$I_1 = 2\int_0^1 \frac{(\ln y)^n}{(1+y)^2}\, dy \quad \text{and} \quad I_2 = 2\int_0^1 \frac{(\ln y)^n}{(1-y)^2}\, dy$$

so that

$$I = 4\int_0^1 (\ln y)^n \cdot \frac{1+y^2}{(1-y^2)^2}\, dy = 4(K-J)$$

where

$$K = \int_0^1 \frac{(\ln y)^n}{(1-y)^2}\, dy \quad \text{and} \quad J = \int_0^1 (\ln y)^n \cdot \frac{2y}{(1-y^2)^2}\, dy.$$

Since the substitution $y = \sqrt{u}$ in J yields $J = 2^{-n}K$, we obtain $I = 4(1-2^{-n})K$.

Note that $\frac{1}{(1-y)^2} = \sum\limits_{k=0}^{\infty}(k+1)y^k$ for $y \in [0,1)$, hence

$$K = \int_0^1 \left(\sum_{k=0}^{\infty}(k+1)y^k(\ln y)^n\right) dy \qquad (1).$$

Now, we shall use the following result (see a quick proof at the end): if m, k are integers such that $k \geq 0$ and $m \geq 1$, then

$$\int_0^1 x^k(\ln x)^m\, dx = \frac{(-1)^m m!}{(k+1)^{m+1}} \qquad (2).$$

Since

$$\sum_{k=0}^{\infty}\int_0^1 |(k+1)y^k(\ln y)^n|\, dy = (-1)^n\sum_{k=0}^{\infty}(k+1)\int_0^1 y^k(\ln y)^n\, dy = n!\sum_{k=0}^{\infty}\frac{1}{(k+1)^n} < \infty$$

we may interchange $\sum$ and $\int$ in (1) and obtain

$$K = \sum_{k=0}^{\infty}\int_0^1 (k+1)y^k(\ln y)^n\, dy = \sum_{k=0}^{\infty}(k+1)\frac{(-1)^n n!}{(k+1)^{n+1}} = (-1)^n n!\sum_{k=0}^{\infty}\frac{1}{(k+1)^n} = (-1)^n n!\zeta(n).$$

and the claimed result follows.

Proof of (2). Fix k and let $U_m = \int_0^1 x^k(\ln x)^m\, dx$. The proof is by induction on m. First, integrating by parts,

$$U_1 = \int_0^1 x^k \ln x\, dx = \left[\frac{x^{k+1}}{k+1}(\ln x)\right]_0^1 - \frac{1}{k+1}\int_0^1 x^k\, dt = \frac{-1}{(k+1)^2}$$

so that (2) holds when $m = 1$.

As for the induction step, we integrate U_m (where $m > 1$) by parts again:

$$U_m = \left[\frac{x^{k+1}}{k+1}(\ln x)^m\right]_0^1 - \frac{m}{k+1}\int_0^1 x^k(\ln x)^{m-1}\, dx = -\frac{m}{k+1}U_{m-1}$$

and so (2) holds if $U_{m-1} = \frac{(-1)^{m-1}(m-1)!}{(k+1)^m}$.

Solution 3 by Moti Levy, Rehovot, Israel.

Let $I_n := \int_0^\infty \ln^n \left|\frac{1-x}{1+x}\right| dx$.
We get rid of the absolute value by splitting the integration range,

$$I_n = \int_0^1 \ln^n \left(\frac{1-x}{1+x}\right) dx + \int_1^\infty \ln^n \left(\frac{x-1}{x+1}\right) dx.$$

By substitution of $x = \frac{1}{y}$ in the second integral,

$$I_n = \int_0^1 \ln^n \left(\frac{1-x}{1+x}\right) dx + \int_0^1 \frac{1}{x^2} \ln^n \left(\frac{1-x}{1+x}\right) dx$$

By the following substitution,

$$u = -\ln \frac{1-x}{1+x}; \quad x = -\frac{e^{-u}-1}{e^{-u}+1}, \; dx = 2\frac{e^{-u}}{\left(e^{-u}+1\right)^2}\; du$$

$$\begin{aligned} I_n &= 2\int_0^\infty u^n \frac{e^{-u}}{\left(1+e^{-u}\right)^2} du + 2\int_0^\infty u^n \frac{e^{-u}}{\left(1-e^{-u}\right)^2} du \\ &= 2\left(-1\right)^n \int_0^\infty u^n \frac{e^u}{\left(e^u+1\right)^2} du + 2\left(-1\right)^n \int_0^\infty u^n \frac{e^u}{\left(e^u-1\right)^2} du \end{aligned}$$

After integration by parts

$$\begin{aligned} I_n &= 2\left(-1\right)^n n \int_0^\infty u^{n-1} \frac{1}{e^u+1} du + 2\left(-1\right)^n n \int_0^\infty u^{n-1} \frac{1}{e^u-1} du \\ &= 4\left(-1\right)^n n \int_0^\infty u^{n-1} \frac{e^u}{e^{2u}-1} du. \end{aligned}$$

Now,

$$\frac{e^u}{e^{2u}-1} = \frac{1}{e^u-1} - \frac{1}{e^{2u}-1}.$$

Hence,

$$\begin{aligned} I_n &= 4\left(-1\right)^n n \left(\int_0^\infty u^{n-1} \frac{1}{e^u-1} du - \int_0^\infty u^{n-1} \frac{1}{e^{2u}-1} du\right) \\ &= 4\left(-1\right)^n n \left(\int_0^\infty u^{n-1} \frac{1}{e^u-1} du - \frac{1}{2^n} \int_0^\infty v^{n-1} \frac{1}{e^v-1} dv\right) \\ &= 4\left(-1\right)^n n \left(1 - \frac{1}{2^n}\right) \int_0^\infty u^{n-1} \frac{1}{e^u-1} du. \end{aligned}$$

An integral representation of the Zeta function is

$$\Gamma\left(s\right) \zeta\left(s\right) = \int_0^\infty v^{s-1} \frac{1}{e^v-1} dv, \quad \mathrm{Re}\left(s\right) > 1.$$

We conclude that

$$\int_0^\infty \ln^n \left|\frac{1-x}{1+x}\right| dx = \left(-1\right)^n 4\left(1 - \frac{1}{2^n}\right) n!\zeta\left(n\right).$$

Also solved by Ramya Dutta, Chennai Mathematical Institute (student)India; Albert Stadler, Switzerland and the proposer.

120. *Proposed by Anastasios Kotronis, Athens, Greece and Haroun Meghaichi (student), Algiers, Algeria, (Jointly.)* **(Corrected.)** Let $I(A) = \int_1^A A^{1/x}\,dx$. For n a non-negative integer compute the following limit, if it exists

$$\lim_{A\to+\infty} \frac{\ln^n A}{A}\left(I(A) - \sum_{k=0}^{n-1} k!\frac{A}{\ln^k A}\right),$$

where the sum over an empty set of indices is interpreted as zero.

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

Let $F(t)$ be defined by $F(t) = \int\limits_1^{e^t} e^{t/x}dx$ so that $I(A) = F(\ln A)$. A simple manipulation shows that

$$\begin{aligned} F(t) &= t\int_{te^{-t}}^t \frac{e^u}{u^2}du = t\left[-\frac{e^u}{u}\right]_{te^{-t}}^t + t\int_{te^{-t}}^t \frac{e^u}{u}du \\ &= (e^{te^{-t}} - 1)e^t + t(G(t) - G(te^{-t})) \end{aligned} \tag{1}$$

Where $G(v) = \int_1^v \frac{e^u}{u}du$. Since $\lim\limits_{t\to\infty} te^{-t} = 0$ we are interested in the asymptotic expansion of G in the neighborhoods of 0 and $+\infty$.
Now, for small values of $v > 0$ we have

$$\begin{aligned} G(v) &= \ln v + \int_1^v \frac{e^u - 1}{u}du = \ln v + \int_0^1 \frac{e^u - 1}{u}du - \int_0^v \frac{e^u - 1}{u}du \\ &= \ln v + \ell - \sum_{n=1}^{\infty} \frac{v^n}{n\cdot(n!)}, \qquad \text{with } \ell = \int\limits_0^1 \frac{e^u-1}{u}du. \end{aligned}$$

Thus, for $t > 0$ we have

$$tG(te^{-t}) = -t^2 + t\log t + \ell t - \sum_{n=1}^{\infty} \frac{t^{n+1}e^{-nt}}{n\cdot(n!)} \tag{2}$$

Also, since

$$\left(\sum_{k=0}^{n} \frac{k!}{u^{k+1}}e^u\right)' = \frac{e^u}{u} - \frac{(n+1)!}{u^{n+2}}e^u$$

we see that for $t > 0$ we have

$$G(t) = \sum_{k=0}^{n} \frac{k!}{t^{k+1}}e^t - e\sum_{k=0}^{n} k! + \int_1^t \frac{(n+1)!}{u^{n+2}}\,e^u du \tag{3}$$

But, if $t > s \geq 1$ we have

$$\begin{aligned} \int_1^t \frac{e^u}{u^m}du &= \int_1^s \frac{e^u}{u^m}du + \int_s^t \frac{e^u}{u^m}du \\ &\leq \int_1^s e^u du + \left[\frac{e^u}{u^m}\right]_s^t + \int_s^t \frac{me^u}{u^{m+1}}du \\ &\leq e^s + e^t\int_s^t \frac{m}{u^{m+1}}du \leq e^s + \frac{e^t}{s^m} \end{aligned}$$

So, choosing $s = t/2$, we see that, in the neighborhood $+\infty$ and for every positive integer m we have

$$\int_1^t \frac{e^u}{u^m}\,du = \mathcal{O}\left(\frac{e^t}{t^m}\right)$$

So, from (3) we obtain the following asymptotic expansion of G the neighborhood $+\infty$

$$\forall\, n \geq 0, \qquad G(t) = \sum_{k=0}^{n} \frac{k!}{t^{k+1}} e^t + \mathcal{O}\left(\frac{e^t}{t^{n+2}}\right) \tag{4}$$

Clearly, for large t, we have $(e^{te^{-t}} - 1)e^t) = \mathcal{O}(t)$, and $tG(te^{-t}) = \mathcal{O}(t^2)$ (according to (2)), but in the neighborhood $+\infty$, $t^k = o\left(\frac{e^t}{t^m}\right)$ for any m. So from the above we conclude that for large t, and any $n \geq 0$ we have

$$F(t) = \sum_{k=0}^{n} \frac{k!}{t^k} e^t + \mathcal{O}\left(\frac{e^t}{t^{n+1}}\right).$$

Or, in terms of I we have, for every non-negative integer n and large A,

$$I(A) = \sum_{k=0}^{n} \frac{k!\,A}{\ln^k A} + \mathcal{O}\left(\frac{A}{\ln^{n+1} A}\right).$$

This proves that

$$\forall\, n \geq 0, \qquad \lim_{A\to\infty} \frac{\ln^n A}{A}\left(I(A) - \sum_{k=0}^{n-1} \frac{k!\,A}{\ln^k A}\right) = n!$$

which is the desired conclusion.

Also solved by Albert Stadler Switzerland; Moti Levy, Rehovot, Israel; Ramya Dutta, Chennai Mathematical Institute (student), India and the proposers.

121. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Le $m > 0$, L_k be k-th Lucas number and $\Gamma : (0,\infty) \to (0,\infty)$ be the Gamma function. Calculate

$$\lim_{n\to\infty} \int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} \Gamma\Big(\frac{x}{n}\sqrt[n]{L_n^m}\Big)\,dx.$$

Solution 1 by Haroun Meghaichi (student), Algeria.

The answer is $\frac{1}{e}\Gamma\left(\frac{\varphi^m}{e}\right)$, where φ is the golden ratio. we'll use the following lemma: Let $f : [a,b] \to \mathbb{R}$ continuous, and $(x_n)_n, (y_n)_n$ two convergent sequences of $[a,b]$ that have the same limit α, then

$$\int_{x_n}^{y_n} f(t)\ \mathrm{d}t = f(\alpha)(y_n - x_n) + o(y_n - x_n).$$

Proof. Let $\varepsilon > 0$, then $\exists\delta > 0$, such that $|f(t) - f(\alpha)| < \varepsilon$, whenever $|x - \alpha| < \delta$. since $x_n, y_n \to \alpha$, then there is $n_0 \in \mathbb{N}$ such that $x_n, y_n \in]\alpha - \delta, \alpha + \delta, [$ wherever $n > n_0$. Therefore

$$\left| \int_{x_n}^{y_n} f(t)\ \mathrm{d}t - f(\alpha)(y_n - x_n) \right| \leq \int_{x_n}^{y_n} |f(t) - f(\alpha)|\ \mathrm{d}t \leq \epsilon |y_n - x_n|.$$

□

Note that the given integral equals

$$I_n = \frac{n}{\sqrt[n]{L_n^m}} \int\limits_{\frac{\sqrt[n]{n!}}{n}\sqrt[n]{L_n^m}}^{\frac{\sqrt[n+1]{(n+1)!}}{n}\sqrt[n]{L_n^m}} \Gamma(t)\ \mathrm{d}t,$$

this comes directly from the sub $t = \frac{x}{n}\sqrt[n]{L_n^m}$, let x_n, y_n be the lower, upper bound of the last integral respectively, then $x_n, y_n \to \varphi^m e^{-1}$, since

$$\sqrt[n]{L_n^m} = \left(\varphi^n + (-\varphi)^{-n}\right)^{m/n} = \varphi^m \left(1 + (-\varphi)^{-2n}\right)^{m/n} = \varphi^m + \mathcal{O}(\varphi^{-2n}).$$

and $\frac{\sqrt[n]{n!}}{n!} \to e^{-1}$, and thus

$$\frac{\sqrt[n+1]{(n+1)!}}{n} = \frac{n+1}{n}\frac{\sqrt[n+1]{(n+1)!}}{n+1} \overset{n\to\infty}{\longrightarrow} e^{-1}.$$

now, note that by Stolz theorem

$$\frac{n}{\sqrt[n]{L_n^m}}(y_n - x_n) = \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right) \overset{n\to\infty}{\longrightarrow} e^{-1}.$$

By the lemma we get

$$I_n = \frac{n}{\sqrt[n]{L_n^m}} \left(\Gamma\left(\frac{\varphi^m}{e}\right)(y_n - x_n) + o(y_n - x_n) \right) = \frac{1}{e}\Gamma\left(\frac{\varphi^m}{e}\right) + o(1).$$

which proves that the limit equals $\frac{1}{e}\Gamma\left(\frac{\varphi^m}{e}\right)$.

Solution 2 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria.

Let $I(n)$ be defined by

$$I(n) = \int_{\sqrt[n]{n!}}^{\sqrt[n+1]{(n+1)!}} \Gamma\left(\frac{x}{n}\sqrt[n]{L_n^m}\right) dx$$

A simple change of variables shows that

$$I(n) = a_n \int_{b_n}^{c_n} \Gamma(x)\, dx = a_n(F(c_n) - F(b_n))$$

with

$$a_n = \frac{n}{\sqrt[n]{L_n^m}}, \quad b_n = \frac{\sqrt[n]{n!}}{n}\sqrt[n]{L_n^m}, \quad c_n = \frac{\sqrt[n+1]{(n+1)!}}{n}\sqrt[n]{L_n^m}, \quad F(t) = \int_1^t \Gamma(x)\, dx$$

Using Stirling asymptotic expansion we have

$$\frac{\sqrt[n]{n!}}{n} = \frac{1}{e} + \frac{\ln n}{2en} + \frac{\log(2\pi)}{2en} + \mathcal{O}\left(\frac{\ln^2 n}{n^2}\right) \tag{1}$$

$$\frac{\sqrt[n+1]{(n+1)!}}{n} = \frac{1}{e} + \frac{\ln n}{2en} + \frac{2+\log(2\pi)}{2en} + \mathcal{O}\left(\frac{\ln^2 n}{n^2}\right) \tag{2}$$

Now, since $L_n = \varphi^n + (-1)^n\varphi^{-n}$ with $\varphi = \frac{1+\sqrt{5}}{2}$, we see immediately that

$$\sqrt[n]{L_n^m} = \varphi^m\left(1 + \mathcal{O}\left(\frac{1}{n\varphi^{2n}}\right)\right)$$

So, combining the above results we see that

$$a_n = \frac{n}{\varphi^m}\left(1 + \mathcal{O}\left(\frac{1}{n\varphi^{2n}}\right)\right) \tag{3}$$

$$b_n = \varphi^m\left(\frac{1}{e} + \frac{\ln n}{2en} + \frac{\ln(2\pi)}{2en}\right) + \mathcal{O}\left(\frac{\ln^2 n}{n^2}\right) \tag{4}$$

$$c_n = \varphi^m\left(\frac{1}{e} + \frac{\ln n}{2en} + \frac{2+\ln(2\pi)}{2en}\right) + \mathcal{O}\left(\frac{\ln^2 n}{n^2}\right) \tag{5}$$

It follows that

$$F(b_n) = F\left(\frac{\varphi^m}{e}\right) + F'\left(\frac{\varphi^m}{e}\right)\frac{\ln n + \ln(2\pi)}{2en}\varphi^m + \mathcal{O}\left(\frac{\ln^2 n}{n^2}\right)$$

$$F(c_n) = F\left(\frac{\varphi^m}{e}\right) + F'\left(\frac{\varphi^m}{e}\right)\frac{\ln n + 2 + \ln(2\pi)}{2en}\varphi^m + \mathcal{O}\left(\frac{\ln^2 n}{n^2}\right)$$

So, since $F'(t) = \Gamma(t)$ we get

$$F(c_n) - F(b_n) = \Gamma\left(\frac{\varphi^m}{e}\right)\frac{\varphi^m}{en} + \mathcal{O}\left(\frac{\ln^2 n}{n^2}\right)$$

Thus,

$$I(n) = a_n(F(c_n) - F(b_n)) = \frac{1}{e}\Gamma\left(\frac{\varphi^m}{e}\right) + \mathcal{O}\left(\frac{\ln^2 n}{n}\right)$$

That is $\lim\limits_{n\to\infty} I(n) = \frac{1}{e}\Gamma\left(\varphi^m e^{-1}\right)$, which is the desired conclusion.

Also solved by Albert Stadler, Switzerland; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Ángel Plaza, Spain; Ramya Dutta, Chennai Mathematical Institute (student), India; Moti Levy, Rehovot, Israel and the proposers.

122. *Proposed by Mohammed Aassila, Strasbourg, France.* Choose 321 different points inside a unit cube. Prove that 4 of these points lie inside some sphere of radius $\frac{4}{23}$.

Solution by the proposer.

Let n and k be positive integers, let r be a positive real number, and suppose there are n points $X_1, X_2, \cdots, X_n$ in the unit cube such that no $k+1$ of them belong to a ball of radius r. Let B_i be the ball of radius r centered at X_i and $B = B_1 \cup B_2 \cup \cdots \cup B_n$. No point belongs to $k+1$ of the B_i's. For if, say, P belonged to $B_1 \cap B_2 \cap \cdots \cap B_{k+1}$, then $X_1, X_2, \ldots, X_{k+1}$ would all belong to the ball of radius r centered at P, contradicting the hypothesis. Hence we have $k|B| \geq |B_1| + |B_2| + \cdots + |B_n|$, that is :

$$|B| \ \geq \ \frac{4}{3}\pi \cdot r^3 \cdot \frac{n}{k}, \tag{1}$$

where $|B_i|$ denotes the volume of the ball B_i. On the other hand, since B is contained in the "rounded cube" consisting of all points at distance at most r from the unit cube, we have :

$$|B| \ \leq \ 1 + 6r + 3\pi r^2 + \frac{4}{3}\pi r^3. \tag{2}$$

Thus, combining (1) and (2) gives :

$$n \ \leq \ k\left(\frac{3}{4\pi r^3} + \frac{9}{2\pi r^2} + \frac{9}{4r} + 1\right).$$

For $k = 3$ and $r = 0.1739$, we get $n < 320.0988...$, so for 321 points inside a unit cube, 4 of them will lie inside some sphere of radius $0.1739 < \frac{4}{23}$.

Also solved by Ramya Dutta, Chennai Mathematical Institute (student), India.

123. *Proposed by Sava Grozdev and Deko Dekov (Jointly), Bulgaria.* Recall the definition of a hexyl triangle, See [1]. Given a triangle ABC and the Excentral triangle $PaPbPc$ of $\triangle ABC$. Let Ka be the point in which the perpendicular to AB through Pb meets the perpendicular to AC through Pc. Similarly define Kb and Kc. Then triangle $KaKbKc$ is known as the Hexyl Triangle. Prove that the Hexyl triangle is similar to the Pedal Triangle of the inverse of the Incenter in the Circumcircle. The reader may find a number of theorems without proofs about the hexyl triangle in [2]. The reader could find the proofs of these theorems and submit them for publication. This problem and the theorems in [2] are discovered by the computer program "Discoverer", created by the authors.

References

[1]: E. W. Weisstein, *MathWorld - A Wolfram Web Resource*, http://mathworld.wolfram.com/HexylTriangle.html

[2]: S. Grozdev and D, Dekov, *Computer-Discovered Mathematics: Hexyl-Anticevian Triangles*, www.journal-1.eu/2015/01/Grozdev-Dekov-Hexyl-Anticevian-Triangles-pp.60-69.pdf

Solution by the proposers.

We use barycentric coordinates. For a survey on barycentric coordinates see [2]. Given triangle ABC with side lengths $a = BC$, $b = CA$ and $c = AB$. The barycentric coordinates of the inverse of the Incenter in the Circumcircle are given in [2], article X(36). Given a point P, the barycentric coordinates of the pedal triangle of P are given in [2], 4.4, page 50. By using these formulas we easily obtain the barycentric coordinates of the pedal triangle $PaPbPc$ of the inverse of the Incenter in the Circumcircle as follows:

$$Pa = (0, (a+b-c)(a^2+b^2-c^2-2ab+bc), (a+c-b)(a^2+c^2-b^2-2ac+bc)),$$

$$Pb = ((a+b-c)(a^2+b^2-c^2-2ab+ac), 0, (b+c-a)(b^2+c^2-a^2-2bc+ac))$$

$$Pc = ((a+c-b)(a^2+c^2-b^2-2ac+ab), (b+c-a)(b^2+c^2-a^2-2bc+ab), 0).$$

We calculate the side lengths of triangle $PaPbPc$ by using the distance formula (See [2], 7.1) as follows. Denote

$$Q = \sqrt{a^3 + b^3 + c^3 - ab^2 - ba^2 - bc^2 - cb^2 - ac^2 - ca^2 + 3abc}$$

Then we have

$$a1 = |PbPc| = \frac{(b+c-a)\sqrt{a(a+b-c)(a+c-b)}}{2Q},$$

$$b1 = |PcPa| = \frac{(a+c-b)\sqrt{b(b+c-a)(a+b-c)}}{2Q},$$

$$c1 = |PaPb| = \frac{(a+b-c)\sqrt{c(b+c-a)(a+c-b)}}{2Q}.$$

The side lengths of the Hexyl triangle are given in [1], Theorem 5.2, as follows:

$$a2 = |KbKc| = \frac{2a\sqrt{bc}}{\sqrt{(c+a-b)(a+b-c)}},$$

$$b2 = |KcKa| = \frac{2b\sqrt{ca}}{\sqrt{(a+b-c)(b+c-a)}},$$

$$c2 = |KaKb| = \frac{2c\sqrt{ab}}{\sqrt{(b+c-a)(c+a-b)}}.$$

We denote

$$k_a = \frac{a1}{a2}, \quad k_b = \frac{b1}{b2}, \quad k_c = \frac{c1}{c2}.$$

Now it is easy to see that $k_a = k_b$ and $k_b = k_c$. Hence $k_a = k_b = k_c$, that is, the lengths of corresponding sides of the triangles are proportional. Denote by k the ratio of similarity. Then

$$k = k_a = k_b = k_c = \frac{(a+b-c)(b+c-a)(c+a-b)}{4Q\sqrt{abc}}$$

References

[1] S. Grozdev and D. Dekov, Computer Discovered Mathematics: Hexyl-Anticevian Triangles, International Journal of Computer Discovered Mathematics, 2015, vol. 0, no 0, 60-69. `http://www.journal-1.eu/2015/01/Grozdev-Dekov-Hexyl-Anticevian-Triangles-pp.60-69.pdf`.

[2] C. Kimberling, *Encyclopedia of Triangle Centers - ETC*, `http://faculty.evansville.edu/ck6/encyclopedia/ETC.html`.

[3] P. Yiu, *Introduction to the Geometry of the Triangle*, 2001, new version of 2013, `http://math.fau.edu/Yiu/YIUIntroductionToTriangleGeometry130411.pdf`

Also solved Michel Bataille, Rouen, France.

MATHCONTEST SECTION

This section of the Journal offers readers an opportunity to solve interesting and elegant mathematical problems mainly appeared in Math Contest around the world and most appropriate for training Math Olympiads. Proposals are always welcomed. The source of the proposals will appear when the solutions be published.

Proposals

85. Prove that

$$\lim_{n\to\infty} n\left(\frac{\pi}{4} - n\int_0^1 \frac{x^n}{1+x^{2n}}\,dx\right) = \int_0^1 f(x)\,dx,$$

where $f(x) = \frac{\arctan x}{x}$ if $x \in (0,1]$ and $f(0) = 1$.

86. Define the sequence $a_0, a_1, \ldots$ inductively by $a_0 = 1$, $a_1 = \frac{1}{2}$, and

$$a_{n+1} = \frac{na_n^2}{1+(n+1)a_n}, \quad \forall n \geq 1.$$

Show that the series $\displaystyle\sum_{k=0}^{\infty} \frac{a_{k+1}}{a_k}$ converges and determine its value.

87. Let $f : [0,\infty) \to \mathbb{R}$ be a periodical function, with period 1, integrable on $[0,1]$. For a strictly increasing and unbounded sequence $(x_n)_{n\geq 0}$, $x_0 = 0$, with $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$, we denote $r(n) = \max\{k \mid x_k \leq n\}$.
a) Show that:

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{r(n)} (x_k - x_{k+1}) f(x_k) = \int_0^1 f(x)\,dx$$

b) Show that:

$$\lim_{n\to\infty} \frac{1}{\ln n}\sum_{k=1}^{r(n)} \frac{f(\ln k)}{k} = \int_0^1 f(x)\,dx$$

88. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that $(x+y)f(2yf(x)+f(y)) = x^3 f(yf(x))$ for all $x, y \in \mathbb{R}^+$.

89. Find all real positive solutions (if any) to

$$x^3 + y^3 + z^3 = x + y + z, \text{ and}$$
$$x^2 + y^2 + z^2 = xyz.$$

Solutions

80. Given is the square matrix $A = (a_{k,l})$ of order $n \geq 2$ with elements $a_{k,l} = (k-l)^3$. Find the rank of the matrix.

(BULGARIAN NATIONAL UNIVERSITY OLYMPIAD 2015)

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. The answer is 2 for $n = 2, 3$ and 4 for $n \geq 4$.

Indeed, consider the matrices $B_n \in M_{n\times 4}(\mathbb{R})$ and $D \in M_{4\times 4}(\mathbb{R})$ defined as follows

$$D = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 3 & 0 \\ 0 & -3 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad B_n = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 2^2 & 2^3 \\ 1 & 3 & 3^2 & 3^3 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & n & n^2 & n^3 \end{bmatrix}$$

It is just a verification that $A = B_n D B_n^T$. Now, since $\operatorname{rank}(B_n) = \min(n, 4)$ we conclude that $\operatorname{rank}(A) \leq \min(n, 4)$.

- for $n \geq 4$ the matrix $B_4 D B_4^T$ is an *invertible* 4×4 sub-matrix of A so $\operatorname{rank}(A) \geq 4$, and consequently $\operatorname{rank}(A) = 4$ in this case.
- for $n = 2$ we have $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ so $\operatorname{rank}(A) = 2$ in this case.
- for $n = 3$ we have $A = \begin{bmatrix} 0 & -1 & -8 \\ 1 & 0 & -1 \\ 8 & 1 & 0 \end{bmatrix}$ and $\operatorname{rank}(A) = 2$ in this case also. □

81. Find the sum

$$\sum_{n=0}^{\infty} \frac{1}{n!\,(n^4 + n^2 + 1)}.$$

(BULGARIAN NATIONAL UNIVERSITY OLYMPIAD 2015)

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. The answer is $e/2$.

Note that for $n \geq 0$ we have

$$\frac{2}{n!(n^4 + n^2 + 1)} = \frac{1}{(n+1)!} + \frac{n}{(n+1)!(n(n+1)+1)} - \frac{n-1}{n!((n-1)n+1)}.$$

Thus,

$$\sum_{n=0}^{\infty} \frac{2}{n!(n^4 + n^2 + 1)} = e - 1 + 1 = e$$

and the announced conclusion follows. □

Also solved by Rafik Zeraoulia, Algeria; Michel Bataille, Rouen, France and Arkady Alt, San Jose, California, USA.

82. In descartes co-ordinate system with origin O given are the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and a point M_0 on it. If M is a point on the ellipse, compute the maximal area of the triangle OM_0M.

(BULGARIAN NATIONAL UNIVERSITY OLYMPIAD 2015)

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

The answer is $ab/2$. Consider the parametrization $t \mapsto M(t) = \begin{bmatrix} a\cos t \\ b\sin t \end{bmatrix}$ of the ellipse. and suppose that $M_0 = M(t_0)$ for some $t_0 \in \mathbb{R}$. Now the area $\mathcal{A}(t)$ of $\triangle OM_0M(t)$ is given by

$$\mathcal{A}(t) = \frac{1}{2}\left|\det(\overrightarrow{OM_0}, \overrightarrow{OM(t)})\right| = \frac{1}{2}\left|\det\begin{bmatrix} a\cos t_0 & a\cos t \\ b\sin t_0 & b\sin t \end{bmatrix}\right| = \frac{ab}{2}|\sin(t-t_0)|$$

Thus the maximum value of $\mathcal{A}(t)$ is $ab/2$ and it is attained when $t = t_0 + \frac{\pi}{2}$.

Solution 2 by Arkady Alt, San Jose, California, USA.

Let $M_0\,(x_0, y_0)$ and $M\,(x,y)$ .Then $\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}=1$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and by Cauchy Inequality
we have $[OM_0M] = \frac{1}{2}\left|\det\begin{pmatrix} x & y \\ x_0 & y_0 \end{pmatrix}\right| = \frac{1}{2}|xy_0 - x_0y| = \frac{1}{2}\left|\frac{x}{a}\cdot ay_0 + \frac{y}{b}\cdot(-bx_0)\right| \le$
$\frac{1}{2}\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}\cdot\sqrt{a^2y_0^2+b^2y_0^2} = \frac{1}{2}\sqrt{a^2y_0^2+b^2y_0^2} = \frac{ab}{2}\sqrt{\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}} = \frac{ab}{2}$.
Since equality in Cauchy Inequality occurs iff $\left(\frac{x}{a},\frac{y}{b}\right) = k\,(ay_0, -bx_0)$ then
$x = ka^2y_0, y = -kb^2x_0$ then by replacing (x,y) in $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with $\left(ka^2y_0, -kb^2x_0\right)$ we obtain $k^2\left(a^2y_0^2+b^2x_0^2\right)=1 \iff k^2a^2b^2\left(\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}\right)=1 \implies k=\frac{1}{ab} \implies$
$x = \frac{ay_0}{b}, y = -\frac{bx_0}{a}$.

Since for $(x,y) = \left(\frac{ay_0}{b}, -\frac{bx_0}{a}\right)$ we have $\frac{x^2}{a^2}+\frac{y^2}{b^2} = \frac{\frac{a^2y_0{}^2}{b^2}}{a^2}+\frac{\frac{b^2x_0^2}{a^2}}{b^2} = \frac{y_0^2}{b^2}+\frac{x_0^2}{a^2}=1$
and $\frac{1}{2}|xy_0 - x_0y| = \frac{1}{2}\left|\frac{ay_0}{b}\cdot y_0 - x_0\cdot\left(-\frac{bx_0}{a}\right)\right| = \frac{ab}{2}\left|\frac{y_0^2}{b^2}+\frac{x_0^2}{a^2}\right| = \frac{ab}{2}$
then $\max\,[OM_0M] = \frac{ab}{2}$.

Also solved by Michel Bataille, Rouen, France.

83. Let P be the sum of all 2×2 matrices whose elements are the integers $0, 1, 2$ and 3, without repetition. Find the matrices:

a) $S = \frac{1}{36}P$;

b) S^{2015};

c) $S^{2015} - M^{2015}$, where $M = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$.

(BULGARIAN NATIONAL UNIVERSITY OLYMPIAD 2015)

Solution by Michel Bataille, Rouen, France.

a) The matrix P is the sum of $4! = 24$ matrices (as many as permutations of $0, 1, 2, 3$). The $(1, 1)$ entry is 0 (resp. 1, resp. 2, resp. 3) in $3! = 6$ of them. Hence the $(1, 1)$ entry of P is $6 \times 0 + 6 \times 1 + 6 \times 2 + 6 \times 3 = 36$. In the same way, the other entries of P equal 36. It follows that $P = 36J$ where $J = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and so $S = J$.

b) By induction, we readily find that $J^n = 2^{n-1}J$ for every positive integer n. It follows that

$$S^{2015} = \frac{1}{36^{2015}} P^{2015} = \frac{1}{36^{2015}} \cdot 36^{2015} \cdot J^{2015} = 2^{2014}J = \begin{pmatrix} 2^{2014} & 2^{2014} \\ 2^{2014} & 2^{2014} \end{pmatrix}.$$

c) It is easily checked that the matrix M satisfies $M^3 = -8I$ where I is the 2×2 unit matrix. As a result,

$$M^{2015} = (M^3)^{671} \cdot M^2 = (-8I)^{671} \cdot M^2 = -2^{2013}M^2.$$

Thus, using $M^2 = \begin{pmatrix} -2 & -2\sqrt{3} \\ 2\sqrt{3} & -2 \end{pmatrix}$,

$$S^{2015} - M^{2015} = 2^{2013}(2J + M^2) = 2^{2014} \begin{pmatrix} 0 & 1 - \sqrt{3} \\ 1 + \sqrt{3} & 0 \end{pmatrix}.$$

84. The function $f(x)$ has a derivative in the interval $[0, 2015]$ and $f(0) = f(2015) = 0$. Prove the existence of such numbers $x, y \in (0, 2015)$, that $f'(x) = 2015f(x)$ and $f(y) = 2015f'(y)$.

(BULGARIAN NATIONAL UNIVERSITY OLYMPIAD 2015)

Solution by Henry Ricardo, New York Math Circle, USA.

Consider the function $g(t) = e^{-2015t}f(t)$. Then we have $g(0) = g(2015) = 0$. By Rolle's theorem, there exists $x \in (0, 2015)$ such that

$$0 = g'(x) = e^{-2015x}\left(f'(x) - 2015f(x)\right),$$

which implies that $f'(x) = 2015f(x)$.

Similarly, the function $h(t) = e^{-\frac{t}{2015}}f(t)$ satisfies the hypotheses of Rolle's theorem, and so there exists $y \in (0, 2015)$ such that

$$0 = h'(y) = e^{-\frac{y}{2015}}\left(f'(y) - \frac{1}{2015}f(y)\right).$$

Consequently, $f(y) = 2015 f'(y)$.

Comment. More generally, if f is differentiable on $[a, b]$ and $f(a) = f(b) = 0$, then for any real number λ there is an $x \in (a, b)$ such that $\lambda f(x) + f'(x) = 0$.

Proof. The function $P(t) = e^{\lambda t} f(t), t \in [a, b]$, satisfies the conditions of Rolle's theorem. Hence there is a number $x \in (a, b)$ such that $0 = P'(x) = (\lambda f(x) + f'(x))\, e^{\lambda x}$. Consequently, $\lambda f(x) + f'(x) = 0$.

Also solved by Michel Bataille, Rouen, France.

MATHNOTES SECTION

A Generalization of the Griffiths' theorem for conics with intersecting diameters

Sava Grozdev, Veselin Nenkov

Abstract. The paper considers a generalization of the Griffiths' theorem from the geometry of the triangle.
Keywords: Triangle, conic, pedal circle, pedal curve, Feuerbach configuration, Euler curve.

1. Introduction

The present paper generalizes a remarkable fact from the geometry of the triangle, known as Griffiths' theorem. The theorem itself is connected with the notion of a *pedal circle of a point with respect to a given* $\triangle ABC$. Remind that if a point P is not on the circumcircle of $\triangle ABC$, then the pedal circle of P with respect to $\triangle ABC$ is the circle, determined by the feet of the perpendiculars from P to the lines BC, CA and AB. The GRIFFITHS' THEOREM says: *When P moves along a line through the circumcenter of $\triangle ABC$, then the pedal circle of P with respect to $\triangle ABC$ passes through a fixed point (the Griffiths' point) on the nine-point circle (the Euler circle).* Two proofs of the Griffiths' theorem with complex could be found in [11] and [6], while a synthetic proof is included in [5]. The generalization in the present paper is connected with the generalization of certain notions which are recalled in the sequel.

2. Generalization of some notions from the geometry of the triangle

Barycentric coordinates with respect to a given $\triangle ABC$ will be used, namely $A(1,0,0)$, $B(0,1,0)$ and $C(0,0,1)$. The midpoints of the sides BC, CA and AB are $M_a(0,\frac{1}{2},\frac{1}{2})$, $M_b(\frac{1}{2},0,\frac{1}{2})$ and $M_c(\frac{1}{2},\frac{1}{2},0)$, respectively. Instead of the circumcircle of $\triangle ABC$ consider an arbitrary circumscribed conic $\bar{k}(O)$ with center $O(x_0,y_0,z_0)$, $(x_0+y_0+z_0=1)$. As shown in [8] and [1], the points on the conic are described by the equation:

$$\bar{k}(O):(1-2x_0)x_0yz+(1-2y_0)y_0zx+(1-2z_0)z_0xy=0. \tag{1}$$

Four special points from the $\triangle ABC$ plane are connected with the center O, see [9], [10].

If $I(x_I,y_I,z_I)(x_I+y_I+z_I=1)$ is one of them, than the other three determine a harmonic triangle of I [12]. The points are presented by

$$I_A\left(-\frac{x_I}{1-2x_I},\frac{y_I}{1-2x_I},\frac{z_I}{1-2x_I}\right), I_B\left(\frac{x_I}{1-2y_I},-\frac{y_I}{1-2y_I},\frac{z_I}{1-2y_I}\right),$$

$$I_C\left(\frac{x_I}{1-2z_I},\frac{y_I}{1-2z_I},-\frac{z_I}{1-2z_I}\right).$$

Note that I, I_A, I_B and I_C are centers of conics $k(I), k(I_A), k(I_B)$ and $k(I_C)$, respectively, which are inscribed in $\triangle ABC$ and are homothetic to $\bar{k}(O)$ [10]. A connection between the coordinates of O and I is shown in [9], i.e.:

$$x_0 = \frac{(1-2x_I-2y_Iz_I)x_I^2}{(1-2x_I)(1-2y_I)(1-2z_I)},$$

$$(2) \qquad y_0 = \frac{(1-2y_I-2z_Ix_I)y_I^2}{(1-2x_I)(1-2y_I)(1-2z_I)},$$

$$z_0 = \frac{(1-2z_I-2x_Iy_I)z_I^2}{(1-2x_I)(1-2y_I)(1-2z_I)}.$$

Using (1) and (2), another writing of the equation of $\bar{k}(O)$ is possible:

$$(3) \qquad \bar{k}(O) : x_I^2yz + y_I^2zx + z_I^2xy = 0$$

Let G be the gravity center of $\triangle ABC$. The center O determines the point $H(1-2x_0, 1-2y_0, 1-2z_0)$ uniquely, which comes from the vector equation $\overrightarrow{GH} = \frac{1}{2}\overrightarrow{GO}$ [7], [1]. If $H_a = AH \cap BC$, $H_b = BH \cap CA$, and $H_c = CH \cap AB$, then the six points $M_a, M_b, M_c, H_a, H_b, H_c$ and the midpoints of the segments AH, BH and CH (nine points totally) belong to a conic Ω, which is a generalization of the Euler circle of $\triangle ABC$. This conic is called to be *Euler curve of the point H with respect to $\triangle ABC$* [7], [1]. Additionally, we will say that Ω is the *Euler curve of $\triangle ABC$, associated to $\bar{k}(O)$*. As shown in [1], the equation of Ω could be written in the following way:

$$(4) \qquad \Omega : (1-2y_0)(1-2z_0)x^2 + (1-2z_0)(1-2x_0)y^2 + (1-2x_0)(1-2y_0)z^2$$
$$-2(1-2x_0)x_0yz - 2(1-2y_0)y_0zx - 2(1-2z_0)z_0xy = 0$$

Rewrite (4) in the form:

$$(4') \qquad \Omega : 4(1-2x_0)x_0yz + 4(1-2y_0)y_0zx + 4(1-2z_0)z_0xy$$
$$-\,[(1-2y_0)(1-2z_0)x + (1-2z_0)(1-2x_0)y + (1-2x_0)(1-2y_0)z]$$
$$(x+y+z) = 0$$

Substitute the coordinates O from (2) and (4′), to obtain the equation of Ω in the form:

$$(5) \qquad \Omega : 4(x_I^2yz + y_I^2zx + z_I^2xy)$$
$$-\,[(1-2x_I-2y_Iz_I)x + (1-2y_I-2z_Ix_I)y + (1-2z_I-2x_Iy_I)z]$$
$$(x+y+z) = 0$$

The curve Ω touches the inscribed curves $k(I), k(I_A), k(I_B)$ and $k(I_C)$, see [9]. For this reason we say that the curves $\bar{k}(O), k(I).k(I_A), k(I_B)$, $k(I_C)$ and Ω belong to a *Feuerbach configuration*. Additionally, we say that Ω is an Euler curve of the Feuerbach configuration.

Let $P(x_P, y_P, z_P)(x_P + y_P + z_P = 1)$ be a point in the plane of $\triangle ABC$, while the lines p_a, p_b and p_c be parallel to OM_a, OM_b and OM_c, respectively and $P_a = p_a \cap BC$, $P_b = p_b \cap CA$ and $P_c = p_c \cap AB$. If $P \in \bar{k}(O)$, see [7], the points P_a, P_b and P_c are collinear lying on the line s_p, which is called *Simson line of P with respect to $\bar{k}(O)$*. If $P \notin \bar{k}(O)$, then the point $Q(x_Q, y_Q, z_Q)$ is determined uniquely and its coordinates are:

$$(6) \qquad x_Q = \frac{x_I^2y_Pz_P}{\vartheta(P)},\ y_Q = \frac{y_I^2z_Px_P}{\vartheta(P)},\ z_Q = \frac{z_I^2x_Py_P}{\vartheta(P)}.$$

where

$$\vartheta(P) = x_I^2 y_P z_P + y_I^2 z_P x_P + z_I^2 x_P y_P. \tag{7}$$

The point Q is called to be conjugated to P with respect to the Feuerbach configuration under consideration [2]. Let the lines q_a, q_b and q_c be parallel to OM_a, OM_b and OM_c, respectively, and $Q_a = q_a \cap BC$, $Q_b = qcb \cap CA$ and $Q_c = q_a \cap AB$. Then the points P_a, P_b, P_c, Q_a, Q_b and Q_c belong to a conic π_P, which is called to be *pedal curve of the points P and Q with respect to the Feuerbach configuration or with respect to* $\bar{k}(O)$ [2]. As shown in the last paper, the equation of the pedal curve π_P is of the form:

$$4\vartheta(P)(x_I^2 yz + y_I^2 zx + z_I^2 xy) - (a_{11}x + a_{22}y + a_{33}z)(x + y + z) = 0 \tag{8}$$

where

$$a_{11} = \left[(1 - 2x_I - 2y_I z_I)y_P + 2y_I^2 z_P\right]\left[(1 - 2x_I - 2y_I z_I)z_P + 2z_I^2 y_P\right] x_P,$$
$$a_{22} = \left[(1 - 2y_I - 2z_I x_I)z_P + 2z_I^2 x_P\right]\left[(1 - 2y_I - 2z_I x_I)x_P + 2x_I^2 z_P\right] y_P,$$
$$a_{33} = \left[(1 - 2z_I - 2x_I y_I)x_P + 2x_I^2 y_P\right]\left[(1 - 2z_I - 2x_I y_I)y_P + 2y_I^2 x_P\right] z_P,$$

while $\vartheta(P)$ is expressed by (7).

3. A generalization of the Griffiths' theorem

Let $M(\lambda, \mu, \nu)(\lambda + \mu + \nu = 1)$ be a point on the circumcurve $\bar{k}(O)$, M' be the diametrically opposite to M on $\bar{k}(O)$. According to (3) we have:

$$x_I^2 \mu\nu + y_I^2 \nu\lambda + z_I^2 \lambda\mu = 0 \tag{9}$$

It is proved in [3] that the Simson lines s_M and $s_{M'}$ of the points M and M' have a common point T on the Euler curve Ω. The coordinates of this point are:

$$x_T = \frac{(1 - 2x_0)(\mu z_0 + \nu y_0 - \mu\nu)}{2y_0 z_0}$$
$$y_T = \frac{(1 - 2y_0)(\nu x_0 + \lambda z_0 - \nu\lambda)}{2z_0 x_0} \tag{10}$$
$$z_T = \frac{(1 - 2z_0)(\lambda y_0 + \mu x_0 - \lambda\mu)}{2x_0 y_0}$$

Taking into account the relationship between the coordinates of the centers O and I, which is expressed by (2), we find the coordinates of T by means of (10):

$$x_T = \frac{\mu\omega_z z_I^2 + \nu\omega_y y_I^2 - \mu\nu\Delta}{2y_I z_I}$$
$$y_T = \frac{\nu\omega_x x_I^2 + \lambda\omega_z z_I^2 - \nu\lambda\Delta}{2z_I x_I} \tag{11}$$
$$z_T = \frac{\lambda\omega_y y_I^2 + \mu\omega_x x_I^2 - \lambda\mu\Delta}{2x_I y_I}$$

where

$$\omega_x = 1 - 2x_I - 2y_I z_I,\ \omega_y = 1 - 2y_I - 2z_I x_I,\ \omega_z = 1 - 2z_I - 2x_I y_I,$$
$$\Delta = (1 - 2x_I)(1 - 2y_I)(1 - 2z_I).$$

Further, we state the task to find all points $P(x_P, y_P, z_P)(x_P + y_P + z_P = 1)$ from the plane of $\triangle ABC$, whose pedal curves pass through T. Since T is on Ω, (5) and (8) imply that:

$$4\vartheta(P)(x_I^2 y_T z_T + y_I^2 z_T x_T + z_I^2 x_T y_T) - (a_{11}x_T + a_{22}y_T + a_{33}z_T) = 0$$

Substitute (11) in the last equation and use (9). We obtain:

$$\frac{1}{x_I^2 y_I^2 z_I^2}\left[(\mu\omega_z z_I^2 - \nu\omega_y y_I^2)x_P + (\nu\omega_x x_I^2 - \lambda\omega_z z_I^2)y_P + (\lambda\omega_y y_I^2 - \mu\omega_x x_I^2)z_P\right]$$
$$\left[(\mu\omega_z z_I^2 - \nu\omega_y y_I^2)x_I^2 x_P + (\nu\omega_x x_I^2 - \lambda\omega_z z_I^2)y_I^2 y_P + (\lambda\omega_y y_I^2 - \mu\omega_x x_I^2)z_I^2 z_P\right]$$
$$= 0$$

It follows that the pedal curve of P passes through T, when P is on the line

(12) $\quad \chi l_P : (\mu\omega_z z_I^2 - \nu\omega_y y_I^2)x_P + (\nu\omega_x x_I^2 - \lambda\omega_z z_I^2)y_P + (\lambda\omega_y y_I^2 - \mu\omega_x x_I^2)z_P = 0$

or it is on the second degree circumcurve of $\triangle ABC$

(13) $\quad \chi_P : (\mu\omega_z z_I^2 - \nu\omega_y y_I^2)x_I^2 x_P + (\nu\omega_x x_I^2 - \lambda\omega_z z_I^2)y_I^2 y_P + (\lambda\omega_y y_I^2 - \mu\omega_x x_I^2)z_I^2 z_P = 0$

It remains to justify the line l_P and the curve χ_P.
The diameter MO of $\bar{k}(O)$ has an equation, determined by

$$\begin{vmatrix} \lambda & \mu & \nu \\ x_0 & y_0 & z_0 \\ x & y & z \end{vmatrix} = 0$$

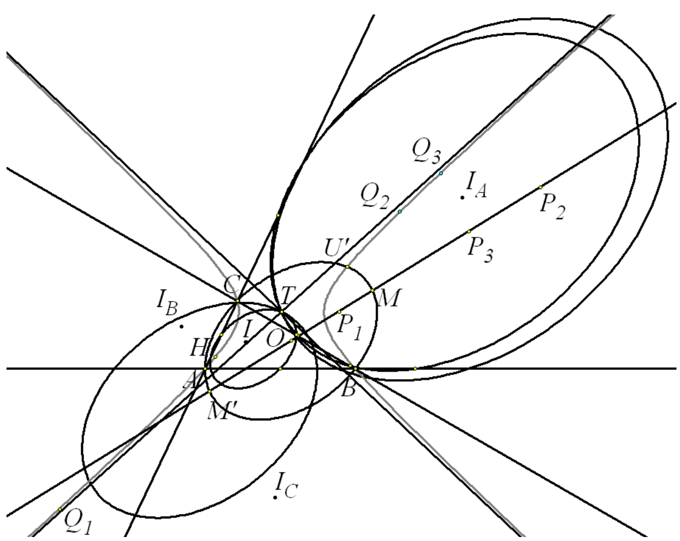

FIGURE 1.

Substitute the coordinates of O from (2). We obtain (12). Consequently, l_P is the diameter MO of $\bar{k}(O)$. Thus, we come to the following generalization of the Griffiths' theorem. See Figure 1.

Theorem 1. *If a point moves on a line through the circumcenter of $\triangle ABC$ with respect to a circumcurve $\bar{k}(O)$, then the pedal curve of this point with respect to $\triangle ABC$ passes through a fixed point on the Euler circle with respect to the Euler curve of $\triangle ABC$, associated to $\bar{k}(O)$.*
Note that if the point P and its conjugate Q with respect to $\bar{k}(O)$ have one and the same pedal curve, then Q also belongs to the set of points, whose pedal curves pass through T. It is of interest to find the locus of the conjugated point Q when P moves on the diameter MO. The two conjugated points could be replaced by each other. For this reason we could consider the point Q as moving on l_P and now the task is to find the locus of P. Further, substitute the coordinates of Q from (6) to (12). After some manipulations we obtain equation (13). In such a way we come to:
Theorem 2. *The set of all points, whose pedal curves pass through a point T on the Euler curve Ω, consists of the diameter l_P of $\bar{k}(O)$ and the circumcurve χ_P of $\triangle ABC$.*
Of course, in theorem 2 each pedal curve should be counted twice - as a pedal curve of the point P on the diameter of $\bar{k}(O)$ and as a pedal curve of the point Q , which is conjugated to P with respect to $\bar{k}(O)$. It is interesting to discover the type of the curve χ_P and to find some of its properties. For the purpose some properties of the second degree curves in barycenric coordinates are revised.

4. Asymptotic directions and centers of second degree curves in barycentric coordinates

Take the equation of a curve in the plane of $\triangle ABC$:

$$k : a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{23}yz + 2a_{31}zx = 0 \tag{14}$$

Putting $z = 1 - x - y$, we get:

$$k : (a_{11} + a_{33} - 2a_{31})x^2 + (a_{22} + a_{33} - 2a_{23})y^2 \tag{15}$$

$$+2(a_{12} - a_{31} - a_{23} + a_{33})xy + 2(a_{31} - a_{33})x + 2(a_{23} - a_{33})y + a_{33} = 0$$

Consider (15) as an equation in an affine coordinate system. A vector $\vec{a}(u, v)$, whose coordinates are determined with respect to the same affine coordinate system as the curve k' with equation

$$k' : a'_{11}x^2 + a'_{22}y^2 + 2a'_{12}xy + 2a'_{31}x + 2a'_{23}y + a'_{33} = 0,$$

is asymptotic to k' if and only if $a'_{11}u^2 + a'_{22}v^2 + 2a'_{12}uv = 0$. From here and from (15) we have

$$(a_{11} + a_{33} - 2a_{23})u^2 + (a_{22} + a_{33} - 2a_{23})v^2 + 2(a_{12} - a_{31} - a_{23} - a_{33})uv = 0$$

With respect to $\triangle ABC$ the vector $\vec{u}$ has the following barycentric coordinates $\vec{a}(u, v, w = -u - v)$. Substitute $u^2 = -uv - wu$ and $v^2 = -uv - vw$ in the last equation to obtain:

$$(a_{22} + a_{33} - 2a_{23})vw + (a_{33} + a_{11} - 2a_{31})wu + (a_{11} + a_{22} - 2a_{12})uv \tag{16}$$

Thus, it is established that the vector $\vec{a}(u, v, w)$ is asymptotic to the curve k with equation (14) if and only if the coordinates of $\vec{a}(u, v, w)$ satisfy (16).

It is shown in [4] that the barycentric coordinates of the center of the curve k with equation (14) are solutions of the following linear system:

$$(17)\qquad \begin{cases} (a_{11}-a_{31})x+(a_{12}-a_{23})y+(a_{31}-a_{33})z \\ (a_{12}-a_{31})x+(a_{22}-a_{23})y+(a_{23}-a_{33})z \\ x+y+z=1 \end{cases}$$

5. Some properties of the curve χ_P

Applying (17) to the equation (13) of the curve χ_P, we obtain that the coordinates of its center are expressed by (11). Consequently, T is the center of χ_P.
According to [3] the vector $\vec{p}_0(\lambda(\nu y_0-\mu z_0),\mu(\lambda z_0-\nu x_0),\nu(\mu x_0-\lambda y_0))$ is collinear with the Simson line s_M of the point M. On the other hand, taking into account (2), we have that the vector

$$\vec{p}_1:\lambda(\mu\omega_z z_I^2-\nu\omega_y y_I^2),\mu(\nu\omega_x x_I^2-\lambda\omega_z z_I^2),\nu(\lambda\omega_y y_I^2-\mu\omega_x x_I^2)$$

is collinear with the line s_M. (13), (16) and the coordinates of $\vec{p}_1$ imply that $\vec{p}_1$ is from the asymptotic direction of χ_P if and only if (9) is satisfied, i.e. $M\in\bar{k}(O)$. But s_M passes through the point T (center of χ_P), thus concluding that this line is an asymptote of χ_P. Analogously, the Simson line $s_{M'}$ of M' is an asymptote of χ_P. In such a way it is established
Theorem 3. *The curve χ_P is hyperbola with center T and with asymptotes the Simson lines of the points M and M'.*
It is shown in [3] that only one more Simson line passes through the point T, namely the Simson line of the point

$$U'(2(x_0+x_T)-1,2(y_0+y_T)-1,2(z_0+z_T)-1)$$

on $\bar{k}(O)$. Substitute (2) and (11) in the coordinates of U' and use the results for the left hand side of (13). It follows that U' is on χ_P and we have the following
Theorem 4. *The hyperbola χ_P and the circumscribed curve $\bar{k}(O)$ have common points A,B,C and U'.*

6. Conclusion

Note, that together with the generalization of the Griffiths' theorem we have determined the whole set of the points, whose pedal curves pass through a fixed point on the Euler curve. In fact it is established that outside the diameters of $\bar{k}(O)$ there is no new pedal curve passing through a fixed point on the Euler curve. Particularly it follows that in case a point P moves on a diameter d of the circumcircle of $\triangle ABC$, then its isogonally conjugate moves on a hyperbola χ_P with a center, which is the common point of the Simson lines of the extremities of the diameter d and the Simson lines are asymptotes of the hyperbola χ_P. Also note, that the proposed proof is not valid for diameters of a circumscribed $\bar{k}(O)$, which have no common point with $\bar{k}(O)$, since it has been applied essentially the existence of the point $M(\lambda,\mu,\nu)$ and all the properties of the circumscribed curves that follow from here. On the other hand, investigations in dynamic geometry (with GEOMETER'S SKATCHPAD for example) show that the generalization is still valid. The hyperbola χ_P is replaced by an ellipse or a parabola but the proof itself needs another approach.

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JUNIOR PROBLEMS

Solutions to the problems stated in this issue should arrive before June 10, 2016.

Proposals

46. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania.* Solve in real numbers the equation

$$\frac{1}{x-1}+\frac{2}{x-2}+\frac{3}{x-3}+\frac{4}{x-4}=2x^2-5x-4.$$

47. *Proposed by Pham–Thanh Hung, Math. Dept. "Can Tho City" Vietnam.* Let a,b,c be positive real numbers. Prove that

$$\left(\frac{a}{b+c}\right)^2+\left(\frac{b}{c+a}\right)^2+\left(\frac{c}{a+b}\right)^2+\frac{10(a+b-c)(b+c-a)(c+a-b)}{(a+b)(b+c)(c+a)}\geq 2.$$

48. *Proposed by Titu Zvonaru, Comăneşti, Romania.* Let P be a point on the hypotenuse BC of the right–angled triangle ABC. If X and Y are the intersections of AP with the external common tangent lines to the circumcircles of the triangles ABP and ACP, prove that $XY = AP\sqrt{2}$ if and only if $BC = AB\sqrt{2}$.

49. *Proposed by Armend Sh. Shabani, University of Prishtina, Department of Mathematics, Republic of Kosova.* Solve the equation

$$3\cdot 5^{x+1}+11\cdot 3^{x-1}+5\cdot 2^x+2^{x-2}=2016.$$

50. *Proposed by Dorlir Ahmeti, University of Prishtina, Department of Mathematics, Republic of Kosova.* Let p be prime number such that $p \equiv 7(\mod 8)$. We define $A=\left\{1,2,...,\frac{p-1}{2}\right\}$ and $f(k)=\left|p\left\lfloor\frac{2^k+p-1}{2p}\right\rfloor-2^{k-1}\right|$ for all $k\in A$ and $\frac{p-1}{2}$ is prime number, where $\lfloor x\rfloor$ is greatest integer not greater than x. Prove that $f(A)=A$.

Solutions

41. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania.* Let a,b,c be positive real numbers. Show that $a+b+c+3\sqrt[3]{abc}\geq 2(\sqrt{ab}+\sqrt{bc}+\sqrt{ac})$.

Solution by Titu Zvonaru, Comăneşti, Romania.

The Schur inequality is

$$x(x-y)(x-z)+y(y-z)(y-z)+z(z-x)(z-y)\geq 0$$

(1) $$\iff x^3+y^3+z^3+3xyz\geq x^2y+xy^2+y^2z+yz^2+x^2z+xz^2$$

Applying the AM - GM inequality we obtain

(2) $$x^2y+xy^2\geq 2\sqrt{x^3y^3}, y^2z+yz^2\geq 2\sqrt{y^3z^3}, x^2z+xz^2\geq 2\sqrt{x^3z^3}$$

By (1) and (2) yields

$$x^3+y^3+z^3+3xyz\geq 2\sqrt{x^3y^3}+2\sqrt{y^3z^3}+2\sqrt{x^3z^3}.$$

Taking $x=\sqrt[3]{a}, y=\sqrt[3]{b}, z=\sqrt[3]{c}$, it results that

$$a+b+c+3\sqrt[3]{abc}\geq 2(\sqrt{ab}+\sqrt{bc}+\sqrt{ac}).$$

The equality holds if and only if $a=b=c$.

Also solved by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain;Michel Bataille, Rouen, France; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Prishtina Math Gymnasium Problem Solving Group, Republic of Kosova and the proposers.

42. *Proposed by Francisco Javier García Capitán, I.E.S. Álvarez Cubero de Priego de Córdoba, Spain* . Let ABC be a triangle and P a point inside the internal part of ABC. Let XYZ be the cevian triangle of P. Find points U,V,W, on the lines YZ, ZX, XY such that the lines UV, VW, WU pass respectively through the points A,B,C.

Solution by Michel Bataille, Rouen, France.

We will say that a point U of the line YZ is *good* if the lines AU and CU intersect ZX and XY respectively at V and W such that B,V,W are collinear. The problem boils down to determining the good points of YZ. We show that there are two good points U_1,U_2. In part 1, we calculate the barycentric coordinates of U_1 and U_2 relatively to ABC and in part 2, we show how to construct the points U_1,U_2.

Part 1. Let $P=(\alpha:\beta:\gamma)$ where $\alpha,\beta,\gamma>0$ and $\alpha+\beta+\gamma=1$. Then, $X=(0:\beta:\gamma), Y=(\alpha:0:\gamma), Z=(\alpha:\beta:0)$, hence the equations of YZ,ZX,XY respectively are

$$\beta\gamma x-\gamma\alpha y-\alpha\beta z=0,\quad -\beta\gamma x+\gamma\alpha y-\alpha\beta z=0,\quad \beta\gamma x+\gamma\alpha y-\alpha\beta z=0.$$

Let $\ell: my+nz=0$ be an arbitrary line through A, intersecting ZY at U and ZX at V. We readily obtain $U=(\alpha(m\beta-n\gamma):-n\beta\gamma:m\beta\gamma)$ and $V=(\alpha(m\beta+n\gamma):n\beta\gamma:-m\beta\gamma)$. Then, $BV: m\beta\gamma x+\alpha(m\beta+n\gamma)z=0$ intersects XY at $W=(-\alpha(m\beta+n\gamma):2m\beta^2+n\beta\gamma:m\beta\gamma)$. Expressing that U is good if and only if C,U,W are collinear, we obtain the condition

$$\begin{vmatrix}\alpha(m\beta-n\gamma) & -\alpha(m\beta+n\gamma) & 0\\ -n\beta\gamma & 2m\beta^2+n\beta\gamma & 0\\ m\beta\gamma & m\beta\gamma & 1\end{vmatrix}=0.$$

This condition easily rewrites as $m^2\beta^2-mn\beta\gamma-n^2\gamma^2=0$, that is, $(m\beta-\tau n\gamma)(m\beta+(1/\tau)n\gamma)=0$ where $\tau=\frac{1+\sqrt{5}}{2}$. Thus, there are two good points U_1,U_2 obtained as the points of intersection with YZ of the lines $\tau\gamma y + \beta z = 0$ and $(-1/\tau)\gamma y + \beta z = 0$. A simple calculation gives

$$U_1 = (\alpha : -\beta\tau : \gamma\tau^2), \qquad U_2 = (\alpha\tau^2 : \beta\tau : \gamma).$$

Part 2. Let Π_A be the perspectivity with centre A from line YZ to line ZX, Π_B the perspectivity with centre B from line ZX to line XY and Π_C the perspectivity with centre C from line XY to line YZ. Clearly, a point U of YZ is good if and only if $\Pi_C \circ \Pi_B \circ \Pi_A(U) = U$, that is, if and only if U is a double point of the projectivity $p = \Pi_C \circ \Pi_B \circ \Pi_A$ from the line YZ to itself. Steiner's construction of the double points of such a projectivity is classical [see for example H. Dorrie, *100 Great Problems of Elementary Mathematics*, Dover, 1965, p. 255-257]. On the figure, we have first constructed $Z' = p(Z), Y' = p(Y)$ and $M' = p(M)$ where M is the point of intersection of AP and YZ.

Also solved by the proposer.

43. *Proposed by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.* In a multiple–choice–test you are asked to answer four questions. Question $i \in \{1, 2, 3, 4\}$ has $i + 1$ possible answers and each question has only one correct answer. Answering randomly, what is the probability of giving at least two correct answers?

Solution by the Prishtina Math Gymnasium Problem Solving Group, Republic of Kosova.

The solver submitted two correct solutions. We present one of them slightly modified by the editor. Let $P(0)$ and $P(1)$ be respectively the probability that one gives no correct answer and just one correct answer. The result is clearly $1 - P(0) - P(1)$. Letting $P_k(0)$ and $P_k(1)$ respectively the probability that the answer of the k–th question be uncorrect or correct, clearly we have:

$$P(0) = P_1(0) \cdot P_2(0) \cdot P_3(0) \cdot P_4(0) = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} = \frac{1}{5}$$

$P(1) = P_1(1) \cdot P_2(0) \cdot P_3(0) \cdot P_4(0) + P_1(0) \cdot P_2(1) \cdot P_3(0) \cdot P_4(0) + P_1(0) \cdot P_2(0) \cdot P_3(1) \cdot P_4(0) + P_1(0) \cdot P_2(0) \cdot P_3(0) \cdot P_4(1)$

$$= \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{3}{4} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{1}{4} \cdot \frac{4}{5} + \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \frac{1}{5} = \frac{5}{12}$$

Finally,

$$1 - P(0) - P(1) = \frac{23}{60}.$$

Also solved by the proposer.

44. *Proposed by Proposed by Armend Sh. Shabani, University of Prishtina, Department of Mathematics, Republic of Kosova.* Let m be an odd integer greater than 3. If $2^{2^n} + 1$ is prime, he can not be expressed as a difference of m-th powers of two positive integers.

Solution by the the Prishtina Math Gymnasium Problem Solving Group, Republic of Kosova.

We assume the contrary $2^{2^n}+1=x^m-y^m, x,y\in\mathbb{N}$.
Then we obtain
$$2^{2^n}+1=(x-y)(x^{m-1}+\dots y^{m-1}).$$
Therefore it should be $x-y=1$. Then,
$$2^{2^n}+1=(y+1)^m-y^m=m(y^{m-1}+\frac{m-1}{2}y^{m-2}+\dots)+1.$$
It means that m divides 2^{2^n} which is contradiction according to unique factorization theorem.

Also solved by the proposer.

45. *Proposed by Marcel Chiriţă, Bucharest, Romania.* Solve in real numbers the following system:
$$\begin{cases}2^{2x^2-1}+2^{y^2-2}=12\\ 3^{2x^2-1}+3^{y^2-2}=36.\end{cases}$$

Solution by Michel Bataille, Rouen, France.

We show that the solutions for the pairs (x,y) are
$$\left(\sqrt{\frac{3}{2}},\sqrt{5}\right),\ \left(-\sqrt{\frac{3}{2}},\sqrt{5}\right),\ \left(\sqrt{\frac{3}{2}},-\sqrt{5}\right),\ \left(-\sqrt{\frac{3}{2}},-\sqrt{5}\right),\ \left(\sqrt{2},2\right),\ \left(-\sqrt{2},2\right),\ \left(\sqrt{2},-2\right),\ \left(-\sqrt{2},-2\right).$$
As it is readily checked, it suffices to show that $(2,3)$ and $(3,2)$ are the solutions for (x,y) of the system
$$\begin{cases}2^x+2^y &= 12\\ 3^x+3^y &= 36\end{cases}$$
The pairs $(2,3)$ and $(3,2)$ are obvious solutions for (x,y). We show that there are no other solutions.
Let (x,y) be an arbitrary solution. Setting $X=e^{x-2},\ Y=e^{y-2}$, we have
$$X^{\ln(2)}+Y^{\ln(2)}=3,\qquad X^{\ln(3)}+Y^{\ln(3)}=4,$$
hence $X\in(0,3^{1/\ln(2)})$ (since $X^{\ln(2)}<3$) and $f(X)=0$ where $f(u)=u^{\ln(3)}+\left(3-u^{\ln(2)}\right)^{\frac{\ln(3)}{\ln(2)}}-4$.
A short calculation gives the derivative of f on $(0,3^{1/\ln(2)})$:
$$f'(u)=(\ln(3))u^{\ln(3)-1}\left[1-\left(\theta(u^{\ln(2)})\right)^{\frac{\ln(3)-\ln(2)}{\ln(2)}}\right]$$
where $\theta(t)=\frac{3-t}{t}$.
On $(0,\infty)$, the function θ is strictly decreasing with $\theta\left(\frac{3}{2}\right)=1$. Thus,
$$f'(u)>0\Longleftrightarrow\theta(u^{\ln(2)})<1\Longleftrightarrow u>\alpha=\left(\frac{3}{2}\right)^{1/\ln(2)}$$
and so f is strictly decreasing on $(0,\alpha]$ and strictly increasing on $[\alpha,3^{1/\ln(2)})$. It is readily checked that $1\in(0,\alpha)$, $e\in(\alpha,3^{1/\ln(2)})$ and $f(1)=0=f(e)$. Therefore $f(u)\neq 0$ if $u\neq 1,e$. It follows that we must have $X=1$ or $X=e$. Since $X=1$ implies $Y=2^{1/\ln(2)}=e$ and $x=2,y=3$ while $X=e$ implies $Y=1$ and $x=3,y=2$, the proof is complete.

Also solved by Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania; Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain and the proposer.