Proposals and solutions must be legible and should appear on separate sheets, each indicating the name of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editors have supplied a solution. The editors encourage undergraduate and pre-college students to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the class and school name. Solutions will be evaluated for publication by a committee of professors according to a combination of criteria. Questions concerning proposals and/or solutions can be sent by e-mail to: mathproblems-ks@hotmail.com

Solutions to the problems stated in this issue should arrive before January 10, 2016

Problems

117. Proposed by Cornel Ioan Vălean, Timiș, Rumania. Calculate

\[ \int_0^\pi \left( \text{Chi}(\cot^2(x)) + \text{Shi}(\cot^2(x)) \right) \csc^2(x) e^{-\csc^2(x)} \, dx. \]

Where \( \text{Shi}(x) = \int_0^x \frac{\sinh(t)}{t} \, dt \) and \( \text{Chi}(x) = \gamma + \log(x) + \int_0^x \frac{\cosh(t)-1}{t} \, dt. \)

118. Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia. Compute the following sum

\[ \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^{m+n} \frac{m \log(m+n)}{(m+n)^3}. \]
119. Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania. Let \( n > 1 \) be an integer. Calculate
\[
\int_0^\infty \ln^n \left| \frac{1-x}{1+x} \right| \, dx.
\]

120. Proposed by Anastasios Kotronis, Athens, Greece (Jointly) and Haroun Meghaichi (student), Algiers, Algeria. Let \( I(A) = \int_1^A \frac{1}{x} \, dx \). For \( n \) a positive integer compute the following limit, if it exists
\[
\lim_{A \to +\infty} \frac{\ln^{n-1} A}{A} \left( I(A) - \sum_{k=1}^{n-1} \frac{k! A}{\ln^{k-1} A} \right),
\]
where the sum over an empty set of indices is interpreted as zero.

121. Proposed by D.M. Bătinețu-Giurgiu, “Matei Basarab” National College, Bucharest, Romania, and Neculai Stanciu, “George Emil Palade” School, Buzău, Romania. Let \( m > 0 \), \( L_k \) be \( k \)-th Lucas number and \( \Gamma: (0, \infty) \to (0, \infty) \) be the Gamma function. Calculate
\[
\lim_{n \to \infty} \int_{\sqrt[n]{n!}}^{n^{+1} \sqrt{(n+1)!}} \Gamma \left( \frac{x}{n} \sqrt{L_n} \right) \, dx.
\]

122. Proposed by Mohammed Aassila, Strasbourg, France. Choose 321 different points inside a unit cube. Prove that 4 of these points lie inside some sphere of radius \( \frac{2}{23} \).

123. Proposed by Sava Grozdev and Deko Dekov (Jointly), Bulgaria. Recall the definition of a hexyl triangle, See [1]. Given a triangle \( ABC \) and the Excentral triangle \( PaPbPc \) of \( \triangle ABC \). Let \( Ka \) be the point in which the perpendicular to \( AB \) through \( Pb \) meets the perpendicular to \( AC \) through \( Pc \). Similarly define \( Kb \) and \( Kc \). Then triangle \( KaKbKc \) is known as the Hexyl Triangle. Prove that the Hexyl triangle is similar to the Pedal Triangle of the inverse of the Incenter in the Circumcircle. The reader may find a number of theorems without proofs about the hexyl triangle in [2]. The reader could find the proofs of these theorems and submit them for publication. This problem and the theorems in [2] are discovered by the computer program “Discoverer”, created by the authors.

References


Solutions

No problem is ever permanently closed. We will be very pleased considering for publication new solutions or comments on the past problems.

110. Proposed by Henry Ricardo, New York Math Circle, New York. Find the sum \( \sum_{n=1}^{\infty} \frac{1}{p(n)} \), where \( p(n) = n(3n - 1)/2 \) is the \( n \)th pentagonal number.

Solution 1 by Haroun Meghaichi(student), University of Science and Technology, Houari Boumediene, Algiers, Algeria.

Note that for any integer \( n \geq 1 \) we have

\[
2 \int_0^1 \int_0^1 x^{3n-2} y^{n-1} \, dy \, dx = \frac{2}{n(3n-1)}.
\]

According to the monotone convergence theorem we can interchange the integral-sum order since all the terms are positive and the series converges, so

\[
\sum_{n=1}^{\infty} \frac{2}{n(3n-1)} = 2 \int_0^1 \int_0^1 \sum_{n=1}^{\infty} x^{3n-2} y^{n-1} \, dy \, dx
= 2 \int_0^1 \int_0^1 \frac{x}{1-x^2 y} \, dy \, dx
= 2 \int_0^1 \frac{-\ln (1-x^2)}{x^2} \, dx
= 2 \left[ \left( \frac{1}{x} - 1 \right) \ln(1-x^3) \right]_0^1 + 6 \int_0^1 \frac{x}{1+x+x^2} \, dx \quad (1)
= 3 \int_0^1 \frac{2x+1}{x^2+x+1} - \frac{4}{(2x+1)^2+3} \, dx
= 3 \ln 3 - \frac{\pi \sqrt{3}}{3}.
\]

In (1) we used an integration by parts.

Solution 2 by Ángel Plaza, Universidad de Las Palmas de Gran Canaria, Spain.

The problem was proposed in [1] and its solution is in [2] and also in [3],[4]. Here, we follow the argument given in [4] for the particular pentagonal numbers.
\[
\sum_{n=1}^{\infty} \frac{1}{p(n)} = \sum_{n=1}^{\infty} \frac{2}{n(3n-1)} = -2 \sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{3}{3n+2} \right). \quad \text{Using the digamma function, we have}
\]
\[
\psi(x) = \frac{d}{dx} \ln \Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)}.
\]

Thus, \[
\sum_{n=0}^{\infty} \left( \frac{1}{n+1} - \frac{3}{3n+2} \right) = \psi(3/2) - \psi(1), \quad \text{and}
\]
\[
\psi(3/2) - \psi(1) = - \ln 3 - \frac{\pi}{2} \cot \frac{\pi}{3} + \frac{1}{2} \sum_{j=1}^{2} \cos \frac{4j\pi}{3} \ln \left( 2 - 2 \cos \frac{2j\pi}{3} \right)
\]
\[
= \frac{1}{6} \left( \sqrt{3} \pi - 9 \log(3) \right)
\]

and, therefore \[
\sum_{n=1}^{\infty} \frac{1}{p(n)} = -2 \left( \frac{1}{6} \left( \sqrt{3} \pi - 9 \log(3) \right) \right) = \log(27) - \frac{\pi}{\sqrt{3}}. \quad \square
\]

References


Solution 3 by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia.

Consider the serie \( f(x) = \sum_{n=1}^{\infty} \frac{2}{n} \cdot \frac{x^{3n-1}}{3n-1} \). Then
\[
f'(x) = \sum_{n=1}^{\infty} \frac{2}{n} \cdot x^{3n-2} = \frac{2}{x^2} \sum_{n=1}^{\infty} \frac{(x^3)^n}{n} = -\frac{2}{x^2} \log(1-x^3),
\]
\[
f(x) = \frac{2(1-x)}{x} \log(1-x) + \left( \frac{2}{x} + 1 \right) \log(1+x+x^2) - 2\sqrt{3} \arctan \frac{2x+1}{\sqrt{3}} + C
\]
\[
= \frac{2}{x} \log(1-x^3) - 2 \log(1-x) + \log(1+x+x^2) - 2\sqrt{3} \arctan \frac{2x+1}{\sqrt{3}} + C
\]

Thus
\[
\lim_{x \to 0+0} f(x) = -2\sqrt{3} \arctan \frac{1}{\sqrt{3}} + C = -2\sqrt{3} \cdot \frac{\pi}{6} + C \Rightarrow C = \frac{\pi}{\sqrt{3}}.
\]

Therefore
\[
f(x) = \frac{2(1-x)}{x} \log(1-x) + \left( \frac{2}{x} + 1 \right) \log(1+x+x^2) - 2\sqrt{3} \arctan \frac{2x+1}{\sqrt{3}} + \frac{\pi}{\sqrt{3}}.
\]

Hence
\[
\lim_{x \to 1-0} f(x) = 0 + 3 \log 3 - 2\sqrt{3} \arctan \sqrt{3} + \frac{\pi}{\sqrt{3}}
\]
\[
= 3 \log 3 - 2\sqrt{3} \cdot \frac{\pi}{\sqrt{3}} + \frac{\pi}{\sqrt{3}} = 3 \log 3 - \frac{\pi}{\sqrt{3}}.
\]
By the Abel’s theorem,
\[
\sum_{n=1}^{\infty} \frac{2}{n(3n-1)} = \lim_{x \to 1-0} f(x) = 3 \log 3 - \frac{\pi}{\sqrt{3}}.
\]

**Solution 4 by Moti Levy, Rehovot, Israel.**

It is well known that (See Costas Efthimiou, "Finding Exact Values For Infinite Sums", 1998),
\[
\sum_{n=1}^{\infty} \frac{1}{(n+a)(n+b)} = \frac{1}{a-b} \int_0^1 \frac{t^b - t^a}{1-t} \, dt.
\]
So,
\[
\sum_{n=1}^{\infty} n \left( \frac{1}{n-\frac{1}{3}} \right) = 3 \int_0^1 \frac{t^{\frac{1}{3}} - 1}{1-t} \, dt = 3 \int_0^1 \frac{1}{t + \left( \frac{\sqrt{3}}{2} \right)^2 + \frac{\sqrt{3}}{2}} \, dt
\]
\[
= 3 \int_0^1 \frac{3u}{u^2 + u + 1} \, du = \frac{9}{2} \ln 3 - \frac{1}{2} \sqrt{3}\pi.
\]
\[
\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}} = 3 \log 3 - \frac{\pi}{\sqrt{3}}.
\]

Also solved by Albert Stadler, Switzerland; Arkady Alt, San Jose, California, USA. Michel Bataille, Rouen, France; Zeraoulia Rafik, Batna, Algeria (Student); Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; Moubinool Omarjee, Lycée Henri IV, Paris, France and the proposer.

111. **Proposed by Moti Levy, Rehovot, Israel.** Let \( m, n \) be integers. Show that if \( n > m \geq 0 \) then
\[
\frac{x^n}{x^m + y^m} + \frac{y^n}{y^m + z^m} + \frac{z^n}{z^m + x^m} \geq 3 \left( \frac{1}{\sqrt{3}} \right)^{n-m}
\]
where real \( x, y, z > 0 \) and \( xy + yz + zx = 1 \).

**Solution 1 by the proposer.**

Since
\[
\frac{x^n}{x^m + y^m} + \frac{y^n}{y^m + z^m} + \frac{z^n}{z^m + x^m} = x^{n-m} - \frac{x^{n-m} y^m}{x^m + y^m} + y^{n-m} - \frac{y^{n-m} z^m}{y^m + z^m} + z^{n-m} - \frac{z^{n-m} x^m}{z^m + x^m}.
\]
Since \( x^m + y^m \geq 2x^{m/2}y^{m/2} \) then
\[
\frac{x^n}{x^m + y^m} + \frac{y^n}{y^m + z^m} + \frac{z^n}{z^m + x^m} \geq x^{n-m} - \frac{x^{n-m}y^m}{2x^{m/2}y^{m/2}} + y^{n-m} - \frac{y^{n-m}z^{m}}{2y^{m/2}z^{m/2}} + z^{n-m} - \frac{z^{n-m}x^{m}}{2z^{m/2}x^{m/2}}
\]
\[
x^{n-m} + y^{n-m} + z^{n-m} - \frac{1}{2} \left( x^{n-m} + y^{n-m} + z^{n-m} \right)
\]
By Chebyshev’s inequality
\[
x^{n-m} + y^{n-m} + z^{n-m} \geq x^{n-m} - \frac{3}{2} \frac{y^{n-m}}{x^{m/2}} + y^{n-m} - \frac{3}{2} \frac{z^{n-m}}{y^{m/2}} + z^{n-m} - \frac{3}{2} \frac{x^{n-m}}{z^{m/2}}
\]
\[
\frac{x^n}{x^m + y^m} + \frac{y^n}{y^m + z^m} + \frac{z^n}{z^m + x^m} \geq \frac{1}{2} \left( x^{n-m} + y^{n-m} + z^{n-m} \right)
\]  
(1)
Now we prove by mathematical induction that for positive integer \( p \),
\[
x^p + y^p + z^p \geq \left( \frac{2}{3} \right)^p.
\]  
(2)

**Solution 2 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.**

By Hölder reversed
\[
\sum \frac{a_k^p}{b_k^q} \geq \left( \frac{\sum a_k}{\sum b_k} \right)^{p/q} 3^{1+q-p}
\]
we get \( p = n, q = 1 \)
\[
\frac{x^n}{x^m + y^m} \geq \frac{(x + y + z)^n}{2(x^m + y^m + z^m)} 3^{2-n}
\]
and we prove that if \( xy + yz + zx = 1 \), then
\[
\frac{(x + y + z)^n}{2(x^m + y^m + z^m)} 3^{2-n} \geq \frac{3}{2} \left( \frac{1}{\sqrt[3]{3}} \right)^{n-m} \iff \frac{(x + y + z)^n}{(x^m + y^m + z^m)} \geq 3^{\frac{n+m-2}{3}}
\]
We adopt the Lagrange multipliers. Let
\[
f(x, y, z) = (x + y + z)^n - 3^{\frac{n+m-2}{3}}(x^m + y^m + z^m)
\]
and \( xy + yz + zx = 1 \). If \( x \to 0 \), then \( y \to \infty \) (or \( z \to \infty \)) and clearly the inequality holds by \( n > m \). So we confine \( (x, y, z) \) to satisfy \( D = \{ (x, y, z) \in \mathbb{R}^3 : a \leq x, y, z \leq 1/a \} \) where \( a \) is so small that \( f \) is positive on the boundary of \( D \) which is compact as well as the set \( D \cap \{ xy + yz + zx = 1 \} \). We want to show that the minimum of \( f \) subject to \( D \cap \{ xy + yz + zx = 1 \} \), which exists by the Weierstrass theorem is zero. To this end we define
\[
F(x, y, z, \lambda) = f(x, y, z) - \lambda(xy + yz + zx - 1), \quad x, y, z > 0
\]
The annihilation of gradient of \( F \) yields \( I = x + y + z, A = 3^{\frac{n+m-2}{3}} \)
\[ nI^{n-1} - Amx^{m-1} - \lambda(y + z) = 0, \quad nI^{n-1} - Amy^{m-1} - \lambda(x + z) = 0, \]
\[ nI^{n-1} - Amz^{m-1} - \lambda(x + y) = 0, \quad xy + yz + zx = 1 \]

It is easy to get that \( x = y = z \) is the only point satisfying the system and then it must be the minimum. Moreover, \( xy + yz + zx = 1 \) yields \( x = 1/\sqrt{3} \) and then by \( f \) we get

\[ \left( \frac{3}{\sqrt{3}} \right)^n - 3 \left( \frac{1}{\sqrt{3}} \right)^m 3^{\frac{n+m-2}{2}} = 0 \]

Now the proof of uniqueness of the gradient’s solution. First of all by the symmetry of \( F(x, y, z, \lambda) \) we can suppose \( x \geq y \geq z \). No subtracting the second by the first we get

\[ \lambda(x - y) = Am(y^{m-1} - x^{m-1}) \]

If \( x = y \) clearly holds. Letting for a moment \( x > y \) we come to

\[ \lambda = -Am \sum_{k=0}^{m-2} x^k y^{m-2-k} \quad \text{and} \quad \lambda = -Am \sum_{k=0}^{m-2} y^k z^{m-2-k} \]

but this is impossible unless \( x = y = z \) because of \( x \geq y \geq z \). This completes the proof.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania and the proposer.

112. Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania. Calculate

\[ \int_0^1 \ln^2 |\sqrt{x} - \sqrt{1-x}| dx. \]

Solution 1 by Michel Bataille, Rouen, France.

Let \( I \) be the proposed integral. We show that

\[ I = \frac{1}{2} + \frac{\pi(1 + \ln 2)}{4} + G \]

where \( G \) denotes the Catalan constant.

From the following expression of \( I \)

\[ I = \int_0^{1/2} \ln^2(\sqrt{1-x} - \sqrt{x}) dx + \int_{1/2}^1 \ln^2(\sqrt{x} - \sqrt{1-x}) dx, \]

and with the help of the change of variables defined by \( x = \sin^2 \left( \frac{\pi}{4} + \frac{y}{2} \right) \), we deduce

\[ I = \frac{1}{8} \int_{-\pi/2}^0 (\ln(1 - \cos y))^2 \cos y dy + \frac{1}{8} \int_{0}^{\pi/2} (\ln(1 - \cos y))^2 \cos y dy \]
so that
\[ I = \frac{1}{8} \int_{-\pi/2}^{\pi/2} (\ln(1 - \cos y))^2 \cos y \, dy = \frac{1}{4} \cdot J \]
where \( J = \int_{0}^{\pi/2} (\ln(1 - \cos y))^2 \cos y \, dy. \)

Integrating by parts, we obtain
\[ J = \left[ \sin y (\ln(1 - \cos y))^2 \right]_{0}^{\pi/2} - \int_{0}^{\pi/2} (\sin y)(2 \ln(1 - \cos y)) \cdot \frac{\sin y}{1 - \cos y} \, dy \]
\[ = -2 \int_{0}^{\pi/2} \sin^2 y \frac{\sin y}{1 - \cos y} \ln(1 - \cos y) \, dy. \]

(note that \((\sin y)(\ln(1 - \cos y))^2 \sim \sqrt{2}(1 - \cos y)^{1/2}(\ln(1 - \cos y))^2\) as \(y \to 0^+\) so that \(\lim_{y \to 0^+} \sin y(\ln(1 - \cos y))^2 = 0\).) It follows that
\[ J = -2 \int_{0}^{\pi/2} (1 + \cos y) \ln(1 - \cos y) \, dy = -2(K + L) \]
where \( K = \int_{0}^{\pi/2} \ln(1 - \cos y) \, dy \) and \( L = \int_{0}^{\pi/2} (\cos y) \ln(1 - \cos y) \, dy \).

The value of \( L \) is quickly obtained by integration by parts:
\[ L = \left[ \sin y \ln(1 - \cos y) \right]_{0}^{\pi/2} - \int_{0}^{\pi/2} (\sin y) \frac{\sin y}{1 - \cos y} \, dy \]
\[ = - \int_{0}^{\pi/2} (1 + \cos y) \ln(1 - \cos y) \, dy = - \left( \frac{\pi}{2} + 1 \right). \]

As for \( K \), we write
\[ K = \int_{0}^{\pi/2} \ln \left( 2 \sin^2 \frac{y}{2} \right) \, dy = \frac{\pi \ln 2}{2} + 2 \int_{0}^{\pi/2} \ln \left( \sin \frac{y}{2} \right) \, dy \]
\[ = \frac{\pi \ln 2}{2} + 4 \int_{0}^{\pi/4} \ln \sin u \, du. \]

\[ \int_{0}^{\pi/4} \ln(\sin x) \, dx = - \frac{G}{2} - \frac{\pi \ln 2}{4} \quad (*) \]
where \( G \) is the Catalan constant [for convenience, the proof is repeated below].

Collecting the previous results, we first obtain \( K = -\frac{\pi \ln 2}{2} - 2G \), then \( J = 4G + 2 + \pi + \pi \ln 2 \) and finally
\[ I = G + \frac{1}{2} + \frac{\pi}{4}(1 + \ln 2). \]

**Proof of (**) Recall that the Catalan constant \( G \) is defined by
\[ G = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^2} = \int_{0}^{1} \tan^{-1}(x) \frac{1}{x} \, dx. \]

Now, let
\[ U = \int_{0}^{\pi/4} \ln(\cos x) \, dx \quad \text{and} \quad V = \int_{0}^{\pi/4} \ln(\sin x) \, dx. \]
Then
\[
V + U = \int_0^{\pi/4} \ln\left(\frac{1}{2} \sin(2x)\right) dx = \frac{\pi}{4} \ln(1/2) + \int_0^{\pi/4} \ln(\sin(2x)) dx
\]
\[
= -\frac{\pi}{4} \ln 2 + \frac{1}{2} \int_0^{\pi/2} \ln(u) du = -\frac{\pi}{2} \ln 2
\]
and
\[
V - U = \int_0^{\pi/4} \ln(\tan x) dx = \left[x \ln(\tan x)\right]_0^{\pi/4} - \int_0^{\pi/4} x \frac{1 + \tan^2 x}{\tan x} dx
\]
\[
= -2 \int_0^{\pi/4} \frac{x}{\sin(2x)} dx = -\frac{1}{2} \int_0^{\pi/2} \frac{u}{\sin(u)} du = -G
\]
where the last equality follows from
\[
\int_0^{\pi/2} \frac{u}{\sin(u)} du = 2 \int_0^1 \frac{\tan^{-1}(x)}{x} dx
\]
[substitution \(u = 2\tan^{-1}(x)\)].
Thus, 2U = \(-\pi/2 \ln(2) + G\) and 2V = \(-\pi/2 \ln(2) - G\).

**Solution 2 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria.**

Let us denote the desired integral by \(I\). We will prove that
\[
I = \frac{1}{2} + K + \frac{\pi}{4} + \frac{\pi \ln 2}{4} \approx 2.745760280
\]
where \(K\) is the Catalan number defined by
\[
K = \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2}.
\]
Clearly we have
\[
I = \frac{1}{4} \int_0^1 \ln^2\left(1 - 2\sqrt{x(1-x)}\right) dx, \quad x \leftarrow \sin^2 t
\]
\[
= \frac{1}{4} \int_0^{\pi/2} \ln^2(1 - \sin(2t)) \sin(2t) dt = \frac{1}{8} \int_0^{\pi} \ln^2(1 - \sin \theta) \sin \theta d\theta
\]
\[
= \frac{1}{8} \int_{-\pi/2}^{\pi/2} \ln^2(1 - \cos \theta) \cos \theta d\theta = \frac{1}{4} \int_0^{\pi/2} \ln^2(1 - \cos \theta) \cos \theta d\theta
\]
\[
= \left[\frac{\ln^2(1 - \cos \theta)}{4} \sin \theta\right]_0^{\pi/2} - \frac{1}{2} \int_0^{\pi/2} \ln(1 - \cos \theta) \sin^2 \theta d\theta
\]
\[
= -\frac{1}{2} \int_0^{\pi/2} \ln(1 - \cos \theta) d\theta - \frac{1}{2} \int_0^{\pi/2} \ln(1 - \cos \theta) \cos \theta d\theta
\]
\[
= -A - \int_0^{\pi/2} \frac{\ln(1 - \cos \theta)}{2} \sin \theta d\theta + \frac{1}{2} \int_0^{\pi/2} \frac{\sin^2 \theta}{1 - \cos \theta} d\theta
\]
\[
= -A + \frac{1}{2} \int_0^{\pi/2} (1 + \cos \theta) d\theta = -A + \frac{\pi}{4} + \frac{1}{2}
\]
Next we will calculate \( A = \int_{0}^{\pi/2} \ln(1 - \cos \theta) d\theta \).

Indeed, we have
\[
A = -\frac{\pi \ln 2}{2} + 2 \int_{0}^{\pi/2} \ln |1 - e^{i\theta}| d\theta = -\frac{\pi \ln 2}{2} + 2 \Re \int_{0}^{\pi/2} \log(1 - e^{i\theta}) d\theta \\
= -\frac{\pi \ln 2}{2} - 2 \int_{0}^{\pi/2} \sum_{n=1}^{\infty} \frac{\cos n\theta}{n} \\
= -\frac{\pi \ln 2}{2} - 2 \sum_{m=0}^{\infty} \frac{(-1)^m}{(2m+1)^2} = -\frac{\pi \ln 2}{2} - 2K
\]

Hence \( I = K + \frac{\pi \ln 2}{4} + \frac{\pi}{4} + \frac{1}{2} \).

Also solved by Haroun Meghaichi (student), University of Science and Technology, Houari Boumediene, Algiers, Algeria; Moti Levy, Rehovot, Israel; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Albert Stadler, Switzerland; Moubinool Omarjee, Lycée Henri IV, Paris, France and the proposer.

113. Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria and Anastasios Kotronis, Athens, Greece (Jointly). Let \( m, n, p \) be positive integers with \( m \geq n \) and \( H_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m} \) the \( m \)-th Harmonic Number with \( H_0 := 0 \).

Show that for the values of \( m, p, n \) for which the denominators do not vanish, the following equalities hold:
\[
\sum_{k=0}^{n} (-1)^{k-1} \binom{n}{m-k} \binom{p}{m+n-k} \frac{H_p + H_m - H_{m-k}}{(m+n-k)(p+n-k)} = \frac{1}{nm(p+n)}
\]
\[
\sum_{k=0}^{n} (-1)^{k-1} \binom{n}{m-k} \binom{p+n-k}{m+n-k} \frac{H_p+n-k + H_m - H_{m-k}}{(m+n-k)(p+n-k)} = \frac{1}{nm(p+n)}.
\]

Solution by Moti Levy, Rehovot, Israel.

The following definitions and results are used in this solution:
1) Let \((\alpha_k)_{k\geq0}\) be a sequence which satisfies the following conditions:
\[
\alpha_0 = 1, \\
\frac{\alpha_{k+1}}{\alpha_k} = \frac{1}{k+1} \frac{(k+a)(k+b)}{(k+c)} z.
\]
Then
\[
\sum_{k=0}^{n} \alpha_k = 2F1 \left[ \begin{array}{c} a & b \\ c & z \end{array} \right],
\]
where \(2F1 \left[ \begin{array}{c} a & b \\ c & z \end{array} \right] \) is hypergeometric function.
2) Similarly, let \((\beta_k)_{k \geq 0}\) be a sequence which satisfies the following conditions:

\[
\beta_0 = 1, \\
\frac{\beta_{k+1}}{\beta_k} = \frac{1}{k+1} \frac{(k + a)(k + b)(k + c)}{(k + d)(k + e)} z.
\]

Then

\[
\sum_{k=0}^{n} \beta_k = 3F2 \left[ \begin{array}{c} a & b & c \\ d & e & z \end{array} \right],
\]

where \(3F2 \left[ \begin{array}{c} a & b & c \\ d & e & z \end{array} \right] \) is hypergeometric function.

3) The Chu-Vandermonde summation formula:

\[
2F1 \left[ \begin{array}{c} -n \\ b \end{array} \right] \left[ \begin{array}{c} c \\ 1 \end{array} \right] = \frac{(c-b)_n}{(c)_n}, \quad n \geq 0,
\]

where \((a)_n\) is the Pochhammer symbol,

\[
(a)_n := a(a + 1) \cdots (a + n - 1).
\]

4) Saalschütz Theorem:

\[
3F2 \left[ \begin{array}{c} -n \\ a \end{array} \right] \left[ \begin{array}{c} b \\ c \end{array} \right] \left[ \begin{array}{c} 1 \\ 1 \end{array} \right] = \frac{(c-b)_n}{(c)_n} \frac{(a-c)_n}{(a-c-a)_n} (x + n)_k \frac{(x + n + k)_k}{(x)_k} (k + n)_k (H_{k+n} - H_n),
\]

5) Let us denote by \(\delta[f(x)] := f'(0)\) and \((x + n + k)_k := \frac{(x + n + 1)_k}{k!}\).

Then

\[
\delta \left[ \binom{x + k + n}{k} \right] = \binom{k + n}{k} (H_{k+n} - H_n),
\]

\[
f(x) = xp(x) \implies \delta[f(x)] = p(0).
\]

First, we prove that

\[
\sum_{k=0}^{n} (-1)^{k-1} \frac{n!}{(m+n-k)(m+n-k)} = \left\{ \begin{array}{ll}
0 & \text{if } n \geq 1 \\
-1 & \text{if } n = 0
\end{array} \right.
\]

Let

\[
c_k = (-1)^{k} \frac{n!}{(m+n-k)(m+n-k)} \frac{(p+n)!}{(m+n-k)(p+n-k)!} \frac{m+n}{m+n-k}
\]

Then \(c_0 = 1\),

\[
\frac{c_{k+1}}{c_k} = \frac{(k+1)(p+n)}{(m+n-k)!} \frac{(m+n-k)}{(m+n-k-1)!} \frac{m+n-k}{m+n-k} = \frac{(k-n)(k-n)}{(k+1)(k-(m+n-1))},
\]

and

\[
\sum_{k=0}^{n} (-1)^{k-1} \frac{n!}{(m+n-k)(m+n-k)} = \frac{n!}{(m+n-k)!} \frac{1}{m+n-k} \sum_{k=0}^{n} c_k.
\]

\[
\sum_{k=0}^{n} (-1)^{k-1} \frac{n!}{(m+n-k)(m+n-k)} = \sum_{k=0}^{n} c_k = 2F1 \left[ \begin{array}{c} -n \\ -m \end{array} \right] \left[ \begin{array}{c} -m \\ 1 \end{array} \right].
\]
By Chu-Vandermonde summation formula (3),
\[ 2\, _2F_1\left[ \begin{array}{c} \frac{-n}{-m} \\ -(m+n-1) \end{array} \right| 1 \right] = \frac{(-n+1)_n}{(-(m+n-1))_n} = \left\{ \begin{array}{ll} 0 & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{array} \right. . \]
This proves equation (7).

Now we similarly prove that
\[ \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \frac{(q-k)}{(m+n-k)} \frac{1}{m+n-k} = \left\{ \begin{array}{ll} 0 & \text{if } n \geq 1 \\ -\frac{1}{m} & \text{if } n = 0 , \quad q \geq m+n. \end{array} \right. \] (8)

Let
\[ g_k = (-1)^{k-1} \frac{(q+n)}{(q)} \binom{m+n}{k^\cdot} \frac{(q-k)}{(m+n-k)} \frac{(n-k)}{(m+n-k)} \]
Then \( g_0 = 1 \) and
\[ \frac{g_{k+1}}{g_k} = \frac{\binom{n+1}{k+1} \binom{q-k-1}{m+n-k-1} \binom{q-k}{m+n-k} \binom{m+n-k}{m+n-k-1}}{\binom{k}{k} \binom{q-k}{m+n-k} \binom{m+n-k}{m+n-k-1}} = \frac{1}{k+1} \frac{(k-n)(k-m)}{(k-m+n)} \]
\[ \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \frac{(q-k)}{(m+n-k)} \frac{1}{m+n-k} = \frac{(-n+1)_n}{(-(m+n-1))_n} = \left\{ \begin{array}{ll} 0 & \text{if } n \geq 1 \\ 1 & \text{if } n = 0 \end{array} \right. . \]

After all the preparations of tools, we prove the first part of the problem, namely,
\[ \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \frac{(p-m-k)}{(p+n)} \frac{H_p + H_m - H_{m-k}}{m+n-k} = \frac{1}{nm} \frac{1}{\binom{n}{n}}. \] (9)

By equation (7), we simplify (9) as follows,
\[ \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \frac{(p-m-k)}{(p+n)} \frac{H_p + H_m - H_{m-k}}{m+n-k} = \frac{1}{n(m+n)} \binom{n-m}{n}. \] (10)

By equation (5), we have
\[ H_m - H_{m-k} = \delta \left[ \binom{m+n}{k} \right] \]
and we rewrite (10).

\[
\sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \frac{(p^n_{m-n-k})}{m+n-k} = 
\sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \frac{\delta \left( \binom{x+m}{k} \right)}{(m+n-k) \binom{n}{k}} = 
\delta \left[ \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \frac{(x+m)_k}{(m+n-k)_k} \frac{1}{m+n-k} \right].
\]

Let

\[
d_k = (-1)^{k-1} \frac{(m+n)_k \binom{n}{k} \binom{x+m}{k}}{(m+n-k)_k} \frac{1}{m+n-k}
\]

Then \(d_0 = 1\) and

\[
d_{k+1} = \frac{(k+1)_k \binom{n}{k} \binom{x+m}{k}}{(m+n)_{k+1} \binom{n}{k+1}} \frac{1}{m+n-k}
\]

\[
= \frac{(k+1)_k \binom{n}{k} \binom{x+m}{k}}{(m+n)_{k+1} \binom{n}{k+1}} \frac{1}{m+n-k}
\]

\[
= \frac{1}{(m+n-k)_k} \frac{1}{(m+n)_k}
\]

This proves equation (10).

Now we prove the second part of the problem,

\[
\sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \binom{q-k}{m-n-k} \frac{H_{q-k}+H_m-H_{m-k}}{m+n-k} = \frac{1}{mn} \binom{n}{q}. \tag{11}
\]

It follows from (8) that we can simplify

\[
\sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \binom{q-k}{m-n-k} \frac{H_{q-k}+H_m-H_{m-k}}{m+n-k}
\]

\[
= \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \binom{q-k}{m-n-k} \frac{H_{q-k}-H_{m-k}}{m+n-k}, \quad q \geq m+n.
\]
By equation (5), we have

\[ H_{q-k} - H_{m-k} = \frac{\delta \left[ \frac{(x+q-k)_{q-m}}{q-m} \right]}{q-m} \]

\[ \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \frac{(q-k)_{m-k}}{(m+n-k)} \frac{H_{q-k} - H_{m-k}}{m+n-k} \]

\[ = \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \frac{(q-k)_{m-k}}{(m+n-k)} \frac{(x+q-k)_{q-m}}{q-m} (m+n-k) \]

\[ = \delta \left[ \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \frac{(q-k)_{m-k}}{(m+n-k)} \frac{(x+q-k)_{q-m}}{q-m} (m+n-k) \right] \]

Let

\[ e_k = (-1)^{k-1} \frac{\binom{n}{m+k}}{(m+n)} \frac{(q-k)_{m-k}}{(q-m)_{q-m}} \frac{(x+q)_{q-m}}{q-m} (m+n-k) \]

Then \( c_0 = 1 \) and

\[ \frac{e_{k+1}}{e_k} = - \frac{\binom{n}{k} \binom{q-k}{m+k} \binom{x+q}{q-m}}{\binom{q-k}{m+n-k}} \frac{H_{q-k} - H_{m-k}}{m+n-k} \]

\[ = \frac{1}{k+1} \frac{(k-n) (k-q) (k-(m+x))}{(k-(m+n-1)) (k-(q+x))} \]

\[ \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \frac{(q-k)_{m-k}}{(m+n-k)} \frac{H_{q-k} - H_{m-k}}{m+n-k} \]

\[ = -\delta \left[ \frac{\binom{q}{m+n}}{(q-m)} \frac{(q-k)_{q-m}}{q-m} (m+n) 3F2 \left[ -n, -q, -(m+x) \left| -m+n-1, -(q+x) \right. \right] \right] \]

Applying Saalschütz Theorem (4) we obtain,

\[ 3F2 \left[ -n, -q, -(m+x) \left| -m+n-1, -(q+x) \right. \right] \]

\[ = \frac{(- (m+n-1) + q)_{n} (- (m+n-1) + (m+x))_{n}}{(- (m+n-1))_{n} (- (m+n-1) + q + m + x)_{n}} \]

\[ = \frac{(- m-n+1+q)_{n} (- n+1+x)_{n}}{(- m-n+1)_{n} (- n+1+q+x)_{n}} \]

\[ \sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \frac{(q-k)_{m-k}}{(m+n-k)} \frac{H_{q-k} - H_{m-k}}{m+n-k} \]

\[ = - \frac{\binom{q}{m+n}}{(q-m)} (m+n) \delta \left[ \frac{H_{x+q} (m+n)}{(-m+n+1+q)_{n} (-n+1+x)_{n}} \right] \]

But

\[ (-n+1+x)_{n} = (-n+1+x) \cdots (-1+x) x, \]
therefore using (6),

\[
\sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \frac{q}{m+k-k} \frac{H_{q-k} - H_{m+k-k}}{m+n-k} \\
= - \frac{\binom{q}{m}}{(m+n)} \frac{q}{m+n} \frac{(-m-n+1+q)_n (-1)^{n-1} (n-1)!}{(-m-n+1)_n (-n+1+q)_n} \\
= - \frac{\binom{q}{m}}{(m+n)} \frac{(-m-n+1+q)_n (-1)^{n-1} (n-1)!}{(-m-n+1)_n (-n+1+q)_n} \\
= - \frac{\binom{q}{m}}{(m+n)} \frac{q! (q-m) (-1)^{n-1} (n-1)!}{(-1)^n n! (m+n+1)!} \\
= - \frac{\binom{q}{m}}{(m+n)} \frac{(q-m)}{n} \\
= - \frac{\binom{q}{m}}{mn} \frac{(m+n)(n)}{n} = \frac{1}{mn}.
\]

by setting \( q = p + n \), we finally obtain

\[
\sum_{k=0}^{n} (-1)^{k-1} \binom{n}{k} \frac{p+n-k}{m+n-k} \frac{H_{p+n-k} - H_{m-k}}{m+n-k} \\
= \frac{1}{mn}.
\]

REFERENCES
[1]: Mortenson Eric, "On Differentiation and Harmonic Numbers".
[2]: Wenchang Chu, Livia De Donno, "Hypergeometric Series and Harmonic Number Identities".

Also solved by the proposers.

114. Proposed by D.M. Bătinețu-Giurgiu, “Matei Basarab” National College, Bucharest, Romania, and Neculai Stanciu, “George Emil Palade” School, Buzău, Romania. For \( m \) a positive integer compute

\[
\lim_{n \to \infty} \left( n \left( me - \sum_{k=1}^{m} \left( 1 + \frac{1}{n} \right)^{n+k} \right) \right).
\]

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria.

Let

\[
I_n = n \left( me - \sum_{k=1}^{m} \left( 1 + \frac{1}{n} \right)^{n+k} \right)
\]
Clearly we have
\[ I_n = n \left( me - \left( 1 + \frac{1}{n} \right)^n \left( n \left( 1 + \frac{1}{n} \right)^{m+1} - n - 1 \right) \right) \]
\[ = n \left( me - e \left( 1 - \frac{1}{2n} + O \left( \frac{1}{n^2} \right) \right) \left( n \left( 1 + \frac{m+1}{n} + \frac{(m+1)n}{2n^2} \right) - n - 1 + O \left( \frac{1}{n^2} \right) \right) \right) \]
\[ = men \left( 1 - \left( 1 - \frac{1}{2n} + O \left( \frac{1}{n^2} \right) \right) \left( 1 + \frac{m+1}{2n} + O \left( \frac{1}{n^2} \right) \right) \right) \]
\[ = men \left( -\frac{m}{2n} + O \left( \frac{1}{n^2} \right) \right) = -\frac{m^2 e}{2} + O \left( \frac{1}{n} \right) \]

Thus
\[ \lim_{n \to \infty} I_n = -\frac{m^2 e}{2} \]

**Solution 2 by Haroun Meghaichi, University of Science and Technology, Houari Boumediene, Algeria.**

In this solution we use \( e^x = 1 + x + o(x) \), and \( \ln(1 + x) = x - \frac{x^2}{2} + o(x^2) \). We have
\[ (n+k) \ln \left( 1 + \frac{1}{n} \right) = (n+k) \left( \frac{1}{n} - \frac{1}{2n^2} + o \left( \frac{1}{n^2} \right) \right) \]
\[ = 1 + \frac{2k - 1}{2n} + o \left( \frac{1}{n} \right) \],
we take the exponential, to get
\[ \left( 1 + \frac{1}{n} \right)^{n+k} = e \left( 1 + \frac{2k - 1}{2n} + o \left( \frac{1}{n} \right) \right) \],
we sum from \( k = 1 \) to \( m \) to get
\[ \sum_{k=1}^{m} \left( 1 + \frac{1}{n} \right)^{n+k} = me + \frac{m^2 e}{2n} + o \left( \frac{1}{n} \right) \].

Now, the answer comes directly
\[ n \left( me - \sum_{k=1}^{m} \left( 1 + \frac{1}{n} \right)^{n+k} \right) = -\frac{m^2 e}{2} + o(1) \xrightarrow{n \to +\infty} -\frac{m^2 e}{2} \]
For any positive integer \( m \).

**Solution 3 by Moti Levy, Rehovot, Israel.**

\[ \sum_{k=1}^{m} \left( 1 + \frac{1}{n} \right)^{n+k} = \left( 1 + \frac{1}{n} \right)^{n+1} + \left( 1 + \frac{1}{n} \right)^{n+2} + \cdots + \left( 1 + \frac{1}{n} \right)^{n+m} \]
\[ = \left( 1 + \frac{1}{n} \right)^{n+1} \left( n \left( 1 + \frac{1}{n} \right)^{m} - n \right) .\]
By l’Hôpital’s rule
\[
\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n+1} \left(n \left(1 + \frac{1}{n}\right)^m - n\right) = \left(\lim_{n \to \infty} \left(1 + \frac{1}{n}\right)^{n+1}\right) \left(\lim_{n \to \infty} \left(\frac{1 + \frac{1}{n}}{\frac{1}{n}}\right)^m - 1\right)
\]
\[
= e \left(\lim_{n \to \infty} \left(m \left(1 + \frac{1}{n}\right)^{m-1} - \frac{1}{n}\right)\right) = me
\]
Again by l’Hôpital’s rule
\[
\lim_{n \to \infty} \left(n \left(me - \sum_{k=1}^{m} \left(1 + \frac{1}{n}\right)^{n+k}\right)\right)
\]
\[
= \lim_{n \to \infty} \left(n \left(me - \left(1 + \frac{1}{n}\right)^{n+1} \left(n \left(1 + \frac{1}{n}\right)^m - n\right)\right)\right)
\]
\[
= \lim_{n \to \infty} \left(n \left(me - \sum_{k=1}^{m} \left(1 + \frac{1}{n}\right)^{n+k} + e - \left(1 + \frac{1}{n}\right)^{n+m}\right)\right)
\]
\[
= \lim_{n \to \infty} \left(n \left(me - \sum_{k=1}^{m} \left(1 + \frac{1}{n}\right)^{n+k}\right)\right) + \lim_{n \to \infty} \left(n \left(e - \left(1 + \frac{1}{n}\right)^{n+m}\right)\right)
\]
\[
= \lim_{n \to \infty} \left(n \left(me - \left(1 + \frac{1}{n}\right)^{n+1} \left(n \left(1 + \frac{1}{n}\right)^m - n\right)\right)\right) = \frac{1}{2} m^2 e
\]

Also solved by Zeraoulia Rafik, Batna, Algeria (student); Michel Bataille, Rouen, France; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Moubinool Omarjee, Lycée Henri IV, Paris, France; Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania and the proposer.

115. Proposed by Florin Stanescu, Serban Cioculescu school, city Gaesti, jud. Dambovita, Romania. Determine the matrices \(X_1, X_2, \ldots, X_9 \in M_2(\mathbb{Z})\) such that
\[
X_1^4 + X_2^4 + \cdots + X_9^4 = X_1^2 + X_2^2 + \cdots + X_9^2 + 18 \cdot I_2
\]
and \(\det X_k = 1\), for \(k = 1, 2, \ldots, 9\).

**Solution 1 by Michel Bataille, Rouen, France.**

Let \(S\) be the set of all matrices of \(M_2(\mathbb{Z})\) with determinant 1 and trace 0 that is,
\[
S = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} : a, b, c \in \mathbb{Z}, \ bc = -a^2 - 1 \right\}.
\]
We show that the matrices \(X_1, \ldots, X_9\) satisfy the required conditions if and only if \(X_1, \ldots, X_9\) are elements of \(S\).

(a) First, suppose that \(X_1, \ldots, X_9 \in S\) and set \(X_k = \begin{pmatrix} a_k & b_k \\ c_k & -a_k \end{pmatrix}\) (\(k = 1, \ldots, 9\)).

Using \(a_k^2 + b_k c_k = -1\), a simple calculation gives \(X_k^2 = -I_2\) and \(X_k^4 = I_2\). It follows that \(X_1^4 + X_2^4 + \cdots + X_9^4 = 9I_2 = X_1^2 + X_2^2 + \cdots + X_9^2 + 18 \cdot I_2\).
(b) Conversely, suppose that \(X_1, \ldots, X_9\) satisfy the conditions. From Hamilton-Cayley’s theorem, any matrix \(X \in \mathbb{Z}\) with determinant 1 satisfies \(X^2 = tX - I_2\) where \(t = \text{tr}(X)\) denotes the trace of \(X\). Note that \(t \in \mathbb{Z}\). It readily follows that \(X^4 = (t^2 - 2)X^2 + (1 - t^2)I_2\) so that \(\text{tr}(X^2) = t^2 - 2\) and \(\text{tr}(X^4) = t^4 - 4t^2 + 2\). Applying these results to \(X_1, \ldots, X_9\) and taking the traces in the relation \(X_1^4 + X_2^4 + \cdots + X_9^4 = X_1^2 + X_2^2 + \cdots + X_9^2 + 18 \cdot I_2\), we obtain
\[
\sum_{k=1}^{9} (t_k^4 - 4t_k^2 + 2) = 36 + \sum_{k=1}^{9} (t_k^2 - 2) \quad (1)
\]
where \(t_1, \ldots, t_9\) denote the traces of \(X_1, \ldots, X_9\). Now, (1) yields
\[
(t_1^4 - 5t_1^2) + \cdots + (t_9^4 - 5t_9^2) = 0 \quad (2).
\]
We remark that if \(t \in \mathbb{Z}\), then \(t^4 - 5t^2 = 0\) if \(t = 0\) and \(t^4 - 5t^2 = -4\) if \(|t| = 1\) or \(|t| = 2\). In addition, \(t^4 - 5t^2 \geq 36\) if \(|t| \geq 3\) (because the function \(u \mapsto u^4 - 5u^2\) is increasing on \([3, \infty)\)). This said, let \(m\) be the number of terms \((t_k^4 - 5t_k^2)\) in (2) such that \(|t_k| \geq 3\) and \(n\) the number of terms in (2) with \(|t_k| = 1\) or \(|t_k| = 2\). Then, we have \(0 \geq 36m - 4n \) with \(m + n \leq 9\). This demands \(m = 0\) since otherwise \(n \leq 8\), hence \(36m - 4n \geq 36 - 32 = 4\). Therefore \(m = 0\), hence \(0 = -4n\) and \(n = 0\) as well. The only possibility is \(t_k = 0\) for \(k = 1, \ldots, 9\). In conclusion, \(\text{tr}(X_1) = \cdots = \text{tr}(X_9) = 0\) and \(X_1, \ldots, X_9 \in \mathcal{S}\).

**Solution 2 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria.**

Suppose that \(X_1, \ldots, X_9\) are \(2 \times 2\) matrices with integer coefficients satisfying
\[
X_1^4 + X_2^4 + \cdots + X_9^4 = X_1^2 + X_2^2 + \cdots + X_9^2 + 18 \cdot I_2 \quad (1)
\]
and \(\det X_k = 1\) for \(k = 1, 2, \ldots, 9\). for \(k = 1, 2, \ldots, 9\), let \(t_k = \text{tr}X_k\). Since \(\lambda^2 - t_k \lambda + 1\) is the characteristic polynomial of \(X_k\) we conclude that, \(X_k^2 = t_k X_k - I_2\) so
\[
\text{tr}(X_k^2) = t_k^2 - 2 \quad (2)
\]
and since \(\det X_k^2 = 1\) we conclude also that \(\lambda^2 - (t_k^2 - 2) \lambda + 1\) is the characteristic polynomial of \(X_k^4\). In particular \(X_k^4 = (t_k^2 - 2)X_k^2 - I_2\) and
\[
\text{tr}(X_k^4) = (t_k^2 - 2)^2 - 2 = t_k^4 - 4t_k^2 + 2 \quad (3)
\]
So taking the trace of both sides of (1) we obtain
\[
\sum_{k=1}^{9} t_k^4 = 5 \sum_{k=1}^{9} t_k^2 \quad (4)
\]
By the Cauchy-Schwarz inequality we have
\[
\left(\sum_{k=1}^{9} t_k^2\right)^2 \leq 9 \sum_{k=1}^{9} t_k^4 = 45 \sum_{k=1}^{9} t_k^2.
\]
Thus \(\sum_{k=1}^{9} t_k^4 = 5 \sum_{k=1}^{9} t_k^2 \leq 225\), and consequently \(|t_k| \leq 3\) for \(k \in \{1, 2, \ldots, 9\}\).
Now, let \(n = \text{card}\{k \in \{1, \ldots, 9\} : t_k \neq 0\}\). It is easy to see that if \(x \neq 0\) (mod 5) then \(x^4 = 1\) (mod 5). So, modulo 5, equation (4) implies that \(n = 0\) (mod 5), and consequently \(n = 0\) or \(n = 5\). Assuming \(n = 5\) we may, without loss of generality, suppose that \(t_6 = t_7 = t_8 = t_9 = 0\), and \(|t_k| \geq 1\) for \(1 \leq k \leq 5\). Repeating
the previous argument with the Cauchy-Schwarz inequality we see that, in fact, 
\[ \sum_{k=1}^{5} t_k^2 \leq 125 \] 
and consequently there is at the most one \( t_k \) such that \( |t_k| = 3 \). But if \( |t_k| \leq 2 \) for every \( k \) then we see immediately that

\[ 5 \sum_{k=1}^{5} t_k^2 > 4 \sum_{k=1}^{5} t_k^2 \geq \sum_{k=1}^{5} t_k^4 \]

which is a contradiction. So, there is exactly one \( t_k \) such that \( |t_k| = 3 \), and we may suppose that \( |t_5| = 3 \). Now, if \( s = \text{card} \left( \{ k \in \{1, \ldots, 9 \} : |t_k| = 1 \} \right) \) then the equality \( \sum_{k=1}^{5} t_k^2 = 5 \sum_{k=1}^{5} t_k^2 \) becomes \( 81 + 16(4-s) + s = 5(9+4(4-s)+s) \) which is absurd. This contradiction came from the assumption that \( n = 5 \). So, we must have \( n = 0 \) and consequently \( t_k = 0 \) for \( 1 \leq k \leq 9 \). This implies that \( X_k^2 = -I_2 \) for \( 1 \leq k \leq 9 \).

Conversely, if \( X_k^2 = -I_2 \) for \( 1 \leq k \leq 9 \), then clearly (1) is satisfied. Finally, Note that

\[ \{ X \in M_2(\mathbb{Z}) : X^2 = -I \} = \left\{ \begin{bmatrix} a & -b \\ c & -a \end{bmatrix} : (a, b, c) \in \mathbb{Z}^3, bc = 1 + a^2 \right\} \]

So, any nine matrices from the previous set yield a solution to the proposed problem.

**Remark:** It was brought to our attention that this problem had previously appeared on: Gazeta Matematica, Seria A, Vol XXXI, No 368. 
Also solved Nicusor Zlota, Traian Vuia Technical College, Focsani, Romania; Michel Bataille, Rouen, France; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Moubinool Omarjee, Lycée Henri IV, Paris, France and the proposer.

116. **Proposed by Mohammed Aassila, Strasbourg, France.** A positive integer \( N \) is called deficient if the sum of its divisors is less than \( 2N \):

\[ \sigma(N) < 2N. \]

It is called abundant if \( \sigma(N) > 2N. \) Prove that there exists an integer \( k \in \mathbb{N} \setminus \{0\} \) such that the numbers

\[ k+1, k+2, k+3, \ldots, k+2015 \]

are all abundant. Prove that there exist infinitely integers \( k \in \mathbb{N} \setminus \{0\} \) such that

\[ k+1, k+2, k+3, k+4 \quad \text{and} \quad k+5 \]

are all deficient.

**Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.**

Recall that, if the prime factorization of an integer \( n \geq 2 \) is \( n = \prod_{j=1}^{\ell} q_j \) (where the primes \( q_1, \ldots, q_\ell \) are not necessarily distinct) then \( \sigma(n) = \prod_{j=1}^{\ell} (1 + q_j) \). In particular, \( \sigma \) is multiplicative and \( 1 < \frac{\sigma(n)}{n} \) for \( n \geq 2 \). It follows that every multiple of an abundant number is abundant.

**Lemma.** There exists a sequence \( (a_n)_{n \geq 1} \) of pairwise co-prime abundant numbers.
Proof. Let us denote the increasing sequence of primes by \((p_n)_{n \geq 1}\). It is well-known that \(\prod_{n=1}^{\infty} \left(1 + \frac{1}{p_n}\right) = +\infty\). So, we may define \(n_1 = 0\) and if \(n_1, \ldots, n_k\) are defined we define \(n_{k+1}\) by

\[ n_{k+1} = \min \left\{ k : \prod_{n=n_k+1}^{k} \left(1 + \frac{1}{p_n}\right) > 2 \right\} \]

Now, if \(a_k = \prod_{n=n_k+1}^{n_{k+1}} p_n\) we see immediately that the terms of the sequence \((a_k)_{k \geq 1}\) are pairwise co-prime. Moreover, for \(k \geq 1\),

\[ \sigma(a_k) = \prod_{n=n_k+1}^{n_{k+1}} \left(1 + \frac{1}{p_n}\right) > 2. \]

So the \(a_k\)'s are all abundant. □

Now, consider a positive integer \(\ell > 1\). Using the Chinese Remainder Theorem there is an integer \(k\) which is a solution of the simultaneous congruences \(k \equiv -n \pmod{a_n}\) for \(n = 1, 2, \ldots, \ell\). It follows that the numbers

\[ k + 1, k + 2, \ldots, k + \ell \]

are all abundant because each one of them is a multiple of an abundant number. This proves the first point by taking \(\ell = 2015\).

For the second point we recall that by the Prime Number Theorem, if \(p_n\) is the \(n\)th prime number then \(\lim_{n \to \infty} \frac{\ln\ln n}{n} = 1\). On the other hand it is straightforward to see that \(\lim_{n \to \infty} \left(1 + \frac{2}{n\ln n}\right)^n = 1\). It follows that the set

\[ N = \left\{ n \geq 60 : p_n > \frac{n\ln n}{2}, \left(1 + \frac{2}{n\ln n}\right)^n < \frac{9}{8} \right\} \]

is infinite (\(\text{card}(N) = +\infty\)).

Now, consider \(n \in N\), and define \(k_n = 60p_1p_2 \cdots p_{n-1}\). We will prove that \(k_n + 1, k_n + 2, k_n + 3, k_n + 4, k_n + 5\) are all deficient. Indeed, let \(q_i = 1 + k_n/i\) for \(i = 1, 2, 3, 4, 5\). If \(t\) is a prime factor of \(q_i\) then clearly \(\gcd(t, p_1p_2 \cdots p_{n-1}) = 1\) and consequently \(t \geq p_n\). Moreover, if the prime factorization of \(q_i\) is \(q_i = t_1t_2 \cdots t_m\) then

\[ (p_n)^m \leq q_i \leq 60p_1 \cdots p_{n-1} < p_n^m \]

because \(p_n > 60\). It follows that \(m \leq n\). Now,

\[ \frac{\sigma(q_i)}{q_i} = \prod_{j=1}^{m} \left(1 + \frac{1}{t_j}\right) \leq \left(1 + \frac{1}{p_n}\right)^m \leq \left(1 + \frac{1}{p_n}\right)^n \leq \left(1 + \frac{2}{n\ln n}\right)^n < \frac{9}{8} \]

So,

\[ \frac{\sigma(k_n + i)}{k_n + i} = \frac{\sigma(i)}{i} \cdot \frac{\sigma(q_i)}{q_i} < \frac{9}{8} \max \left\{ \frac{\sigma(i)}{i} : i = 1, \ldots, 5\right\} = \frac{9}{8} \cdot \frac{\sigma(4)}{4} = \frac{63}{32} \leq 2 \]

we conclude that \(k_n + 1, k_n + 2, k_n + 3, k_n + 4, k_n + 5\) are all deficient, which is the desired conclusion.
Remark. Note that this statement is optimal since any sequence of six consecutive integers contains a multiple of 6 and this one cannot be deficient.

Also solved by the proposer.
This section of the Journal offers readers an opportunity to solve interesting and elegant mathematical problems mainly appeared in Math Contest around the world and most appropriate for training Math Olympiads. Proposals are always welcomed. The source of the proposals will appear when the solutions be published.

Proposals

80. Given is the square matrix $A = (a_{k,l})$ of order $n \geq 2$ with elements $a_{k,l} = (k - l)^3$. Find the rank of the matrix.

81. Find the sum
\[ \sum_{n=0}^{\infty} \frac{1}{n! (n^4 + n^2 + 1)}. \]

82. In Descartes co-ordinate system with origin $O$ given are the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and a point $M_0$ on it. If $M$ is a point on the ellipse, compute the maximal area of the triangle $OM_0M$.

83. Let $P$ be the sum of all $2 \times 2$ matrices, whose elements are the integers 0, 1, 2 and 3, without repetition. Find the matrices:
   a) $S = \frac{1}{36}P$;
   b) $S^{2015}$;
   c) $S^{2015} - M^{2015}$, where $M = \begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$.

84. The function $f(x)$ has a derivative in the interval $[0, 2015]$ and $f(0) = f(2015) = 0$. Prove the existence of such numbers $x, y \in (0, 2015)$, that $f'(x) = 2015f(x)$ and $f'(y) = 2015f'(y)$. 
Solutions

75. Let \( f : \mathbb{N} \to \mathbb{N} \) be a function from the positive integers to the positive integers for which \( f(1) = 1, f(2n) = f(n) \) and \( f(2n + 1) = f(n) + f(n + 1) \) for all \( n \in \mathbb{N} \). Prove that for any natural number \( n \), the number of odd natural numbers \( m \) such that \( f(m) = n \) is equal to the number of positive integers not greater than \( n \) having no common prime factors with \( n \).

(BMO Shortlist 2014)

Official Solution

The crucial observation that solves this problem is that the function \( f \) encodes Euclid’s algorithm, when we view the numbers in binary. To make this precise, we will write \( g(n) \) for \( f(n + 1) \) and consider for each integer \( n \) the pair \((f(n), g(n))\). If we let \( x \) be the binary string representing \( n \) then the recurrence relations give us that

\[
(f(x0), g(x0)) = ((f(x), f(x) + g(x))
\]

and

\[
(f(x1), g(x1)) = ((f(x) + g(x), f(x))
\]

Then we calculate the pair \((f(x), g(x))\) as follows: start form the pair \((1, 1) = f(1), g(1)\). Now read the binary digits of \( x \) from left to right, ignoring the initial 1: whenever you see a 0 add the first coordinate to the second, and whenever you see a 1 add the second coordinate to the first. For example to calculate the pair \((f(27), g(27))\) we write 27 = 11011 in binary, which gives us a sequence of pairs

\[
\begin{align*}
(f(1), g(1)) &= (1, 1) \\
(f(11), g(11)) &= (2, 1) \\
(f(110), g(110)) &= (2, 3) \\
(f(1101), g(1101)) &= (5, 3) \\
(f(11011), g(11011)) &= (8, 3)
\end{align*}
\]

Now from this it follows by induction that \((f(n), g(n))\) are coprime positive integers, with \( f(n) \geq g(n) \) if and only if \( n \) is odd. To complete the proof, we just need to show that each pair \((a, b)\) of coprime positive integers arises as \((f(n), g(n))\) for a unique positive integer \( n \). To show the existence of \( n \), imagine running Euclid’s algorithm on the pair \((a, b)\): that is to say, we successively either subtract the first coordinate from the second or the second from the first (depending on which of the two is larger) until we can’t go any further, which is we reach the pair \((1, 1)\). We can record this as a string of consisting of a 0 for each time we subtracted the first form the second, and a 1 for each time we subtracted the second from the first. Reversing this string and prepending a 1 gives the binary expansion of a number \( n \) which by our method of calculating \((f, g)\) has \((f(n), g(n)) = (a, b)\). To get the uniqueness of \( n \) we just have to note that when we run the Euclid’s algorithm to construct \( n \) in the previous paragraph, we had no choices in any stage: there is a unique series of reductions that takes \((a, b)\) to \((1, 1)\) while remaining in positive
Let $n$ be an integer. This series of reductions corresponds to a unique binary string, so the integer $n$ we constructed is unique.

76. Let $a, b, c, p, q, r$ be positive integers such that $a^p + b^q + c^r = a^q + b^r + c^p = a^r + b^p + c^q$. Prove that $a = b = c$ or $p = q = r$.

(BMO Shortlist 2014)

**Official Solution**

The proof is essentially a size argument. We split into 3 cases, the first two of which are quite straightforward.

Case 1: two of $p, q, r$ are equal, wlog $q = r$. Subtract $a^p + b^q + c^r$ from the given equation to show that

$$a^p - a^q = b^q - b^r = c^r - c^q$$

Now if $p > q$ then $x^p - x^q = x^q(x^{p-q} - 1)$ is a strictly increasing function of $x > 0$, so by the only way the equality can hold is if $a = b = c$. If $p < q$ then the same argument shows that $a = b = c$ also. Yet the only remaining subcase is when $p = q = r$.

Case 2: two of $a, b, c$ are equal, wlog $b = c$. Subtract $b^q + b^q + b^r$ from the given equation to show that $a^p - b^q = a^q - b^p = a^r - b^r$. Exactly as in the previous case, the function of positive integer $s$, $a^s - b^s = (a - b)(a^{s-1} + a^{s-2}2b + ... + b^{s-1})$ is strictly increasing, strictly decreasing or $0$ according as $a > b, a < b$ or $a = b$. In the first two subcases this forces $p = q = r$, and in the last subcase $a = b = c$.

Case 3: the $a, b, c$ are distinct, as are the $p, q, r$. Wlog $a$ is the greatest of $a, b, c$ and (cycling the variables if necessary) $p$ is the greatest of $p, q, r$. In particular, $a, p \geq 3$.

We claim that for such $a, p$ we have

$$a^p \geq (a - 1)^p + 2a^{p-1}$$

Indeed, since $(a - 1)^p + 2a^{p-1} \leq a^{p-1}((a - 1)^2 + 2ap^2)$, it suffices to prove the inequality for $p = 3$, when it rearranges to the inequality $a^2 - 3a + 1 \geq 0$. This certainly holds for $a \geq 3$. As a consequence we have the inequality

$$a^p + b^q + c^r \geq (a - 1)^p + 2a^{p-1} \geq b^p + c^q + a^r$$

which is a contradiction.

77. Let $I$ be the incenter of $\triangle ABC$ and let $H_a, H_b$ and $H_c$ be the orthocenters of $\triangle BIC, \triangle CIA$ and $\triangle AIB$, respectively. The line $H_aH_b$ meets $AB$ at $X$ and the line $H_aH_c$ meets $AC$ at $Y$. If the midpoint $T$ of the median $AM$ of $\triangle ABC$ lies on $XY$. Prove that the line $H_aT$ is perpendicular to $BC$.

(BMO Shortlist 2014)

**Solution by Andrea Fanchini Cantù, Italy.**

We use barycentric coordinates and the usual Conway’s notations.

The bisector $IB$ have equation $IB \equiv cx - a \equiv 0$, so its infinite perpendicular point is $IB_{\infty} (a : c - a : -c)$ and line $CIB_{\infty}$ have equation

$$
\begin{vmatrix}
0 & 0 & 1 \\
a & c - a & -c \\
x & y & z
\end{vmatrix}
= 0 \text{ i.e. } CIB_{\infty} \equiv (a - c)x + ay = 0
$$
The bisector $IC$ have equation $IC \equiv -bx + ay = 0$, so its infinite perpendicular point is $IC_{\infty\perp}(-a : b : a - b)$ and line $BIC_{\infty\perp}$ have equation

$$\begin{vmatrix} 0 & 1 & 0 \\ -a & b & a - b \\ x & y & z \end{vmatrix} = 0 \ i.e. \ BIC_{\infty\perp} \equiv (a-b)x + az = 0$$

Therefore $H_a = CIB_{\infty\perp} \cap BIC_{\infty\perp} = (a : c - a : b - a)$. Now cyclically we have that $H_b(c - b : b : a - b)$, $H_c(b - c : a - c : c)$

Line $H_aH_b$ has the equation

$$\begin{vmatrix} a & c - a & b - a \\ c - b & b & a - b \\ x & y & z \end{vmatrix} = 0 \ i.e. \ H_aH_b \equiv (a-b)(s-a)x - (a-b)(s-b)y + c(s-c)z = 0$$

so $X = H_aH_b \cap AB = (s - b : s - a : 0)$. Line $H_aH_c$ has the equation

$$\begin{vmatrix} a & c - a & b - a \\ b - c & a - c & c \\ x & y & z \end{vmatrix} = 0 \ i.e. \ H_aH_c \equiv (a-c)(s-a)x + b(s-b)y - (a-c)(s-c)z = 0$$

so $Y = H_aH_c \cap AC = (s - c : 0 : s - a)$. Therefore line $XY$ has the equation

$$\begin{vmatrix} s - b & s - a & 0 \\ s - c & 0 & s - a \\ x & y & z \end{vmatrix} = 0 \ i.e. \ XY \equiv (s-a)x - (s-b)y - (s-c)z = 0$$

Now, being $A(1,0,0)$ and $M(0,\frac{1}{2},\frac{1}{2})$, the midpoint $T$ is $T(2 : 1 : 1)$. If $T$ lies on $XY$ we have $2(s-a) - (s-b) - (s-c) = 0 \Rightarrow 2a = b + c, \ (\ast)$. Using ($\ast$) the equation of the line $H_aT$ is

$$\begin{vmatrix} a & c - a & b - a \\ 2 & 1 & 1 \\ x & y & z \end{vmatrix} = 0 \ i.e. \ H_aT \equiv 2(c-b)x + (b-3c)y + (3b-c)z = 0$$

and the infinite point of this line is $H_aT_{\infty}(-2(b+c) : 5b - 3c : -3b + 5c)$. The infinite perpendicular point of the side $BC$ is $BC_{\infty\perp}(-a^2 : SC : SB)$, but using the position ($\ast$) we have

$$BC_{\infty\perp}(-2(b+c) : 5b - 3c : -3b + 5c), \ i.e. \ H_aT_{\infty} = BC_{\infty\perp}.$$
78. Let \( n \in \mathbb{N} \), \( n > 2 \) and suppose \( a_1, a_2, ..., a_{2n} \) is a permutation of the numbers \( 1, 2, ..., 2n \) such that \( a_1 < a_3 < ... < a_{2n-1} \) and \( a_2 > ... > a_{2n} \). Prove that

\[
(a_1 - a_2)^2 + (a_3 - a_4)^2 + ... + (a_{2n-1} - a_{2n})^2 > n^3.
\]

(MO Shortlist 2014)

**Solution by Arber Igrishta, Vushtrri, Republic of Kosova.**

From Cauchy-Schwarz inequality we have: \((a_1 - a_2)^2 + (a_3 - a_4)^2 + ... + (a_{2n-1} - a_{2n})^2\)\(\left(\frac{1}{n} + \frac{1}{n} + ... + \frac{1}{n}\right)\) \(\geq (|a_1 - a_2| + |a_3 - a_4| + ... + |a_{2n-1} - a_{2n}|)^2\).

Now, from Proizvolovs Identity, we have: \(|a_1 - a_2| + |a_3 - a_4| + ... + |a_{2n-1} - a_{2n}| = n^2\)

So, \((a_1 - a_2)^2 + (a_3 - a_4)^2 + ... + (a_{2n-1} - a_{2n})^2\) \(\geq n^3\) and so \((a_1 - a_2)^2 + (a_3 - a_4)^2 + ... + (a_{2n-1} - a_{2n})^2\) \(\geq n^3\), by checking the condition of equality of Cauchy-Schwarz inequality, easily we see that equality never occurs. And we are done.

79. Let \( M = \{1, 2, ..., 2013\} \) and let \( \Gamma \) be a circle. For every nonempty subset \( B \) of the set \( M \), denote by \( S(A) \) sum of elements of the set \( A \), and define \( S(\emptyset) = 0 \) ( \( \emptyset \) is the empty set ). Is it possible to join every subset \( A \) of \( M \) with some point \( A \) on the circle \( \Gamma \) so that following conditions are fulfilled:

1. Different subsets are joined with different points;
2. All joined points are vertices of a regular polygon;
3. If \( A_1, A_2, ..., A_k \) are some of the joined points, \( k > 2 \), such that \( A_1, A_2...A_k \) is a regular \( k-gon \), then 2014 divides \( S(A_1) + S(A_2) + ... + S(A_k) \)?

(BMO Shortlist 2014)

**Official Solution**

We will prove that this is possible. Total number of subsets of the set \( M \) is \( 2^{2013} \). On circle \( \Gamma \) we arbitrary choose \( 2^{2013} \) points which are vertices of a regular \( 2^{2013} \)-gon. We join subsets of the set \( M \) and chosen points in the following manner: if we join subset \( A \) with some point in \( \Gamma \), we join subset \( A^c = M \setminus A \) with a point symmetric to the point joint with \( A \) with respect to the center of \( \Gamma \) (number \( 2^{2013} \) is even, so this is possible). If \( A_1, A_2, ..., A_k \) are some of the points joined with subsets, which are vertices of a regular \( k-gon \), then it follows that \( k \mid 2^{2013} \), so \( k \) is a number divisible by 4, say \( k = 4t \).

That is why all points \( A_1, A_2, ..., A_k \) can be divided in \( \frac{k}{4} = 2t \) pairs of symmetric points with respect to center of \( \Gamma \). Using that

\[ S(A_1) + S(A_2) + ... + 2013 = 1007 \times 2013 \]

we get \( S(A_1) + S(A_2) + ... + S(A_k) = 2t \times 1007 \times 2013 = 2014 \times 2013 \times t \), so all conditions are fulfilled.
The definite integral $\int_0^1 x^n \psi(x) \, dx$ and the generalized Glashier-Kinkelin constants

Moti Levy

Abstract. The moments of the Log Gamma function over the interval $[0, 1]$ are expressed by the generalized Glashier-Kinkelin constants and a recurrence formula for Bernoulli’s numbers is derived.

1. Introduction

In Problem 103 in Mathproblems Journal, Volume 4, Issue 3 (2014), Ovidiu Furdui, the proposer, asked for a closed form of $\int_0^1 x^n \psi(x) \, dx = \int_0^1 x^n d(\ln \Gamma(x))$ for $n \geq 3$. A closed form of this definite integral was conjectured by the AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia in the Mathproblems Journal Volume 4, Issue 4 (2014).

$$\int_0^1 x^n \psi(x) \, dx = -\ln \sqrt{2\pi} + \sum_{k=1}^{n-1} (-1)^{k+1} \binom{n}{k} \ln A_k,$$

where $\psi$ denotes the Digamma function and $A_k$ are the generalized Glashier–Kinkelin constants.

2. Evaluation of the moments

Actually, the conjecture is a corollary of results in [1]. Professor Victor S. Adamchik evaluated the integral, $\int_0^z x^n \psi(x) \, dx$ in [1]. Proposition 3 in [1] states that

$$\int_0^z x^n \psi(x) \, dx = \left(-1\right)^{n-1} \zeta'(-n) + \left(-\frac{1}{n+1}\right) B_{n+1} H_n - \sum_{k=0}^{n} \left(-1\right)^k \binom{n}{k} \frac{1}{k+1} B_{k+1}(z) H_k$$

$$+ \sum_{k=0}^{n} \left(-1\right)^k \binom{n}{k} \frac{z^{n-k}}{k+1} \zeta'(-k, z)$$

(2)

By substituting $z = 1$ and noting that $B_n(1)$ are the Bernoulli numbers $B_n$, and that $\zeta'(-k, 1) = \zeta'(-k),$

$$\int_0^1 x^n \psi(x) \, dx = \sum_{k=0}^{n-1} \left(-1\right)^k \binom{n}{k} \left( \zeta'(-k) - \frac{1}{k+1} B_{k+1} H_k \right)$$

$$= \zeta'(0) + \sum_{k=1}^{n-1} (-1)^{k+1} \binom{n}{k} \left( \frac{1}{k+1} B_{k+1} H_k - \zeta'(-k) \right).$$

(3)
Professor V. S. Adamchik also showed that the generalized Glaisher–Kinkelin constants satisfy the following formula,

\[ \ln A_k = \frac{1}{k+1} B_{k+1} H_k - \zeta' (-k), \quad (4) \]

where \( B_n \) are Bernoulli numbers and \( H_n \) are harmonic numbers (see Proposition 4 in [1]).

\[ \zeta' (0) = -\ln \sqrt{2\pi}. \quad (5) \]

The proof of (1) follows when equations (4) and (5) are plugged into equation (3).

3. Recurrence formula for Bernoulli’s numbers

There is another expression of (1) (see solution to Problem 103 in the Mathproblems Journal Volume 4, Issue 4 (2014),

\[ \int_0^1 x^n \psi (x) \, dx = -\frac{1}{n} + \frac{H_n}{n+1} + \sum_{k=0}^{n-1} (-1)^k \binom{n}{k} \zeta' (-k) - \sum_{k=0}^{n-1} \frac{(-1)^k}{n-k} \zeta (-k). \quad (6) \]

We repeat here briefly, for the convenience of the reader, the derivation of (6). The following expression for the Digamma function is well known:

\[ \psi (x) = -\gamma - \frac{1}{x} + \sum_{k=1}^{\infty} \frac{x}{k (x+k)}. \]

\[ \int_0^1 x^n \psi (x) \, dx = -\gamma \int_0^1 x^n \, dx - \int_0^1 x^n \psi (x) \, dx + \sum_{k=1}^{\infty} \int_0^1 \frac{x^{n+1}}{k (x+k)} \, dx. \]

After interchanging the order of summation and integration,

\[ \int_0^1 x^n \psi (x) \, dx = -\gamma \int_0^1 x^n \, dx - \int_0^1 x^{n-1} \, dx + \sum_{k=1}^{\infty} \int_0^1 \frac{x^{n+1}}{k (x+k)} \, dx \]

\[ = -\frac{1}{n+1} \gamma - \frac{1}{n} + \sum_{k=1}^{\infty} \int_0^1 \left( \sum_{j=0}^{n} (-1)^j k^{-1} x^{n-j} + (-1)^{n+1} \frac{k^n}{k + x} \right) \, dx. \]

Now we change again the order of integration and summation,

\[ \int_0^1 x^n \psi (x) \, dx \]

\[ = -\frac{1}{n+1} \gamma - \frac{1}{n} + \sum_{k=1}^{\infty} \int_0^1 \left( \sum_{j=0}^{n} (-1)^j k^{-1} x^{n-j} + (-1)^{n+1} k^n \ln \left( 1 + \frac{1}{k} \right) \right) \]

\[ = -\frac{1}{n+1} \gamma - \frac{1}{n} + \sum_{m=2}^{\infty} (-1)^m \frac{1}{m + n} \zeta (m) \]

The definite integral (6) is expressed by an alternating sum related to the Zeta function.

Using a closed form of \( \sum_{m=2}^{\infty} \frac{(-1)^m}{m + n} \zeta (m) \), which can be found in [2], we obtain:

\[ \int_0^1 x^n \psi (x) \, dx = -\frac{1}{n} + \frac{H_n}{n+1} + \sum_{l=0}^{n-1} (-1)^l \binom{n}{l} \zeta' (-l) - \sum_{l=0}^{n-1} \frac{(-1)^l}{n-l} \zeta (-l). \quad (7) \]
It is well known that
\[ \zeta(-n) = -\frac{B_{n+1}}{n+1}, \quad B_1 = \frac{1}{2}, \] (8)
and from (4),
\[ \zeta'(-k) = \frac{B_{k+1}}{k+1} H_k - \ln A_k. \] (9)

By substituting (8) and (9) in (7), we get
\[ -\frac{1}{n} + \frac{H_n}{n+1} + \sum_{k=0}^{n-1} (-1)^k \frac{B_{k+1}}{k+1} \left( \binom{n}{k} H_k + \frac{1}{n} \right) = 0. \] (10)

After some manipulations of (10) we arrived at a recurrence formula for the Bernoulli numbers:
\[ (-1)^n B_n H_n = -\frac{1}{n} + \frac{H_n}{n+1} - \sum_{k=1}^{n-1} (-1)^k \frac{B_k}{k} \left( \binom{n}{k-1} H_{k-1} + \frac{1}{n-k+1} \right), \quad B_1 = \frac{1}{2}, \]
or
\[ (-1)^n B_n H_n = -\frac{1}{2n} + \frac{H_n}{n+1} - \sum_{k=2}^{n-1} (-1)^k \frac{B_k}{k} \left( \binom{n}{k-1} H_{k-1} + \frac{1}{n-k+1} \right), \quad n \geq 2. \]

References

Moti Levy:
2 Hachita st., Rehovot, Israel.
E-mail: mordechailevy@gmail.com
Solutions to the problems stated in this issue should arrive before
January 10, 2016

Proposals

41. Proposed by D.M. Bătinețu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania. Let \( a, b, c \) be positive real numbers. Show that
\[
a + b + c + 3\sqrt[3]{abc} \geq 2(\sqrt{ab} + \sqrt{bc} + \sqrt{ca}).
\]

42. Proposed by Francisco Javier García Capitán, I.E.S. Álvarez Cubero de Priego de Córdoba, Spain. Let \( ABC \) be a triangle and \( P \) a point inside the internal part of \( ABC \). Let \( XYZ \) be the cevian triangle of \( P \). Find points \( U, V, W \), on the lines \( YZ \), \( ZX \), \( XY \) such that the lines \( UV \), \( VW \), \( WU \) pass respectively through the points \( A, B, C \).

43. Proposed by Paolo Perfetti, "Tor Vergata", Roma, Italy. In a multiple-choice-test you are asked to answer four questions. Question \( i \in \{1, 2, 3, 4\} \) has \( i+1 \) possible answers and each question has only one correct answer. Answering randomly, what is the probability of giving at least two correct answers?

44. Proposed by Armend Sh. Shabani, University of Prishtina, Department of Mathematics, Republic of Kosova. Let \( m \) be an odd integer greater than 3. If \( 2^{2^m} + 1 \) is prime, he can not be expressed as a difference of \( m \)-th powers of two positive integers.

45. Proposed by Marcel Chiriță, Bucharest, Romania. Solve in real numbers the following system:
\[
\begin{align*}
2^{2x^2-1} + 2y^2-2 &= 12 \\
3^{2x^2-1} + 3y^2-2 &= 36.
\end{align*}
\]
Solutions

36. **Proposed by D.M. Bătinețu-Giurgiu, Matei Basarab National College, Bucharest, Romania and Neculai Stanciu, George Emil Palade General School, Buzău, Romania.** Prove that if \( p \) is a prime number \( (p > 3) \), then the number \( p^2 + 2015 \) is multiple of 24.

**Solution by Arkady Alt, San Jose, California, USA.** First note that \( p^2 - 1 \) divisible by 8 for any odd \( p \). Indeed, for \( p = 2k + 1 \) we have \( p^2 - 1 = 4k(k + 1) \) where \( k(k + 1) \) is even for any integer \( k \). Since \( p > 3 \) is prime then \( p \) isn’t divisible by 3 and, therefore, \( p \) can be represented in the form \( p = 3k ± 1 \) for some integer \( k \). Hereof, \( p^2 - 1 = 9k^2 ± 6k = 3k(3k ± 2) \). Since 3 and 8 are relatively prime and \( p^2 - 1 \) divisible by 8 and by 3 then \( p^2 - 1 \) divisible by 24. Noting that 2016 is divisible by 24 as well we can conclude that \( p^2 + 2015 = p^2 - 1 + 2016 \) is divisible by 24.


37. **Proposed by Trașcă Iuliana, Olt, Romania.** Let \( x, y, z > 0 \). Prove that
\[
\frac{2x + 2y + 4z}{4x + 4y + 3z} + \frac{2x + 4y + 2z}{4x + 3y + 4z} + \frac{4x + 2y + 2z}{3x + 4y + 4z} \geq \frac{24}{11}
\]

**Solution by Corneliu Mănescu–Avram, Transportation High School, Ploiești, Romania** (expanded by the editor). Simplify by 2 and sum 1 to each fraction al left. We get
\[
\frac{5x + 5y + 5z}{4x + 4y + 3z} + \frac{5y + 5z + 5x}{4y + 4z + 3x} + \frac{5z + 5x + 5y}{4z + 4x + 3y} \geq \frac{12}{11} + 3
\]
that is
\[
5(x + y + z) \left( \frac{1}{4x + 4y + 3z} + \frac{1}{4y + 4z + 3x} + \frac{1}{4z + 4x + 3y} \right) \geq \frac{45}{11}
\]
The AM–HM inequality yields
\[
5(x + y + z) \left( \frac{1}{4x + 4y + 3z} + \frac{1}{4y + 4z + 3x} + \frac{1}{4z + 4x + 3y} \right) \geq \frac{9}{4x + 4y + 3z + 4y + 4z + 3x + 4z + 4x + 3y} = \frac{45}{11}
\]
and the proof is complete.

Also solved by Adnan Ali (two proofs), Student in A.E.C.S-4, Mumbai, India, Alexandru–Andrei Cioc and Daniel Văcaru (jointly), Lucia Ma Li (student) IES Isabel de España, Las Palmas de Gran Canaria, Spain and Ángel Plaza Universidad de Las Palmas de Gran Canaria, Spain (jointly), Henry Ricardo (two proofs), New York Math Circle, New York,
38. Proposed by Stanescu Florin, Serban Cioculescu school, jud. Dambovita, Romania. Prove that in a triangle $ABC$ the following inequalities hold.

$$\frac{81}{4} \frac{rrR}{p} \leq \frac{w_a^2}{a} + \frac{w_b^2}{b} + \frac{w_c^2}{c} \leq \frac{p(p^2 + r^2 - 8rR)}{4rR}$$

where $p$ is the semiperimeter, $r$ is the incircle, $R$ is the excircle, $w_a, w_b, w_c$ are the bisectors.

Solution by Ioan Viorel Codreanu, Satulung, Maramureș, Romania (expanded by the editor) The upper bound. We have

$$w_a = \frac{2\sqrt{bc}}{b + c} \sqrt{p(p - a)} \leq \sqrt{p - a}$$

and similarly $w_b \leq \sqrt{p(p - b)}$, $w_c \leq \sqrt{p(p - c)}$. Using the well known equality

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{p^2 + r^2 + 4rR}{4pR}$$

we get

$$\sum_{cyc} \frac{w_a^2}{a} \leq \sum_{cyc} \frac{p(p - a)}{a} = -3p + \sum_{cyc} \frac{p^2}{a} = \frac{p(p^2 + r^2 - 8rR)}{4rR}$$

The lower bound.

$$w_a^2 = \frac{bc((b + c)^2 - a^2)}{(b + c)^2} = bc - \frac{bca^2}{(b + c)^2}$$

and then

$$\frac{w_a^2}{a} = abc \left( \frac{1}{a^2} - \frac{1}{(b + c)^2} \right)$$

Using the equality $abc = 4prR$ we get

$$\sum_{cyc} \frac{w_a^2}{a} = 4prR \sum_{cyc} \left( \frac{1}{a^2} - \frac{1}{(b + c)^2} \right)$$

and it suffices to prove

$$\sum_{cyc} \frac{1}{a^2} - \sum_{cyc} \frac{1}{(b + c)^2} \geq \frac{81}{16p^2}$$

By the AM–GM inequality we obtain

$$\sum_{cyc} \frac{1}{a^2} = \sum_{cyc} \frac{1}{2} \left( \frac{1}{b^2} + \frac{1}{c^2} \right) \geq \sum_{cyc} \frac{1}{bc} \geq \sum_{cyc} \frac{4}{(b + c)^2}$$

so we get

$$\sum_{cyc} \frac{4}{(b + c)^2} - \sum_{cyc} \frac{1}{(b + c)^2} = \sum_{cyc} \frac{3}{(b + c)^2} \geq \frac{81}{16p^2}$$

Now consider the convex function $f(x) = 1/x$, $x > 0$. Using the Jensen inequality
\[ \sum_{cyc} \frac{1}{3} \left( \frac{3}{(b+c)^2} \right) \geq \frac{3}{3} \left( \frac{1}{(a+b)^2} + \frac{1}{(b+c)^2} + \frac{1}{(c+a)^2} \right) = \frac{27}{16p^2} \]

Also solved by Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Titu Zvonaru, Comănești, Romania and Neculai Stanciu, Buzău, Romania (jointly); Arkady Alt, San Jose, California, USA and the proposer.

39. Proposed by Roberto Tauraso, Dipartimento di Matematica, Università degli studi “Tor Vergata”, Roma, Italy. Find the sum of the third power of the solutions of the equation

\[ \frac{1}{x} + \frac{2}{x-1} + \frac{3}{x-2} + \frac{4}{x-3} = 1 \]

Solution by Neculai Stanciu, Buzău, and Titu Zvonaru, Comănești, Romania (jointly). (expanded by the editor) The given equation is actually

\[ x^4 - 16x^3 + 51x^2 - 46x + 6 = 0 \]

The identity

\[ (x_1 + x_2 + x_3 + x_4)^3 = x_1^3 + x_2^3 + x_3^3 + x_4^3 + 3(x_1 + x_2 + x_3 + x_4)(x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4) + 3(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4) \]

and Vieta’s formulae

\[ \begin{align*}
x_1x_2x_3x_4 &= 16, \\
x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 &= 46 \\
x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 &= 51, \\
x_1 + x_2 + x_3 + x_4 &= 16
\end{align*} \]

yield that

\[ x_1^3 + x_2^3 + x_3^3 + x_4^3 = (16)^3 - 3 \cdot 16 \cdot 51 + 3 \cdot 46 = 1786 \]

and we are done!

Also solved by Nicușor Zlota, “Traian Vuia” Technical College, Focșani, Romania; Ionuț Voicu, Pitești, Romania; Arkady Alt, San Jose, California, USA and the proposer.

40. Proposed by Valmir Krasniqi, Department of Mathematics, University of Prishtina, Republic of Kosova. Find all functions \( f : \mathbb{R} \to \mathbb{R} \), such that \( f(f(x)) + y) = x + f(f(y) - x) \) for all \( x, y \in \mathbb{R} \).

Solution by Henry Ricardo, New York Math Circle, New York, USA

We will show that there is no solution to this functional equation. If we set \( y = -f(x) \) in the given equation, we get

\[ -x + f(0) = f(f(-f(x)) - x). \]
Since the function $-x + f(0)$ takes all possible real values as $x$ ranges over $\mathbb{R}$, we see that the function $f$ is surjective. Thus there exists $a \in \mathbb{R}$ such that $f(a) = 0$. Setting $x = a$ in the given equation yields

$$f(y) = a + f(f(y) - a) \quad \text{for all } y \in \mathbb{R}.$$ 

Now set $z = f(y) - a$. Then $f(z) = z$ for all $z \in \mathbb{R}$ since the function $f(y) - a$ assumes all possible real values.

However, it is easy to verify that the function $f(x) = x$ does not satisfy the original equation.

**Comment:** We note that the similar functional equation $f(f(x) + y) = 2x + f(f(y) - x)$ for all $x, y \in \mathbb{R}$, a shortlisted contribution of the Czech Republic to the 2002 IMO, has the solution $f(x) = x - a$, where $a$ is any real number.

Also solved by Dorlir Ahmeti, Prishtina, Republic of Kosova and the proposer.
Binomial identities with Harmonic Numbers

Solution to Problem 113
MathProblems (4) 4 (2014)

OMRAN KOUBA and ANASTASIOS KOTRONIS

ABSTRACT: We propose to give general identities which are valid for non-integer arguments that generalize the identities proposed by the authors in Problem 113 of issue 4 of volume 4.

We start by formulating another problem, that, as we will see, implies the proposed identities of Problem 113.

The problem.
Let \( n \) be a positive integer. Show that, in \( \mathbb{Q}(X,Y) \); the field of rational fractions in two variables with rational coefficients, the following identity holds:

\[
\sum_{j=0}^{n} (-1)^j \binom{n}{j} \left( \frac{X - j}{n - 1} \right) \left( \sum_{k=0}^{j} \frac{1}{Y + k} \right) = \frac{1}{X + 1 - n} - \binom{X + Y + 1}{n} \frac{1}{(X + 1 - n) \binom{Y + n}{n}} \quad (\ast)
\]

Solution. We will use the next lemmas and corollary:

Lemma 1. Let \( \mathbb{F} \) be a field of characteristic 0. Let \( n \) be a positive integer, and let \( P(X) \in \mathbb{F}[X] \) be a polynomial with \( \deg P < n \), then

\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} P(k) = 0.
\]

Proof. This a well-known and classical result. \( \square \)

Applying this Lemma to the particular case \( P(X) = \binom{Z + n - 1 - X}{n - 1} \in \mathbb{F}[X] \), (with \( \mathbb{F} = \mathbb{K}(Z) \),) which is of degree \( n - 1 \) we obtain the next corollary:
Corollary 2. Let \( K \) be a field of characteristic 0. For a positive integer \( n \), we have the following identity in \( K[Z] \):
\[
\sum_{k=0}^{n} (-1)^k \binom{n}{k} \left( \frac{Z + n - 1 - k}{n - 1} \right) = 0.
\]

Lemma 3. Let \( \mathbb{F} \) be a field of characteristic 0. Let \( n \) be a positive integer, and let \( Q(Y) \in \mathbb{F}[Y] \) be a polynomial with \( \deg Q < n \), then
\[
\frac{Q(Y)}{(Y+1)(Y+2) \cdots (Y+n)} = \frac{1}{(n-1)!} \sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} \frac{Q(-k)}{Y+k}.
\]

Proof. This is just simple fraction decomposition. \( \square \)

Now, let us define
\[
F_n(X, Y) = \sum_{j=0}^{n} (-1)^j \binom{n}{j} \left( \frac{X-j}{n-1} \right) \left( \sum_{k=0}^{j} \frac{1}{Y+k} \right)
\]
and consider, for \( 0 \leq j \leq n \) the following rational fraction from \( K(X) \):
\[
U_{j,n}(X) = (-1)^j \frac{n}{X+1-n} \left( \frac{n-1}{j-1} \right) \left( \frac{n}{n} \right) = \frac{n}{X-1-n} \left( \frac{n-1}{j-1} \right) \left( \frac{n}{n} \right)
\]
Clearly we have
\[
U_{j,n}(X) - U_{j+1,n}(X) = (-1)^{j-1} \binom{n}{j-1} \left( \frac{n-1}{j-1} \right) \left( \frac{n}{n} \right) + (-1)^{j+1} \binom{n}{j+1} \left( \frac{n-1}{j+1} \right) \left( \frac{n}{n} \right) = (-1)^j \binom{n}{j} \left( \frac{n-1}{n-1} \right)
\]
It follows that for \( 0 \leq k \leq n \)
\[
\sum_{j=k}^{n} (-1)^j \binom{n}{j} \left( \frac{n-1}{n-1} \right) = U_{k,n}(X)
\]
Thus, going back to (1), interchanging the order of summation, and using (2) we see that
\[
F_n(X, Y) = \frac{1}{Y+k} \sum_{j=k}^{n} (-1)^j \binom{n}{j} \left( \frac{n-1}{n-1} \right) = \frac{U_{k,n}(X)}{Y+k}.
\]
or, according to the (2),
\[
F_n(X, Y) = \frac{n}{X+1-n} \sum_{k=1}^{n} (-1)^k \binom{n-1}{k-1} \left( \frac{X+1-k}{Y+k} \right)
\]
Applying Lemma 3 to \( Q(Y) = \binom{X+1}{n} - \binom{Y+n-1}{n} \) from \( \mathbb{F}[Y] \), (with \( \mathbb{F} = K(X) \)), which is of degree smaller than \( n \), we see that
\[
\frac{1}{(n-1)!} \sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} \left( \frac{X+1-k}{Y+k} \right) = \frac{\binom{X+1}{n} - \binom{Y+n-1}{n}}{(Y+1)(Y+2) \cdots (Y+n)}
\]
or
\[
\sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} \frac{(X+1-k)}{k-1} \frac{1}{Y+k} + \frac{1}{Y+n} = \frac{(X+Y+1)}{n(Y+n)} - \frac{Y}{n(Y+n)}
\]
and finally
\[
\sum_{k=1}^{n} (-1)^{k-1} \binom{n-1}{k-1} \frac{(X+1-k)}{k-1} \frac{1}{Y+k} = \frac{(X+Y+1)}{n(Y+n)} - \frac{1}{n}
\]
which is equivalent to \[4\] according to \[3\].

**Corollary 4.** Let \(n\) be a positive integer. In \(\mathbb{Q}(X,Y)\); the field of rational fractions in two variables with rational coefficients, the following identity holds:
\[
\sum_{j=0}^{n} (-1)^{j} \binom{n}{j} \frac{X+n-j-1}{n-1} \left( \sum_{k=j}^{n} \frac{1}{Y+n-k} \right) = \frac{1}{X} - \frac{(Y-X+n)}{X(Y+n)}
\]

**Proof.** Indeed, substituting \(X\) by \(-X+n-1\) in \([4]\), and using the fact that \((-a/m) = (-1)^{n} \left( a + m - 1 \right)\),
yields
\[
\sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} \frac{X+j-1}{n-1} \left( \sum_{k=j}^{n} \frac{1}{Y+k} \right) = \frac{1}{X} - \frac{(Y-X+n)}{X(Y+n)}
\]
and the change of summation index \(j \leftarrow n-j\), gives the desired formula \([\dagger]\). \(\square\)

And substituting \(Y\) by \(X-n\) in \([\dagger]\) we obtain also,

**Corollary 5.** Let \(n\) be a positive integer. In \(\mathbb{Q}(X)\); the field of rational fractions in one variable with rational coefficients, the following identity holds:
\[
\sum_{j=0}^{n} (-1)^{j} \binom{n}{j} \frac{X+n-j-1}{n-1} \left( \sum_{k=j}^{n} \frac{1}{X+k} \right) = \frac{1}{X}
\]

**Introducing The Digamma Function.** In what follows we consider the digamma function \(\psi\) defined by \(\psi = \Gamma'/\Gamma\) where \(\Gamma\) is the well-known Eulerian Gamma function. \(\psi\) satisfies the functional equation \(\psi(z+1) - \psi(z) = 1/z\), and it is analytic in \(\mathbb{C} \setminus \mathbb{N}\).

For \(z \in \mathbb{C}\) which is not an integer smaller or equal to \(n\) we have
\[
\psi(z-k+1) - \psi(z-k) = \frac{1}{z-k} \quad \text{for} \quad j \leq k \leq n
\]
hence, using Corollary \[2\] we conclude from Corollary \[5\] that
\[
\sum_{j=0}^{n} (-1)^{j} \binom{n}{j} \frac{z+n-1-j}{n-1} \psi(z-j+1) = \frac{1}{z}.
\]
(6)

Similarly, from Corollary \[4\] and using Corollary \[2\] again, we obtain
\[
\sum_{j=0}^{n} (-1)^{j} \binom{n}{j} \frac{z+n-j-1}{n-1} \psi(w+n-j+1) = \frac{1}{z} - \frac{w-z+n}{z(Y+n)}
\]
for every complex numbers \(z\) and \(w\) for which both sides of the above formula are defined.
Now, combining (6) and (7) we get
\[
\sum_{j=0}^{n} (-1)^{j-1} \binom{n}{j} \left( \frac{z + n - j - 1}{n - 1} \right) (\psi(w + n - j + 1) - \psi(z - j + 1)) = \frac{(w - z + n)}{z(z + n)} \quad (8)
\]
for every complex numbers \(z\) and \(w\) for which both sides of the above formula are defined.

**Extending The Binomial Coefficients.** Now, in order to obtain more Binomial coefficients identities, let us extend the definition of the binomial coefficients \(\binom{a}{b}\) for complex \(a\) and \(b\) by the formula
\[
\binom{a}{b} = \lim_{z \to a} \left( \lim_{w \to b} \frac{z!}{w!(z-w)!} \right), \quad \text{with } t! = \Gamma(t+1). \quad (9)
\]
whenever this is defined, (note that the order of the limits is important.)

Before we proceed, let us give this definition some thought. The limits in this definition help us to handle the cases where \(a\), \(b\) or \(a-b\) are singularities of the function \(t \mapsto t!\), these are the elements of \(Z^* = \{-1, -2, \ldots\}\).

- If \(b \in Z^*\) the inner limit is 0, and we do not need to be concerned about the value of \(a\). So, (9) yields \(\binom{a}{b} = 0\) in this case.
- If \(b \in \mathbb{N}\), then clearly the considered limit gives the standard definition:
  \[
  \binom{a}{b} = \frac{1}{b!} \prod_{0 \leq k < b} (a-k)
  \]
- If \(b \notin \mathbb{Z}\) and \(a \notin Z^*\), then the inner limit in (9) is trivial, and the outer limit yields either \(\frac{a!}{b!(a-b)!}\) if \(a-b\) is not a negative integer, or \(\binom{a}{a-b}\) if \(a-b\) is an integer.
- Finally, \(\binom{a}{b}\) is not defined if \(b \notin \mathbb{Z}\) and \(a \in Z^*\).

Now, let us come back to the situation that we are considering. Firs, let us assume that none of the complex numbers \(w\), \(z\) or \(w-z\) is an integer. Then, clearly we have
\[
\frac{\binom{w}{z-k}}{(z+n-k)\binom{w+n}{z+n-k}} = \frac{w!}{(z-k)!(w-z+k)!} \cdot \frac{(w+n)!}{n!w!} \cdot \frac{(z+n-k-1)!(w-z+k)!}{(w+n)!} = \frac{1}{n} \frac{(z+n-k-1)!}{(z-k)!} \cdot \frac{1}{n} \frac{(z+n-k-1)}{n-1}
\]
and by taking limits as it is explained above, this remains true for every \(w\) and \(z\) for which this is defined. That is
\[
\frac{\binom{w}{z-k}}{(z+n-k)\binom{w+n}{z+n-k}} = \frac{1}{n\binom{w+n}{n}} \binom{z+n-k-1}{n-1}. \quad (10)
\]
Thus, multiplying both sides of (6) by \(\frac{1}{n^{(w+n)}}\), we obtain
\[
\sum_{j=0}^{n} \frac{(-1)^j (n)_{w} (z+j)_{w+n} \psi(z-j+1)}{(z+n-j)(z+n-j)} = \frac{1}{nz^{(w+n)}}
\]  
(11)
for every complex numbers \(z\) and \(w\) for which both sides of the above formula are defined.

Similarly, we have
\[
\binom{w+n-k}{z-k} \binom{w+n-z}{z+n-k} = \frac{(w+n-k)!}{(z-k)!(w+n-z)!} \cdot \frac{(w+n-z)!}{n!(w-z)!} \cdot \frac{(z+n-k-1)!(w-z)!}{(w+n-k)!}
\]
\[
= \frac{1}{n} \frac{(w+n-k-1)!}{(z-k)!} \frac{1}{n} \begin{pmatrix} z+n-k-1 \\ n-1 \end{pmatrix}.
\]
That is
\[
\binom{w+n-k}{z-k} \binom{w+n-z}{z+n-k} = \frac{1}{n} \binom{w+n-z}{z+n-k} \begin{pmatrix} z+n-k-1 \\ n-1 \end{pmatrix}.
\]  
(12)

Thus, multiplying both sides of (8) by \(\frac{1}{n^{(w+n-z)}}\), we obtain
\[
\sum_{j=0}^{n} \frac{(-1)^j (n)_{w} (z+j)_{w+n} \psi(w+n-j+1) - \psi(z-j+1)}{(z+n-j)(z+n-j)} = \frac{1}{nz^{(w+n)}}
\]  
(13)
for every complex numbers \(z\) and \(w\) for which both sides of the above formula are defined.

Equations (11) and (13) are the natural generalizations of the proposed identities of Problem 113, and in fact the only mathematical content resides in (w).

Omran KOUBA
Higher Institute for Applied Sciences and Technology,
Damascus, Syria.
e-mail: omran_kouba@hiast.edu.sy
or: kouba.omran@gmail.com

Anastasios KOTRONIS
Athens, Greece,
e-mail: akotronis@gmail.com