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PROBLEMS AND SOLUTIONS

Proposals and solutions must be legible and should appear on separate sheets, each indicating the name of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editors have supplied a solution. The editors encourage undergraduate and pre-college students to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the class and school name. Solutions will be evaluated for publication by a committee of professors according to a combination of criteria. Questions concerning proposals and/or solutions can be sent by e-mail to: *mathproblems-ks@hotmail.com*

Solutions to the problems stated in this issue should arrive before July 15, 2015

Problems

110. *Proposed by Henry Ricardo, New York Math Circle, New York.* Find the sum $\sum_{n=1}^{\infty} \frac{1}{p(n)}$, where $p(n) = n(3n-1)/2$ is the nth pentagonal number.

111. *Proposed by Moti Levy, Rehovot, Israel.* Let m, n be integers. Show that if $n > m \geq 0$ then

$$\frac{x^n}{x^m+y^m} + \frac{y^n}{y^m+z^m} + \frac{z^n}{z^m+x^m} \geq \frac{3}{2}\left(\frac{1}{\sqrt{3}}\right)^{n-m}$$

where real $x, y, z > 0$ and $xy + yz + zx = 1$.

112. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Calculate

$$\int_0^1 \ln^2 |\sqrt{x} - \sqrt{1-x}| dx.$$

113. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria and Anastasios Kotronis, Athens, Greece (Jointly).* Let m, n, p be positive integers with $m \geq n$ and $H_m = 1 + \frac{1}{2} + \cdots + \frac{1}{m}$ the m-th Harmonic Number with $H_0 := 0$.
Show that for the values of m, p, n for which the denominators do not vanish, the following equalities hold:

$$\sum_{k=0}^{n}(-1)^{k-1}\frac{\binom{n}{k}\binom{p}{m-k}(H_p + H_m - H_{m-k})}{(m+n-k)\binom{p+n}{m+n-k}} = \sum_{k=0}^{n}(-1)^{k-1}\frac{\binom{n}{k}\binom{p+n-k}{m-k}(H_{p+n-k} + H_m - H_{m-k})}{(m+n-k)\binom{p+n-k}{m+n-k}}$$
$$= \frac{1}{nm\binom{p+n}{n}}.$$

114. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* For m a positive integer compute

$$\lim_{n\to\infty}\left(n\left(me - \sum_{k=1}^{m}\left(1+\frac{1}{n}\right)^{n+k}\right)\right).$$

115. *Proposed by Florin Stanescu, Serban Cioculescu school, city Gaesti, jud. Dambovita, Romania.* Determine the matrices $X_1, X_2, \ldots, X_9 \in M_2(\mathbb{Z})$ such that

$$X_1^4 + X_2^4 + \cdots + X_9^4 = X_1^2 + X_2^2 + \cdots + X_9^2 + 18 \cdot I_2$$

and $\det X_k = 1$, for $k = 1, 2, \ldots, 9$.

116. *Proposed by Mohammed Aassila, Strasbourg, France .* A positive integer N is called deficient if the sum of its divisors is less than $2N : \sigma(N) < 2N$. It is called abundant if $\sigma(N) > 2N$. Prove that there exists an integer $k \in \mathbb{N} \setminus \{0\}$ such that the numbers

$$k+1,\ k+2,\ k+3,\ \cdots,\ k+2015$$

are all abundant. Prove that there exist infinitely many integers $k \in \mathbb{N} \setminus \{0\}$ such that

$$k+1,\ k+2,\ k+3,\ k+4 \quad \text{and} \quad k+5$$

are all deficient.

Solutions

No problem is ever permanently closed. We will be very pleased considering for publication new solutions or comments on the past problems.

103. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* The Glaisher–Kinkelin constant A is defined by

$$A = \lim_{n\to\infty} n^{-\frac{n^2}{2}-\frac{n}{2}-\frac{1}{12}} \mathrm{e}^{\frac{n^2}{4}} \prod_{k=1}^{n} k^k = 1.28242\,71291\,00622\ldots.$$

Prove that

(a) $\displaystyle\sum_{n=1}^{\infty}\left(n^2\ln\frac{n+1}{n} - n + \frac{1}{2} - \frac{1}{3n}\right) = -\frac{1}{2} + \ln\sqrt{2\pi} - 2\ln A - \frac{\gamma}{3}$;

(b) $\displaystyle\int_0^1 x^2\psi(x)\mathrm{d}x = \ln\frac{A^2}{\sqrt{2\pi}}$,

where ψ denotes the Digamma function.

Open problem. Calculate, if possible, in terms of well-known constants the integral $\displaystyle\int_0^1 x^k\psi(x)\mathrm{d}x$, where $k \geq 3$ is an integer.

Solution 1 by Haroun Meghaichi(student), University of Science and Technology, Houari Boumediene, Algiers, Algeria. Note that by using the Digamma series representation, we get

$$\begin{aligned}\int_0^1 x^2\psi(x)\ \mathrm{d}x &= \int_0^1 x^2\left(-\gamma - \frac{1}{x} + \sum_{n=1}^{\infty}\frac{1}{n} - \frac{1}{n+x}\right)\mathrm{d}x\\ &= \frac{-\gamma}{3} - \frac{1}{2} + \sum_{n=1}^{\infty}\left(\int_0^1 \frac{x^2}{n} - \frac{x^2}{x+n}\ \mathrm{d}x\right)\\ &= \frac{-\gamma}{3} - \frac{1}{2} - \sum_{n=1}^{\infty}\left(n^2\ln\frac{n+1}{n} - n + \frac{1}{2} - \frac{1}{3n}\right) \qquad (1)\end{aligned}$$

the integral-sum interchange is justified by the normal convergence of the series. Then we only need to solve the part (a) to deduce the solution to the part (b). let the partial sum (of the proposed series) be S_n, we have

$$\begin{aligned}\sum_{k=1}^{n} k^2\ln\frac{k+1}{k} &= \sum_{k=1}^{n}(k+1)^2\ln(k+1) - k^2\ln k - 2(k+1)\ln(k+1) + \ln(k+1)\\ &= (n+1)^2\ln(n+1) + \ln(n+1)! - 2\sum_{k=1}^{n+1} k\ln k\end{aligned}$$

Let us return to the definition of the Glaisher constant and apply the natural logarithm on both sides, we get

$$a_n = \sum_{k=1}^{n} k \ln k = \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12}\right) \ln n - \frac{n^2}{4} + A + o(1)$$

And Stirling approximation asserts that

$$\ln n! = (n + 1/2) \ln n - n + \ln \sqrt{2\pi} + o(1),$$

and we know that $H_n = \ln(n+1) + \gamma + o(1)$, combining this three asymptotic expansions in the expression of S_n we get

$$\begin{aligned} S_n &= -\frac{H_n}{3} - \frac{n^2}{2} + (n+1)^2 \ln(n+1) + \ln(n+1)! - 2a_{n+1} \\ &= -\frac{\ln(n+1) + \gamma}{3} - \frac{n^2}{2} + (n+1)^2 \ln(n+1) + \left(n + \frac{3}{2}\right) \ln(n+1) - n - 1 \\ &\quad + \ln \sqrt{2\pi} - \left(n^2 + 3n + \frac{13}{6}\right) \ln(n+1) + \frac{(n+1)^2}{2} - 2 \ln A + o(1) \\ &= \frac{-1}{2} + \ln \sqrt{2\pi} - 2 \ln A - \frac{\gamma}{3} + o(1). \end{aligned}$$

Which proves (a), now we use (1) to prove (b) we have

$$\int_0^1 x^2 \psi(x)\ \mathrm{d}x = \frac{-\gamma}{3} - \frac{1}{2} - \left(\frac{-1}{2} + \ln \sqrt{2\pi} - 2 \ln A - \frac{\gamma}{3}\right) = \ln \frac{A^2}{\sqrt{2\pi}}.$$

Solution 2 by Moti Levy, Rehovot, Israel.
Reference: H. M. Srivastava, Junesang Choi, "*Zeta and q-Zeta Functions and Associated Series and Integrals*", Chapter 3, Elsevier, 2012.
(a) By the Taylor series expansion of $\ln(1+x)$,

$$\ln\left(\frac{n+1}{n}\right) = \ln\left(1 + \frac{1}{n}\right) = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{1}{n^k}.$$

$$\begin{aligned} &\sum_{n=1}^{\infty} \left(n^2 \ln\left(\frac{n+1}{n}\right) - n + \frac{1}{2} - \frac{1}{3n}\right) \\ &= \sum_{n=1}^{\infty} \left(n^2 \left(\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k} \frac{1}{n^k}\right) - n + \frac{1}{2} - \frac{1}{3n}\right) \\ &= \sum_{n=1}^{\infty} \left(n^2 \left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3}\right) + n^2 \left(\sum_{k=4}^{\infty} \frac{(-1)^{k+1}}{k} \frac{1}{n^k}\right) - n + \frac{1}{2} - \frac{1}{3n}\right) \\ &= \sum_{n=1}^{\infty} n^2 \left(\sum_{k=4}^{\infty} \frac{(-1)^{k+1}}{k} \frac{1}{n^k}\right) = \sum_{n=1}^{\infty} \sum_{k=4}^{\infty} \frac{(-1)^{k+1}}{k} \frac{1}{n^{k-2}} = \sum_{k=4}^{\infty} \frac{(-1)^{k+1}}{k} \sum_{n=1}^{\infty} \frac{1}{n^{k-2}} \\ &= \sum_{k=4}^{\infty} \frac{(-1)^{k+1}}{k} \zeta(k-2) = \sum_{k=4}^{\infty} \frac{(-1)^{k+1}}{k} \zeta(k-2) = -\sum_{k=2}^{\infty} \frac{(-1)^k}{k+2} \zeta(k). \end{aligned}$$

Interchanging the order of summation is justified since the series is absolutely convergent.

Closed forms of $\sum_{k=2}^{\infty}(-1)^k \frac{\zeta(k)}{k+2}$ can be found in the referred book:

$$\sum_{k=2}^{\infty}\frac{(-1)^k}{k+2}\zeta(k)=\zeta'(0)-2\zeta'(-1)-\frac{1}{2}\zeta(0)+\zeta(-1)+\frac{1}{3}\gamma+\frac{1}{2}H_2. \qquad (1)$$

Now, $H_2=\frac{1}{1}+\frac{1}{2}=\frac{3}{2}$ and the particular values of the Zeta function are:

$$\begin{aligned}\zeta'(0)&=-\ln\sqrt{2\pi},\\ \zeta'(-1)&=\frac{1}{12}-\ln A,\\ \zeta'(-2)&=-\frac{1}{4\pi^2}\zeta(3),\\ \zeta(0)&=-\frac{1}{2},\\ \zeta(-1)&=-\frac{1}{12}.\end{aligned}$$

$$\sum_{k=2}^{\infty}\frac{(-1)^k}{k+2}\zeta(k)=-\ln\sqrt{2\pi}+2\ln A+\frac{1}{3}\gamma+\frac{1}{2}.$$

(b)
The following series expression for the Digamma function is well known:

$$\psi(x)=-\gamma-\frac{1}{x}+\sum_{k=1}^{\infty}\frac{x}{k(x+k)}.$$

$$\int_0^1 x^p\psi(x)\,dx=-\gamma\int_0^1 x^p dx-\int_0^1 x^{p-1}dx+\int_0^1\left(\sum_{k=1}^{\infty}\frac{x^{p+1}}{k(x+k)}\right)dx.$$

After interchanging the order of summation and integration,

$$\begin{aligned}\int_0^1 x^p\psi(x)\,dx&=-\gamma\int_0^1 x^p dx-\int_0^1 x^{p-1}dx+\sum_{k=1}^{\infty}\int_0^1\frac{x^{p+1}}{k(x+k)}dx\\ &=-\frac{1}{p+1}\gamma-\frac{1}{p}++\sum_{k=1}^{\infty}\int_0^1\left(\sum_{j=0}^{p}(-1)^j k^{j-1}x^{p-j}+(-1)^{p+1}\frac{k^p}{k+x}\right)dx.\end{aligned}$$

Now we change again the order of integration and summation,

$$\begin{aligned}
&\int_0^1 x^p \psi(x)\,dx \\
&= -\frac{1}{p+1}\gamma - \frac{1}{p} + \sum_{k=1}^{\infty}\left(\sum_{j=0}^{p}\frac{(-1)^j k^{j-1}}{p-j+1} + (-1)^{p+1} k^p \ln\left(1+\frac{1}{k}\right)\right) \\
&= -\frac{1}{p+1}\gamma - \frac{1}{p} + \sum_{k=1}^{\infty}\left(\sum_{n=0}^{p}\frac{(-1)^n k^{n-1}}{p-n+1} + (-1)^{p+1} k^p \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\frac{1}{k^n}\right) \\
&= -\frac{1}{p+1}\gamma - \frac{1}{p} + \sum_{k=1}^{\infty}\left((-1)^{p+1}\sum_{n=p+2}^{\infty}\frac{(-1)^{n+1}}{n}\frac{1}{k^{n-p}}\right) \\
&= -\frac{1}{p+1}\gamma - \frac{1}{p} + (-1)^{p+1}\sum_{n=p+2}^{\infty}\frac{(-1)^{n+1}}{n}\sum_{k=1}^{\infty}\frac{1}{k^{n-p}} \\
&= -\frac{1}{p+1}\gamma - \frac{1}{p} + (-1)^{p+1}\sum_{m=2}^{\infty}\frac{(-1)^{n+1}}{m+p}\sum_{k=1}^{\infty}\frac{1}{k^{m}} \\
&= -\frac{1}{p+1}\gamma - \frac{1}{p} + \sum_{m=2}^{\infty}\frac{(-1)^m}{m+p}\zeta(m)
\end{aligned}$$

The definite integral is expressed by an alternating sum related to the Zeta function. Again, we use a closed form of $\sum_{m=2}^{\infty}\frac{(-1)^m}{m+p}\zeta(m)$, which can be found in the reference, and obtain:

$$\int_0^1 x^p\psi(x)\,dx = -\frac{1}{p} + \frac{H_p}{p+1} + \sum_{l=0}^{p-1}(-1)^l\binom{p}{l}\zeta'(-l) - \sum_{l=0}^{p-1}\frac{(-1)^l}{p-l}\zeta(-l).$$

For $p=2$,

$$\begin{aligned}
\int_0^1 x^2\psi(x)\,dx &= -\frac{1}{2} + \frac{H_2}{2+1} + \sum_{l=0}^{1}(-1)^l\binom{2}{l}\zeta'(-l) - \sum_{l=0}^{1}\frac{(-1)^l}{2-l}\zeta(-l) \\
&= -\frac{1}{2} + \frac{\frac{3}{2}}{2+1} + \zeta'(0) - 2\zeta'(-1) - \frac{1}{2}\zeta(0) + \zeta(-1) \\
&= -\frac{1}{2} + \frac{\frac{3}{2}}{2+1} - \ln\sqrt{2\pi} - 2\left(\frac{1}{12} - \ln A\right) - \frac{1}{2} * \left(-\frac{1}{2}\right) + \left(-\frac{1}{12}\right) \\
&= -\ln\sqrt{2\pi} + 2\ln A \cong -0.42143.
\end{aligned}$$

For $p=3$,

$$\begin{aligned}
\int_0^1 x^3\psi(x)\,dx &= -\frac{1}{3} + \frac{H_3}{3+1} + \sum_{l=0}^{2}(-1)^l\binom{3}{l}\zeta'(-l) - \sum_{l=0}^{2}\frac{(-1)^l}{3-l}\zeta(-l) \\
&= -\frac{1}{3} + \frac{\frac{11}{6}}{4} + \zeta'(0) - 3\zeta'(-1) + 3\zeta'(-2) - \frac{1}{3}\zeta(0) + \frac{1}{2}\zeta(-1) - \zeta(-2) \\
&= -\frac{1}{3} + \frac{11}{24} - \ln\sqrt{2\pi} - 3\left(\frac{1}{12} - \ln A\right) + 3\left(-\frac{1}{4\pi^2}\zeta(3)\right) - \frac{1}{3}\left(-\frac{1}{2}\right) + \frac{1}{2}\left(-\frac{1}{12}\right) \\
&= -\ln\sqrt{2\pi} + 3\ln A - \frac{3}{4\pi^2}\zeta(3) \cong -0.26402.
\end{aligned}$$

Solution 3 by Omran Kouba, Higher Institute for Applied sciences and Technology, Damascus, Syria. Let us define S_m by

$$S_m = \sum_{n=1}^{m} \left(n^2 \ln \frac{n+1}{n} - n + \frac{1}{2} - \frac{1}{3n} \right) \tag{1}$$

It easily seen that

$$\begin{aligned} S_m &= \sum_{n=1}^{m} n^2 \ln(n+1) - \sum_{n=1}^{m} n^2 \ln n - \frac{1}{2} \sum_{n=1}^{m} (2n-1) - \frac{1}{3} \underbrace{\sum_{k=1}^{m} \frac{1}{n}}_{H_m} \\ &= m^2 \ln(m+1) - \frac{m^2}{2} - \frac{1}{3} H_m + \sum_{n=1}^{m} (n-1)^2 \ln n - \sum_{n=1}^{m} n^2 \ln n \\ &= m^2 \ln(m+1) - \frac{m^2}{2} - \frac{1}{3} H_m + \sum_{k=1}^{m} (1-2n) \ln n \\ &= m^2 \ln(m+1) - \frac{m^2}{2} - \frac{1}{3} H_m + \ln(m!) - 2 \ln \left(\prod_{k=1}^{m} k^k \right) \end{aligned} \tag{2}$$

Now, recalling that

$$\begin{aligned} H_m &= \ln m + \gamma + o(1) \\ \ln(m!) &= \left(m + \frac{1}{2} \right) \ln m - m + \ln \sqrt{2\pi} + o(1) \\ \ln \left(\prod_{k=1}^{m} k^k \right) &= \left(\frac{m^2}{2} + \frac{m}{2} + \frac{1}{12} \right) \ln m - \frac{m^2}{4} + \ln A + o(1) \end{aligned}$$

we conclude from (2) that

$$\begin{aligned} S_m &= m^2 \ln \left(1 + \frac{1}{m} \right) - m - \frac{\gamma}{3} + \ln \sqrt{2\pi} - 2 \ln A + o(1) \\ &= -\frac{1}{2} - \frac{\gamma}{3} + \ln \sqrt{2\pi} - 2 \ln A + o(1) \end{aligned}$$

This proves that

$$\sum_{n=1}^{\infty} \left(n^2 \ln \frac{n+1}{n} - n + \frac{1}{2} - \frac{1}{3n} \right) = \lim_{m \to \infty} S_m = -\frac{1}{2} - \frac{\gamma}{3} + \ln \sqrt{2\pi} - 2 \ln A$$

Which is (a).

Recall that the digamma function ψ satisfies

$$\psi(x+1) = \psi(x) + \frac{1}{x}, \quad \psi(x+1) = -\gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+x} \right)$$

with uniform convergence on $[0, 1]$ hence

$$\begin{aligned} \int_0^1 x^2 \psi(x) dx &= -\frac{1}{2} + \int_0^1 x^2 \psi(x+1) dx \\ &= -\frac{1}{2} - \frac{\gamma}{3} + \sum_{n=1}^{\infty} \left(\frac{1}{3n} - \int_0^1 \frac{x^2}{x+n} dx \right) \end{aligned}$$

But

$$\int_0^1 \frac{x^2}{x+n}dx = n^2\int_0^1 \frac{1}{x+n}dx + \int_0^1 (x-n)dx = n^2 \ln\left(\frac{1+n}{n}\right) + \frac{1}{2} - n$$

Thus

$$\begin{aligned}\int_0^1 x^2\psi(x)dx &= -\frac{1}{2} - \frac{\gamma}{3} - \sum_{n=1}^{\infty}\left(n^2\ln\left(\frac{1+n}{n}\right) + \frac{1}{2} - n - \frac{1}{3n}\right)\\ &= -\ln\sqrt{2\pi} + 2\ln A = \ln\frac{A^2}{\sqrt{2\pi}}\end{aligned}$$

which is (b).

Solution 4 by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia. (a) Let's compute the nth partial sum:

$$\begin{aligned}S_n &= \sum_{k=1}^{n}\left(k^2\log\left(1+\frac{1}{k}\right) - k + \frac{1}{2} - \frac{1}{3k}\right)\\ &= \sum_{k=1}^{n} k^2\log(k+1) - \sum_{k=1}^{n} k^2\log k - \sum_{k=1}^{n} k + \frac{n}{2} - \frac{1}{3}H_n\\ &= \sum_{k=1}^{n}\left((k+1)^2 - 2(k+1) + 1\right)\log(k+1) - \sum_{k=2}^{n} k^2\log k - \frac{n(n+1)}{2} + \frac{n}{2} - \frac{1}{3}H_n\\ &= (n^2+2n+1)\log(n+1) - 2\sum_{k=2}^{n+1} k\log k + \log n! + \log(n+1) - \frac{n^2}{2} - \frac{1}{3}H_n\\ &= n^2\log(n+1) - 2\sum_{k=2}^{n} k\log k + \log n! - \frac{n^2}{2} - \frac{1}{3}H_n\\ &= n^2\log(n+1) - 2\sum_{k=2}^{n} k\log k + \left(\log\sqrt{2\pi} + \left(n+\frac{1}{2}\right)\log n - n + O\left(\frac{1}{n}\right)\right) - \frac{n^2}{2} - \frac{1}{3}H_n\\ &= \left(n^2\log\left(1+\frac{1}{n}\right) - n\right) + \log\sqrt{2\pi} - 2\left(\frac{n^2}{4} + \sum_{k=2}^{n} k\log k + \left(-\frac{n^2}{2} - \frac{n}{2} - \frac{1}{12}\right)\log n\right)\\ &\quad - \frac{1}{3}\left(H_n - \log n\right) + O\left(\frac{1}{n}\right).\end{aligned}$$

Hence,

$$\sum_{n=1}^{\infty}\left(n^2\log\left(1+\frac{1}{n}\right) - n + \frac{1}{2} - \frac{1}{3n}\right) = \lim_{n\to\infty} S_n = -\frac{1}{2} + \log\sqrt{2\pi} - 2\log A - \frac{1}{3}\gamma.$$

(b) Using the estimate in (a) and the formula $\psi(x) = -\gamma + \sum_{n=0}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+x}\right)$, we have

$$\begin{aligned}
\int_0^1 x^2\psi(x)dx &= -\gamma\int_0^1 x^2dx + \int_0^1 (x^2-x)dx + \sum_{n=1}^{\infty}\left(\frac{1}{3(n+1)} - \int_0^1 \frac{x^2}{n+x}dx\right)\\
&= -\frac{\gamma}{3} - \frac{1}{6} + \sum_{n=1}^{\infty}\left(\frac{1}{3(n+1)} - \frac{1}{2} + n - n^2\log\frac{n+1}{n}\right)\\
&= -\frac{\gamma}{3} - \frac{1}{6} - \frac{1}{3}\sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \sum_{n=1}^{\infty}\left(n^2\log\frac{n+1}{n} - n + \frac{1}{2} - \frac{1}{3n}\right)\\
&= -\frac{\gamma}{3} - \frac{1}{2} - \left(-\frac{1}{2} + \log\sqrt{2\pi} - 2\log A - \frac{\gamma}{3}\right)\\
&= 2\log A - \log\sqrt{2\pi} = \log\frac{A^2}{\sqrt{2\pi}}.
\end{aligned}$$

Open problem. We propose the following conjecture.

Conjecture 1. *Let $k \geq 2$. Then*

$$\int_0^1 x^k\psi(x)dx = -\log\sqrt{2\pi} + \sum_{j=1}^{k-1}(-1)^{j+1}C_k^j\log A_j,$$

where $A_j, j \geq 1$ is the generalized Glaisher-Kinkelin constant.

Also solved by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Anastasios Kotronis, Athens, Greece; Moubinool OMARJEE , Lyce Henri IV, Paris , France; Albert Stadler, Switzerland. and the proposer.

104. *Proposed by Marcel Chiriţă, Bucharest, Romania.* Consider $f : [0,1] :\to \mathbb{R}$ a differentiable function and let $a, b \in (0,1)$ with $a < b$ such that $\int_0^a f(x)dx = 0$, and $\int_b^1 f(x)dx = 0$. We denote by $M = \sup_{x\in(0,1)}|f'(x)|$. Show that

$$\left|\int_0^1 f(x)dx\right| \leq \frac{1-a+b}{4}M.$$

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy. If $f \equiv 0$ or $\sup\{|f'(x)|: x \in (0,1)\} = +\infty$ there is nothing to prove. Let $\sup\{|f'(x)|: x \in (0,1)\} = M < \infty$.
The conditions

$$\int_0^a f(x)dx = \int_b^1 f(x)dx = 0$$

imply that there exist $\xi \in (0,a)$ and $\eta \in (b,1)$ such that $f(\xi) = f(\eta) = 0$. Now we observe that

$$\left|\int_a^b f(x)dx\right| \leq M\frac{(\xi-\eta)^2}{4}$$

This comes from the fact that the graph of f between a and b is inside the quadrilateral determined by the four straight lines

$$y = M(x-\xi), \quad y = -M(x-\xi), \quad y = -M(x-\eta), \quad y = M(x-\eta)$$

and whose vertices are the four points $(\xi, 0)$, $(\frac{\xi+\eta}{2}, M\frac{\eta-\xi}{2})$, $(\eta, 0)$, $(\frac{\xi+\eta}{2}, M\frac{\xi-\eta}{2})$. Therefore $\int_a^b f(x)dx$ at most equals the area of the upper half or lower half of the quadrilateral whose area is

$$\left((\eta-\xi)M\frac{\eta-\xi}{2}\frac{1}{2}\right) = \frac{M(\eta-\xi)^2}{4}$$

Now it is easy to observe that

$$\frac{M(\eta-\xi)^2}{4} \le \frac{M}{4} \le \frac{1+b-a}{4}M$$

The upper bound $(1+b-a)M/4$ cannot be lowered because we can give examples where $a \sim b$ and $\xi \sim 0$, $\eta \sim 1$.

Solution 2 by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Consider $g_{a,b} : [0,1] \to \mathbb{R}$ defined by

$$g_{a,b}(t) = \begin{cases} \left(\frac{1}{2a} - 1\right)x & \text{if } 0 \le x \le a \\ \left(\frac{1}{2} - x\right) & \text{if } a \le x \le b \\ \left(1 - \frac{1}{2(1-b)}\right)(1-x) & \text{if } b \le x \le 1 \end{cases}$$

One checks easily that

$$\begin{aligned} \int_0^1 g_{a,b}(x)f'(x)dx &= \left(\frac{1}{2a} - 1\right)\int_0^a xf'(x)dx + \int_a^b \left(\frac{1}{2} - x\right)f'(x)dx \\ &\quad + \left(1 - \frac{1}{2(1-b)}\right)\int_b^1 (1-x)f'(x)dx \\ &= \left(\frac{1}{2} - a\right)f(a) - \left(\frac{1}{2a} - 1\right)\underbrace{\int_0^a f(x)dx}_{0} \\ &\quad + \left[\left(\frac{1}{2} - x\right)f(x)\right]_{x=a}^{x=b} + \int_a^b f(x)dx \\ &\quad - \left(\frac{1}{2} - b\right)f(b) + \left(1 - \frac{1}{2(1-b)}\right)\underbrace{\int_b^1 f(x)dx}_{0} \\ &= \int_a^b f(x)dx = \int_0^1 f(x)dx \end{aligned}$$

It follows that

$$\left|\int_0^1 f(x)dx\right| \le \sup_{x\in(0,1)} |f'(x)| \int_0^1 |g_{a,b}(x)|\, dx \tag{1}$$

Now, it is an easy task to calculate $\int_0^1 |g_{a,b}(x)|\,dx$ distinguishing the different cases $0 < a < b \le \frac{1}{2}$, $0 < a \le \frac{1}{2} \le b < 1$, and $\frac{1}{2} < a < b < 1$. We find

$$\int_0^1 |g_{a,b}(x)|\,dx = \frac{1}{4}\left(\left|\frac{1}{2} - a\right| + \left|\frac{1}{2} - b\right|\right)$$

So, inequality (1) becomes

$$\left|\int_0^1 f(x)dx\right| \le \frac{|1-2a| + |1-2b|}{8} \sup_{x\in(0,1)} |f'(x)|$$

which is sharper than the proposed inequality since clearly for $0 < a < b < 1$ we have

$$\left|\frac{1}{2} - a\right| + \left|\frac{1}{2} - b\right| \le 1 \le 1 + b - a.$$

Solution 3 by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia. From the given condition,

$$\exists x_0 \in [0,a] : f(x_0) = 0,\ \exists x_1 \in [b,1] : f(x_1) = 0.$$

Then using Taylor's formula, we have

$$\exists c_{x_0} \in (0,b) : f(x) = f(x_0) + f'\left(c_{x_0}\right)(x - x_0) = f'\left(c_{x_0}\right)(x - x_0),$$
$$\exists c_{x_1} \in (a,1) : f(x) = f(x_1) + f'\left(c_{x_1}\right)(x - x_1) = f'\left(c_{x_1}\right)(x - x_1).$$

Hence,

$$\forall x \in (0,b) : |f(x)| \le M|x - x_0|,$$
$$\forall x \in (a,1) : |f(x)| \le M|x - x_1|.$$

Thus, we have

$$\begin{aligned}
\left|\int_0^1 f(x)dx\right| &= \left|\int_0^a f(x)dx + \int_a^b f(x)dx + \int_b^1 f(x)dx\right| = \left|\int_a^b f(x)dx\right| \\
&= \frac{1}{2}\left|\int_a^1 f(x)dx + \int_0^b f(x)dx\right| \\
&\le \frac{1}{2}\left(\int_a^1 |f(x)|dx + \int_0^b |f(x)|dx\right) \\
&\le \frac{M}{2}\left(\int_a^1 |x - x_1|dx + \int_0^b |x - x_0|dx\right) \\
&\le \frac{M}{2}\left(\frac{1-a}{2} + \frac{b}{2}\right) = \frac{1-a+b}{4}\cdot M.
\end{aligned}$$

Also solved by Moti Levy, Rehovot, Israel; Florin Stanescu, Romania and the proposer.

105. *Proposed by Serafeim Tsipelis Ioannina, Greece and Anastasios Kotronis, Athens, Greece.* Evaluate

$$\sum_{k=1}^{\infty} \arctan\left(\frac{1}{k^2+1}\right).$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. For $k \geq 1$, let

$$\theta_k = \arctan\left(\frac{1}{k^2+1}\right) = \arctan\left(\frac{1/k^2}{1+1/k^2}\right) \quad \text{and} \quad r_k = \sqrt{\left(1+\frac{1}{k^2}\right)^2 + \frac{1}{k^4}}$$

and define $w = \sqrt[4]{2}e^{i\pi/8} = \frac{1}{\sqrt{2}}\sqrt{\sqrt{2}+1} + \frac{i}{\sqrt{2}}\sqrt{\sqrt{2}-1}$, so that

$$1 + \frac{w^2}{k^2} = 1 + \frac{1+i}{k^2} = r_k e^{i\theta_k}$$

Now, noting that $r_k = 1 + O(k^{-2})$ and $\theta_k = O(k^{-2})$ we know that the product $R = \prod_{k=1}^{\infty} r_k$ and the sum $\Theta = \sum_{k=1}^{\infty} \theta_k$ are both convergent. Moreover, using the well-known product expansion for $z \mapsto \sinh z$, we get

$$\sinh(\pi w) = \pi w \prod_{k=1}^{\infty}\left(1 + \frac{w^2}{k^2}\right) = \pi w R e^{i\Theta} \tag{1}$$

But

$$0 < \Theta < \sum_{k=1}^{\infty} \frac{1}{1+k^2} < \sum_{k=1}^{3} \frac{1}{1+k^2} + \sum_{k=4}^{\infty} \frac{1}{k(k-1)} = \frac{103}{90} < \frac{3\pi}{8}$$

Thus (1) implies

$$\frac{\pi}{8} + \Theta = \operatorname{Arg}(\sinh(\pi w)) = \arctan\left(\frac{\Im(\sinh(\pi w))}{\Re(\sinh(\pi w))}\right)$$

and finally

$$\Theta = \arctan\left(\tan\left(\frac{\pi}{\sqrt{2}}\sqrt{\sqrt{2}-1}\right)\coth\left(\frac{\pi}{\sqrt{2}}\sqrt{\sqrt{2}+1}\right)\right) - \frac{\pi}{8} \approx 1.037287.$$

Also solved by Haroun Meghaichi(student), University of Science and Technology, Houari Boumediene, Algiers, Algeria; Moubinool Omarjee, Lycée Henri IV, Paris, France; Albert Stadler, Switzerland; and the proposer.

106.*Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* For $a > 0$ and $x \in [0, 1]$ we consider

$$F(a, x) = \frac{ax + \sqrt{a}(1-x)}{x + \sqrt{a}(1-x)} - a^x.$$

Find the largest subset $D \subset (0, +\infty) \times [0, 1]$ such that $F(a, x) \geq 0$ for $(a, x) \in D$.
Solution by Proposer. We will prove that

$$D = (0, 1] \times \left[\frac{1}{2}, 1\right] \cup [1, +\infty) \times \left[0, \frac{1}{2}\right].$$

Indeed, consider $f:\mathbb{R}\to\mathbb{R}, f(u)=u\coth u$, (with $f(0)=1$ by continuity.) Clearly, f is even and for $u>0$ we have

$$f'(u)=\coth u-\frac{u}{\sinh^2 u}=\frac{u}{\sinh^2 u}\left(\frac{\sinh u}{u}\cosh u-1\right)>0$$

because $\sinh u>u$ and $\cosh u>1$. Thus f is increasing on $[0,+\infty)$ it follows that for $t\in\mathbb{R}$ and $u\in[-1,1]$ we have

$$f(ut)=f(|ut|)\le f(|t|)=f(t)$$

or equivalently

$$ut\frac{e^{2ut}+1}{e^{2ut}-1}\le t\frac{e^{2t}+1}{e^{2t}-1}$$

this is equivalent to

$$\frac{t(1+u+(1-u)e^{2t})}{e^{2t}(e^{2t}-1)(e^{2ut}-1)}\left(e^{2(u+1)t}-\frac{(1+u)e^{4t}+(1-u)e^{2t}}{1+u+(1-u)e^{2t}}\right)\ge 0$$

Or, by removing the positive terms:

$$\frac{1}{e^{2ut}-1}\left(e^{2(u+1)t}-\frac{(1+u)e^{4t}+(1-u)e^{2t}}{1+u+(1-u)e^{2t}}\right)\ge 0$$

Thus,

$$e^{2(u+1)t}\ge\frac{(1+u)e^{4t}+(1-u)e^{2t}}{1+u+(1-u)e^{2t}}\iff ut\ge 0$$

Now take $u=2x-1$, $t=\frac{\ln a}{4}$ for some $x\in[0,1]$ and $a>0$ to obtain the announced conclusion.

107. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.*
Let $a,b\in\mathbb{R}_+$ and $f:\mathbb{R}\to\mathbb{R}$ be an even continuous function on $\mathbb{R}$. Prove that:

$$\int_{-a}^{a}\frac{f(x)}{b+\arctan x+\sqrt{b^2+\arctan^2 x}}dx=\frac{1}{b}\int_0^a f(x)dx.$$

Solution 1 by Ángel Plaza, University of Las Palmas, Gran Canaria, Spain. Notice that summing function $g(x)=\frac{f(x)}{b+\arctan x+\sqrt{b^2+\arctan^2 x}}$ in the left-hand side integral at points x and $-x$, and since function $f(x)$ is even and $\arctan x$ is odd we have

$$\begin{aligned} g(x)+g(-x) &= \frac{f(x)}{b+\arctan x+\sqrt{b^2+\arctan^2 x}}+\frac{f(x)}{b-\arctan x+\sqrt{b^2+\arctan^2 x}} \\ &= \frac{f(x)\left(2b+2\sqrt{b^2+\arctan^2 x}\right)}{2b^2+2b\sqrt{b^2+\arctan^2 x}} \\ &= \frac{f(x)}{b}. \end{aligned}$$

Therefore,

$$\int_{-a}^{a}\frac{f(x)}{b+\arctan x+\sqrt{b^2+\arctan^2 x}}dx=\frac{1}{b}\int_0^a f(x)dx.$$

Solution 2 by Haroun Meghaichi, University of Science and Techonology, Houari Boumediene, Algeria. Note that

$$\frac{1}{b+\arctan x+\sqrt{b^2+\arctan^2 x}} = \frac{b+\arctan x-\sqrt{b^2+\arctan^2 x}}{2b\arctan x}$$
$$= \frac{1}{2b}+\frac{b-\sqrt{b^2+\arctan^2 x}}{2b\arctan x}$$

Note that the $g:\mathbb{R}\mapsto\mathbb{R}$, defined by

$$g(x)=\begin{cases}0 & \text{if } x=0\\ \frac{b-\sqrt{b^2+\arctan^2 x}}{2b\arctan x} & \text{else.}\end{cases}$$

is continous and odd then

$$\int_{-a}^{a}\frac{f(x)}{b+\arctan x+\sqrt{b^2+\arctan^2 x}}\ \mathrm{d}x = \frac{1}{2b}\int_{-a}^{a} f(x)\ \mathrm{d}x+\underbrace{\int_{-a}^{a} f(x)g(x)\ \mathrm{d}x}_{=0}$$
$$= \frac{1}{b}\int_{0}^{a} f(x)\ \mathrm{d}x$$

Also solved by Omran Kouba, Higher Institute for Applied sciences and Technology, Damascus, Syria; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania; Albert Stadler, Switzerland; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Moti Levy, Rehovot, Israel; Moubinool Omarjee, Lycée Henri IV, Paris, France; and the proposer.

108. *Proposed by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.* Let $(a_n)_{n\geq 1}$ be the sequence defined inductively by setting $a_1=1$ and

$$a_{n+1}=\frac{n+1}{2n}\sum_{k=1}^{n} a_k a_{n+1-k}, \quad \text{for } n\geq 1.$$

Find $\sum_{n=1}^{\infty} a_n e^{-n}$.

Solution 1 by Anastasios Kotronis, Athens, Greece.

We will show that $a_n=\frac{n^{n-1}}{n!}$. For $n=1$ it is obviously true. Now assume that $a_k=\frac{k^{k-1}}{k!}$ for all k with $1\leq k\leq n$. We have

$$a_{n+1}=\frac{n+1}{2n}\sum_{k=1}^{n}\frac{k^{k-1}}{k!}\cdot\frac{(n+1-k)^{n-k}}{(n+1-k)!}=\frac{1}{2nn!}\sum_{k=1}^{n}\binom{n+1}{k}k^{k-1}(n+1-k)^{n-k},$$

so it suffices to prove that

$$\sum_{k=1}^{n}\binom{n+1}{k}k^{k-1}(n+1-k)^{n-k}=2n(n+1)^{n-1}.$$

For this, we start from Abel's binomial identity

$$\sum_{k=0}^{n}\binom{n}{k}a(a+bk)^{k-1}(c-bk)^{n-k}=(a+c)^n, \tag{1}$$

(see [1] for a proof) and we have

$$\begin{aligned}
(a+c)^n &= \sum_{k=0}^{n}\binom{n}{k}a(a+bk)^{k-1}(c-bk)^{n-k}\\
&= \sum_{k=0}^{n}\binom{n}{k}a(a+bk)^{k-1}(c-bk)^{n-k-1}\left(c-nb+b(n-k)\right)\\
&= \sum_{k=0}^{n}\binom{n}{k}a(a+bk)^{k-1}(c-bk)^{n-k-1}(c-nb)+b\sum_{k=0}^{n}\binom{n}{k}(n-k)a(a+bk)^{k-1}(c-bk)^{n-k-1}\\
&= \sum_{k=0}^{n}\binom{n}{k}a(a+bk)^{k-1}(c-bk)^{n-k-1}(c-nb)+nb\sum_{k=0}^{n-1}\binom{n-1}{k}a(a+bk)^{k-1}(c-bk)^{n-k-1}\\
&\overset{(1)}{=} \sum_{k=0}^{n}\binom{n}{k}a(a+bk)^{k-1}(c-bk)^{n-k-1}(c-nb)+nb(a+c)^{n-1}
\end{aligned}$$

which yields

$$\sum_{k=0}^{n}\binom{n}{k}(a+bk)^{k-1}(c-bk)^{n-k-1}=\left(\frac{1}{a}+\frac{1}{c-nb}\right)(a+c)^{n-1}$$

or equivalently

$$\sum_{k=1}^{n-1}\binom{n}{k}(a+bk)^{k-1}(c-bk)^{n-k-1}=\frac{1}{a}\left((a+c)^{n-1}-c^{n-1}\right)+\frac{1}{c-nb}\left((a+c)^{n-1}-(a+bn)^{n-1}\right).$$

Now we take the limit as $a\to 0$ to get

$$\sum_{k=1}^{n-1}\binom{n}{k}(bk)^{k-1}(c-bk)^{n-k-1}=\frac{d}{dx}x^{n-1}\bigg|_{x=c}+\frac{1}{c-nb}\left(c^{n-1}-(bn)^{n-1}\right)$$

and again the limit as $c\to nb$ to get

$$b^{n-2}\sum_{k=1}^{n-1}\binom{n}{k}k^{k-1}(n-k)^{n-k-1}=(n-1)(nb)^{n-2}+\frac{d}{dx}x^{n-1}\bigg|_{x=nb}=2(n-1)(nb)^{n-2}.$$

For $n+1$ in the place of n the above will give

$$\sum_{k=1}^{n}\binom{n+1}{k}k^{k-1}(n+1-k)^{n-k}=2n(n+1)^{n-1}$$

which is what we wanted. After these, what we seek is $\sum_{n\geq 1}\frac{n^{n-1}}{n!}e^{-n}$.
But it is known that for Lambert's W function, the inverse of $f(x):=xe^x$ on $[-1,+\infty)$, it is $W(x)=\sum_{n\geq 1}\frac{(-n)^{n-1}}{n!}x^n$ for $|x|\leq e^{-1}$ (see `http://en.wikipedia.org/wiki/Lambert_W_function#Asymptotic_expansions`, (13)), so our sum equals $-W(-e^{-1})=1$.

References

[1] W. Chu, *The Electronic Journal of Combinatorics*, **17**, (2010), #N24, Elementary Proofs for Convolution Identities of Abel and Hagen-Rothe. (online: `http://www.combinatorics.org/Volume_17/PDF/v17i1n24.pdf`)

Solution 2 by Proposer. First, we will prove by induction that $a_n = n^{n-1}/n!$ for every $n \geq 1$. Indeed, suppose that $a_k = k^{k-1}/k!$ for $1 \leq k \leq n$ for some $n \geq 1$. Noting that

$$\frac{1}{n}\sum_{k=1}^{n} k a_k a_{n+1-k} = \frac{1}{n}\sum_{k=1}^{n}(n+1-k)a_{n+1-k}a_k$$

we conclude that

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n}(n+1-k)a_k a_{n+1-k} &= \frac{1}{2n}\left(\sum_{k=1}^{n}(n+1-k)a_k a_{n+1-k} + \sum_{k=1}^{n} k a_k a_{n+1-k}\right)\\ &= \frac{n+1}{2n}\sum_{k=1}^{n} a_k a_{n+1-k} = a_{n+1}.\end{aligned}$$

Hence,

$$\begin{aligned}a_{n+1} =& \frac{1}{n}\sum_{k=1}^{n}(n+1-k)a_k a_{n+1-k} = \frac{1}{n}\sum_{k=1}^{n}\frac{k^{k-1}}{k!}\cdot\frac{(n+1-k)^{n+1-k}}{(n+1-k)!}\\ &= \frac{1}{n\cdot(n+1)!}\underbrace{\sum_{k=1}^{n}\binom{k}{n+1}k^{k-1}(n+1-k)^{n+1-k}}_{A_n} \qquad (1)\end{aligned}$$

But

$$\begin{aligned}A_n &= \sum_{k=1}^{n}\sum_{\ell=0}^{n+1-k}\binom{k}{n+1}\binom{\ell}{n+1-k}k^{k-1}(n+1)^{\ell}(-k)^{n+1-k-\ell}\\ &= \sum_{\substack{1\leq k\leq n\\ 0\leq \ell\leq n+1-k}}\binom{\ell}{n+1}\binom{k}{n+1-\ell}(-1)^{n+1-k-\ell}(n+1)^{\ell}k^{n-\ell}\\ &= (-1)^{n+1}\sum_{k=1}^{n}\binom{k}{n+1}(-1)^k k^n\\ &+ \sum_{\ell=1}^{n}\binom{\ell}{n+1}(-1)^{n+1-\ell}(n+1)^{\ell}\left(\sum_{k=1}^{n+1-\ell}\binom{k}{n+1-\ell}(-1)^k k^{n-\ell}\right)\end{aligned}$$

Now, we use the fact that for $m \geq 0$ we have

$$\sum_{r=0}^{m+1}\binom{r}{m+1}(-1)^r r^m = \left((1-e^{-z})^{m+1}\right)^{(m)}\Big]_{z=0} = 0,$$

since $z_0 = 0$ is a zero of order $m+1$ to the entire function $z \mapsto (1-e^{-z})^{m+1}$.

Hence,

$$\sum_{k=1}^{n}\binom{k}{n+1}(-1)^k k^n = -(-1)^{n+1}(n+1)^n$$

and

$$\sum_{k=1}^{n+1-\ell}\binom{k}{n+1-\ell}(-1)^k k^{n-\ell}=\begin{cases}0 & \text{if} \quad \ell<n,\\ -1 & \text{if} \quad \ell=n\end{cases}$$

So,

$$A_n=-(n+1)^n+(n+1)^{n+1}=n\cdot(n+1)^n$$

Replacing back in (1) we conclude that

$$a_{n+1}=\frac{(n+1)^n}{(n+1)!}.$$

This concludes the proof of the fact that $a_n=n^{n-1}/n!$ for $n\geq 1$.

Now, since $a_n\sim\frac{e^{-n}}{n\sqrt{2\pi n}}$ we see that the power series $\sum_{n=1}^{\infty}a_nx^n$ defines an analytic function $x\mapsto T(x)$ on $(-1/e,1/e)$ which is continuous on $[-1/e,1/e]$. The problem asks for the value of $T(1/e)$.

Since $xT'(x)=\sum_{n=1}^{\infty}na_nx^n$ then

$$xT'(x)T(x)=\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}ka_ka_{n+1-k}\right)x^{n+1}=\sum_{n=1}^{\infty}na_{n+1}x^{n+1},$$

or,

$$xT'(x)T(x)=\sum_{n=1}^{\infty}(n-1)a_nx^n=xT'(x)-T(x)$$

If $f(x)=T(x)e^{-T(x)}/x$ for $0<x<1/e$, then one checks easily that $f'=0$. Hence f is constant on the interval $(0,1/e)$. This constant is 1 since clearly $\lim_{x\to 0}f(x)=a_1=1$. We conclude that $1=\lim_{x\to 1/e}f(x)=T(1/e)e^{1-T(1/e)}$. Therefore, $T(1/e)$ is a solution of the equation $ye^{1-y}=1$.

Now, the function $y\mapsto ye^{1-y}$ is strictly increasing on $[0,1]$ and strictly decreasing on $[1,\infty)$, it attains its maximum for $y=1$ which is 1. So 1 is the unique solution of the equation $ye^{1-y}=1$, and consequently $T(1/e)=1$, which is the desired answer.

Remark 1. In fact, we have proved that $T(x)e^{-T(x)}=x$ for every $x\in[0,1/e]$, and since clearly T is increasing on $[0,1/e]$, we conclude that $0\leq T(x)\leq T(1/e)=1$. Hence $T(x)$ is the only solution in the interval $[0,1]$ of the equation $ye^{-y}=x$.

Remark 2. This solution has the advantage to be elementary, in the sense that one re-discovers the series expansion of the *Tree function* or *Lambert's function* T without using Lagrange's Theorem about the series development of the inverse function.

Solution 3 by Moti Levy, Rehovot, Israel. Let

$$a_n=\frac{n}{2\,(n-1)}\sum_{k=1}^{n-1}a_ka_{n-k},\quad n>1.$$

$$(n-1)\,a_n=\frac{1}{2}n\sum_{k=1}^{n-1}a_ka_{n-k},\quad n>1.$$

Let $A(z) = \sum_{n=1}^{\infty} a_n z^n$ be the generating function of the sequence $(a_n)_{n\geq 1}$.

$$\sum_{n=1}^{\infty} n a_n z^n - \sum_{n=1}^{\infty} a_n z^n = \frac{1}{2}\sum_{n=2}^{\infty} n \left(\sum_{k=1}^{n-1} a_k a_{n-k}\right) z^n$$

Let $B(z) = \sum_{n=2}^{\infty} b_n z^n$ be the generating function of the sequence $(b_n)_{n\geq 2}$ where

$$b_n = \sum_{k=1}^{n-1} a_k a_{n-k}.$$

$$\begin{aligned}\sum_{n=2}^{\infty} b_n z^n &= \sum_{n=2}^{\infty}\sum_{k=1}^{n-1} a_k a_{n-k} z^n = \sum_{k=1}^{\infty}\sum_{n=k+1}^{\infty} a_k a_{n-k} z^n \\ &= \sum_{k=1}^{\infty} a_k z^k \sum_{n=k+1}^{\infty} a_{n-k} z^{n-k} = \sum_{k=1}^{\infty} a_k z^k \sum_{m=1}^{\infty} a_m z^m = A^2(z).\end{aligned}$$

Differentiation rule for generating functions is

$$zA'(z) = \sum_{n=1}^{\infty} n a_n z^n.$$

$$\begin{aligned}zA'(z) - A(z) &= \frac{1}{2} z \left(A^2(z)\right)' \\ zA'(z) - A(z) &= zA(z)A'(z)\end{aligned}$$

The differential equation

$$z(1 - A(z))A'(z) = A(z)$$

is solved by standard technique,

$$A(z)e^{-A(z)} = kz, \quad A'(0) = 1, \quad A(0) = 0$$

$$A'(z)e^{-A(z)} - A(z)e^{-A(z)} = k.$$

Setting $z = 0$, we get $k = 1$.

$$A(z)e^{-A(z)} = z.$$

Clearly, $\sum_{n=1}^{\infty} a_n e^{-n} = A\left(\frac{1}{e}\right)$.

$$A\left(e^{-1}\right)e^{-A\left(e^{-1}\right)} = e^{-1}$$

implies $A\left(e^{-1}\right) = 1$. We conclude that $\sum_{n=1}^{\infty} a_n e^{-n} = 1$.

Remarks:

1) The generating function $A(z)$ is known as the "Tree function" $T(z)$. It can be expressed by the Lambert W-function $W(z)$.

$$A(z) = -W(-z) = T(z).$$

2) The power series expansion of the Lambert W-function, leads to

$$a_n = \frac{n^{n-1}}{n!}, \quad n \geq 1.$$

Also solved by Albert Stadler, Buchenrain 61, Herrliberg, Switzerland and AN-anduud Problem Solving Group,Ulaanbaatar, Mongolia .

109. *Proposed by Sava Grozdev, Institute of Mathematics and Informatics - BAS, Sofia, Bulgaria, and Deko Dekov, Stara Zagora, Bulgaria (Jointly).* Let H be the orthocenter of $\triangle ABC$. Denote by O_a, O_b and O_c the circumcenters of triangles HBC, HCA and HAB, respectively. For any point P in the plane of $\triangle ABC$, we denote by P_a, P_b and P_c the reflections of P in sidelines BC, CA and AB, respectively, and we define the following property:

(P) The lines O_aP_a, O_bP_b and O_cP_c concur in a point.

Find five named notable points of $\triangle ABC$ which satisfy property (P).

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. We will use the notation $\odot(XYZ)$ to denote the circumcircle of $\triangle XYZ$, and $\mathcal{S}_{XY}$ to denote the symmetry with respect to the line XY.

It is well-known that the $A' = \mathcal{S}_{BC}(H) \in \odot(ABC)$, so $\odot(ABC) = \odot(A'BC)$ and consequently $\mathcal{S}_{BC}(\odot(ABC)) = \odot(HBC)$. This implies that O_a, the center of $\odot(HBC)$, is $\mathcal{S}_{BC}(O)$ where O is the circumcenter of $\odot(ABC)$. Similarly, $O_b = \mathcal{S}_{CA}(O)$ and $O_c = \mathcal{S}_{AB}(O)$.

Thus, if ℓ denotes the line OP then

$$O_aP_a = \mathcal{S}_{BC}(\ell), \quad O_bP_b = \mathcal{S}_{CA}(\ell), \quad O_cP_c = \mathcal{S}_{AB}(\ell).$$

Now, a famous theorem by Collings [S.N. Collings, *Reflections on a triangle, part 1.* Math. Gazette, 57 (1973) 291–293], asserts that the lines $\mathcal{S}_{BC}(\ell)$, $\mathcal{S}_{CA}(\ell)$ and $\mathcal{S}_{AB}(\ell)$ concur in a point if and only if ℓ passes through the orthocenter H of $\triangle ABC$. So, (P) holds if and only if $\ell = OH$ which is the Euler line. This reduces the problem to finding five named notable points of $\triangle ABC$ that belong to the Euler line of this triangle. Well, there are the *centroid*, the *orthocenter*, the *nine point center*, the *Longchamps point*, the *Shiffler point* and the *Exeter point.*

Also solved by Adnan Ali, India; Andrea Fanchini Cantú, Italy and the proposers.

MATHCONTEST SECTION

This section of the Journal offers readers an opportunity to solve interesting and elegant mathematical problems mainly appeared in Math Contest around the world and most appropriate for training Math Olympiads. Proposals are always welcomed. The source of the proposals will appear when the solutions be published.

Proposals

75. Let $f : \mathbb{N} \to \mathbb{N}$ be a function from the positive integers to the positive integers for which $f(1) = 1, f(2n) = f(n)$ and $f(2n+1) = f(n) + f(n+1)$ for all $n \in \mathbb{N}$. Prove that for any natural number n, the number of odd natural numbers m such that $f(m) = n$ is equal to the number of positive integers not greater than n having no common prime factors with n.

76. Let a, b, c, p, q, r be positive integers such that $a^p + b^q + c^r = a^q + b^r + c^p = a^r + b^p + c^q$. Prove that $a = b = c$ or $p = q = r$.

77. Let I be the incenter of $\triangle ABC$ and let H_a, H_b and H_c be the orthocenters of $\triangle BIC, \triangle CIA$ and $\triangle AIB$, respectively. The line H_aH_b meets AB at $X$and the line H_aH_c meets AC at Y. If the midpoint T of the median AM of $\triangle ABC$ lies on XY. Prove that the line H_aT is perpendicular to BC.

78. Let $n \in \mathbb{N}$, $n > 2$ and suppose $a_1, a_2, ..., a_{2n}$ is a permutation of the numbers $1, 2, ..., 2n$ such that $a_1 < a_3 < .... < a_{2n-1}$ and $a_2a_4... > a_{2n}$. Prove that

$$(a_1 - a_2)^2 + (a_3 - a_4)^2 + ... + (a_{2n-1} - a_{2n})^2 > n^3.$$

79. Let $M = \{1, 2, ..., 2013\}$ and let Γ be a circle. For every nonempty subset B of the set M, denote by $S(B)$ sum of elements of the set B, and define $S(\varnothing) = 0$ ($\varnothing$ is the empty set). Is it possible to join every subset B of M with some point A on the circle Γ so that following conditions are fulfilled:

1. Different subsets are joined with different points;
2. All joined points are vertices of a regular polygon;
3. If $A_1, A_2, ..., A_k$ are some of the joined points, $k > 2$, such that $A_1A_2...A_k$ is a regular $k - gon$, then 2014 divides $S(A_1) + S(A_2) + ... + S(A_k)$?

Solutions

70. *Find the minimum real number k such that the inequality*

$$(ab+ac+ad+bc+bd+cd)+4abcd \leq 2(abc+bcd+cda+dab)+k$$

holds for all positive real numbers a, b, c, d whichs are satisfying the equality

$$a+b+c+d=1$$

(Proposed by Ilker Can iek, Turkey)

Solution by by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy.
The right value is $17/64$ and corresponds to taking $a=b=c=d=1/4$. (Computer assisted)
Letting $A=a+b+c+d$ we come to

$$A^2(ab+ac+ad+bc+bd+cd)+4abcd \leq 2(abc+bcd+cda+dab)A+\frac{17}{64}A^4$$

The symmetry allows us to set $a \geq b \geq c \geq d$ and then $c=d+z$, $b=c+z=d+z+y$, $a=b+x=d+z+y+x$. Upon expanding we get

$$\begin{aligned}
&d^2(96x^2+128zy+96z^2+128y^2+128yx+64zx)+\\
&+d(224yx^2+112z^2x+192y^2x+224z^2y+80x^3+112z^3+128y^3+\\
&+320zyx+320zy^2+208zx^2)+\\
&+17x^4+33z^4+16y^4+72yx^3+32y^3x+152z^2yx+96zy^3+76zx^3+\\
&+88z^3y+44z^3x+152z^2y^2+86z^2x^2+144zy^2x+200zyx^2+88y^2x^2
\end{aligned}$$

Also solved by Albert Stadler, Switzerland and Neculai Stanciu, Romania and Titu Zvonaru, Romania and the proposer.

71. *Let ABC be a triangle with $AB > BC > CA$. Let D be a point on AB such that $CD = BC$, and let M be the midpoint of AC. Show that $BD = AC$ and that $\angle BAC = 2\angle ABM$.* (Mexico National Olympiad, 2007)

Solution. We have received no solution of this problem.

72. *The subsets $A_1, A_2, \ldots, A_{2000}$ of a finite set M satisfy $|A_i| > \frac{2}{3}|M|$ for each $i = 1, 2, \ldots, 2000$. Prove that there exists $m \in M$ which belongs to at least* 1334 *of the subsets A_i.* (Greece National Olympiad 2000)

Solution by Neculai Stanciu, Romania and Titu Zvonaru, Romania. Let $x_1, x_2, ..., x_n$ be the elements of the set M. We consider a matrix B with n rows and 2000 columns, such that $b_{ij}=1$ if $x_i \in A_j$ and $b_{ij}=0$ otherwise. By the statement yields that the sum of the elements of matrices B is greater than $2000 \cdot \frac{2n}{3}$. We deduce that there exists at least one of n rows has a sum S, with $S > \left\lfloor \frac{\frac{4000n}{3}}{n} \right\rfloor = \left\lfloor \frac{4000}{3} \right\rfloor = 1333$ so there exists one element whics belongs to at least 1334 of the sets $A_1, A_2, ..., A_{2000}$.

73. *Let n be a positive integer and let k be an odd positive integer. Moreover, let a, b and c be integers (not necessarily positive) satisfying the equations $a^n + kb = b^n + kc = c^n + ka$. Prove that $a = b = c$.*

(Beneluz, 2007)

Solution by Neculai Stanciu, Romania and Titu Zvonaru, Romania.
If n is odd, wlog $b \geq c$, then $a \leq b$ and $c \geq a$. Now, we also have $b \leq c$ so $a = b = c$.
If n is even, then $n = 2m \Rightarrow a^{2m} - b^{2m} = k(c-b) \Leftrightarrow b^{2m} - c^{2m} = k(a-c) \Leftrightarrow c^{2m} - a^{2m} = k(b-a)$.
It is easy to see that a, b, c are congruent modulo 2. Let be $v \geq 1$ the maximum integer such that 2^v divides $a-b$, then 2^{v+1} divides $b-c$, 2^{v+2} divides $c-a$ and 2^{v+3} divides $b-a$, so $a-b$ is divisible by every power of 2, so $a = b$. In a similar way we obtain that $b = c$ so $a = b = c$.

74. *Consider a triangle ABC. Let S be a circumference in the interior of the triangle that is tangent to the sides BC, CA, AB at the points D, E, F respectively. In the exterior of the triangle we draw three circumferences S_A, S_B, S_C. The circumference S_A is tangent to BC at L and to the prolongation of the lines AB, AC at the points M, N respectively. The circumference S_B is tangent to AC at E and to the prolongation of the line BC at P. The circumference S_C is tangent to AB at F and to the prolongation of the line BC at Q. Show that the lines EP, FQ and AL meet at a point of the circumference S.*

(Lusophon Mathematical Olympiad 2013)

Solution by Neculai Stanciu, Romania and Titu Zvonaru, Romania. We use usual notations. We have
$AF = AE = s-a$, $BF = BQ = s-b$, $CE = CP = s-c$, , $BL = s-c$, $CL = s-b$, $LQ = s-c+s-b = a$ and $LP = s-b+s-c = a$. We denote, $E' = AP \cap FQ$, $F' = AQ \cap EP$ and $T = EP \cap FQ$.

Now, by Menelaus theorem in triangle ABP with the transversal $Q-F-E'$ and in triangle ACQ with the transversal $P-E-F'$ we obtain :

$$\frac{QB}{QP} \cdot \frac{E'P}{E'A} \cdot \frac{FA}{FB} = 1 \Rightarrow \frac{E'P}{E'A} = \frac{2a}{s-a}$$

and

$$\frac{PC}{PQ} \cdot \frac{F'Q}{F'A} \cdot \frac{EA}{Ec} = 1 \Rightarrow \frac{F'Q}{F'A} = \frac{2a}{s-a}.$$

Since,

$$\frac{LQ}{LP} \cdot \frac{E'P}{E'A} \cdot \frac{F'A}{F'Q} = \frac{a}{a} \cdot \frac{2a}{s-a} \cdot \frac{s-a}{2a} = 1$$

and by converse of Ceva's theorem yields that the lines EP, FQ, AI are concurrent.

Now, by Van-Aubel theorem we obtain,

$$\frac{AT}{AL} = \frac{AE'}{E'P} + \frac{AF'}{F'Q} \Rightarrow \frac{AT}{AL} = \frac{s-a}{a} \Rightarrow \frac{AL}{TL} = \frac{s}{a}.$$

Let T' be the projection of the point T' on BC. We deduce that, $\frac{AL}{TL} = \frac{h_a}{TT'} \Rightarrow TT' = \frac{ah_a}{s} = 2r$, hence the point T is on the circumference S .

MATHNOTES SECTION

Computer-Discovered Mathematics: Anticevian Corner Products

Sava Grozdev Deko Dekov

Abstract. By using the computer program "Discoverer", we give theorems about anticevian corner products.

Keywords. anticevian corner product, triangle geometry, remarkable point, computer-discovered mathematics, Euclidean geometry, Discoverer.
AMS Subject Classification. 51-04, 68T01, 68T99.

1. Introduction

The computer program "Discoverer", created by the authors, is the first computer program, able easily to discover new theorems in mathematics, and possibly, the first computer program, able easily to discover new knowledge in science. See e.g. [2],[3],[4]. In this paper, by using the "Discoverer", we investigate the anticevian corner products. The paper contains more than 100 theorems about anticevian corner products. We expect that the majority of these theorems are new, discovered by a computer.

Given $\triangle ABC$, the side lengths are denoted $a = BC$, $b = CA$ and $c = AB$. The labeling of triangle centers follows [7]. Hence, X(1) denotes the Incenter, X(2) denotes the Centroid, X(37) is the Grinberg Point, etc. We refer the reader to ([7], Glossary), for the definition of a triangle center.

Let P and Q be triangle centers and let $JaJbJc$ be the anticevian triangle of P. We say that $JaCB$, $CJbA$ and $BAJc$ are the anticevian corner triangles of P. Denote by Ha the Q-triangle center of $JaCB$, by Hb the Q-triangle center of $CJbA$, and by Hc is the Q-triangle center of $BAJc$. If the lines AHa, BHb and CHc concur in a point, we say that the *anticevian corner product of P and Q exists*, and we call the point of concurrence of the lines the *anticevian corner product of P and Q*.

The computer program "Discoverer" has discovered the following theorems:

Theorem 1. *The Anticevian Corner Product of the Incenter and the Orthocenter exists and it is the Spieker Center.*

Theorem 2. *The Triangle of the Orthocenters of the Anticevian Corner Triangles of the Incenter is the Triangle of the Incenters of the Anticevian Corner Triangles of the Centroid.*

Theorem 3. *The Triangle of the Orthocenters of the Anticevian Corner Triangles of the Symmedian Point is the Triangle of the Circumcenters of the Anticevian Corner Triangles of the Centroid.*

In this paper we give a proof of theorem 1 by using barycentric coordinates. Also, we give examples of anticevian corner products, discovered by the "Discoverer". The enclosed files contain 197 examples of anticevian corner products. Of these 107 are points which are included in [7]. Every item in the enclosed List K (or equivalently, row in Table P-X, or equivalently, row in Table X-P) can be rewritten as a theorem. For example, the item 2 of List K rewrites to theorem 1. The rest of 90 examples (available in the enclosed List D) are not included in [7]. We recommend to the reader to prove theorems 2 and 3 and the examples of the anticevian corner products, given in the enclosed files.

2. Preliminaries

In this section we review some basic facts about barycentric coordinates. We refer the reader to [5],[6],[8],[9],[1]. We use barycentric coordinates. A point is an element of $\mathbb{R}^3$ defined up to a proportionality factor, that is,

$$\text{for } \forall k \in \mathbb{R} \setminus \{0\} : P = (u, v, w) \text{ means that } P = (u, v, w) = (ku, kv, kw).$$

The reference triangle ABC has vertices $A = (1,0,0)$, $B = (0,1,0)$ and $C = (0,0,1)$. A point $P = (u, v, w)$ is normalized if $u + v + w = 1$. Given two normalized points $P = (u_1, v_1, w_1)$ and $Q = (u_2, v_2, w_2)$, then ([8], 15, Proposition 1):

$$|PQ|^2 = -a^2vw - b^2wu - c^2uv \tag{1}$$

where $u = u_1 - u_2, v = v_1 - v_2,$ and $w = w_1 - w_2$.

Let DEF be a triangle whose vertices have normalized barycentric coordinates wrt $\triangle ABC$ as follows: $D = (p_1, q_1, r_1)$, $E = (p_2, q_2, r_2)$ and $F = (p_3, q_3, r_3)$. Let P be a point with normalized barycentric coordinates $P = (p, q, r)$ wrt $\triangle DEF$. Then the barycentric coordinates of $P = (u, v, w)$ wrt $\triangle ABC$ are as follows ([8], 30):

$$\begin{array}{rcl} u & = & p_1p + p_2q + p_3r \\ v & = & q_1p + q_2q + q_3r \\ w & = & r_1p + r_2q + r_3r \end{array} \tag{2}$$

The equation of the line joining two points with coordinates (u_1, v_1, w_1) and (u_2, v_2, w_2) is

$$\begin{vmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ x & y & z \end{vmatrix} = 0. \tag{3}$$

Three lines $L_i : p_1x + q_1y + r_1z = 0, i = 1, 2, 3$, are concurrent if and only if

$$\begin{vmatrix} p_1 & q_1 & r_1 \\ p_2 & q_2 & r_2 \\ p_3 & q_3 & r_3 \end{vmatrix} = 0. \tag{4}$$

and the intersection of L_1 and L_2 is the point

$$(q_1r_2 - q_2r_1, r_1p_2 - r_2p_1, p_1q_2 - p_2q_1) \tag{5}$$

3. Proof of Theorem 1

Proof. The Incenter I of $\triangle ABC$ has barycentric coordinates $I = (a, b, c)$, and the anticevian triangle of I have barycentric coordinates $Ja = (-a, b, c)$, $Jb = (a, -b, c)$ and $Jc = (a, b, -c)$.
The side lengths a_1, b_1 and c_1 of $\triangle JaCB$ are as follows (see (1)):

$$a_1 = |BC| = a,\ b_1 = |JaB| = \frac{ac(a+b-c)}{b+c-a},\ c_1 = |JaC| = \frac{ab(a+c-b)}{b+c-a}.$$

The barycentric coordinates of the orthocenter Ha of $\triangle JaCB$ wrt $\triangle JaCB$ are as follows:

$$Ha = \left(\frac{1}{b_1^2 + c_1^2 - a_1^2}, \frac{1}{c_1^2 + a_1^2 - b_1^2}, \frac{1}{a_1^2 + b_1^2 - c_1^2}\right).$$

Hence, the barycentric coordinates of Ha wrt $\triangle JaCB$ are as follows:

$$Ha = (b + c - a, a + b - c, a + c - b).$$

By using (2), we find the barycentric coordinates of Ha wrt $\triangle ABC$ as follows:

$$Ha = (-a, a + c, a + b).$$

The side lengths a_2, b_2 and c_2 of $\triangle CJbA$ are as follows (see (1)):

$$a_2 = |JbA| = \frac{bc(a+b-c)}{a+c-b},\ b_2 = |CA| = b,\ c_2 = |JbC| = \frac{ab(b+c-a)}{a+c-b}.$$

The barycentric coordinates of the orthocenter Hb of $\triangle CJbA$ wrt $\triangle CJbA$ are as follows: $Ha = (a+b-c, a+c-b, b+c-a)$. By using (2), we find the barycentric coordinates of Hb wrt $\triangle ABC$ as follows:

$$Hb = (b + c, -b, a + b).$$

The side lengths a_3, b_3 and c_3 of $\triangle BAJc$ are as follows (see (1)):

$$a_3 = |JcA| = \frac{bc(a+c-b)}{a+b-c},\ b_3 = |JcB| = \frac{ac(b+c-a)}{a+b-c},\ c_3 = |AB| = c.$$

The barycentric coordinates of the orthocenter Hc of $\triangle BAJc$ wrt $\triangle BAJc$ are as follows: $Ha = (a+c-b, b+c-a, a+b-c)$. By using (2), we find the barycentric coordinates of Hc wrt $\triangle ABC$ as follows:

$$Hc = (b + c, a + c, -c).$$

Now by using (3) we find the barycentric equations of the lines AHa, BHb and CHc as follows:

$$\begin{aligned} AHa: \ (a+b)y - (a+c)z &= 0 \\ BHb: \ (a+b)x - (b+c)z &= 0 \\ CHc: \ (a+c)x - (b+c)y &= 0 \end{aligned}$$

By using (4), we prove that these lines concur in a point. Then, by using (5), we find the point of intersection of the lines AHa, BHb and CHc as the point of intersection R of the lines AHa and BHb as follows: $R = (b+c, c+a, a+b)$. Point R is the anticevian corner product of the incenter and the orthocenter. It is easy to see that R is the Spieker center. This completes the proof.

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JUNIOR PROBLEMS

Solutions to the problems stated in this issue should arrive before July 15, 2015

Proposals

36. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" General School, Buzău, Romania*
Prove that if p is a prime number ($p > 3$), then the number $p^2 + 2015$ is multiple of 24.

37. *Proposed by Traşcă Iuliana, Olt, Romania .* Let $x, y, z > 0$. Prove that

$$\frac{2x+2y+4z}{4x+4y+3z} + \frac{2x+4y+2z}{4x+3y+4z} + \frac{4x+2y+2z}{3x+4y+4z} \geq \frac{24}{11}$$

38. *Proposed by Stanescu Florin, Serban Cioculescu school, jud. Dambovita, Romania.* Prove that in a triangle ABC the following inequalities hold.

$$\frac{81}{4}\frac{rR}{p} \leq \frac{w_a^2}{a} + \frac{w_b^2}{b} + \frac{w_c^2}{c} \leq \frac{p(p^2+r^2-8rR)}{4rR}$$

where p is the semiperimeter, r is the incircle, R is the exircle, w_a, w_b, w_c are the bisectors.

39. *Proposed by Roberto Tauraso, Dipartimento di Matematica, Università degli studi " Tor Vergata", Roma, Italy.* Find the sum of the third power of the solutions of the equation

$$\frac{1}{x} + \frac{2}{x-1} + \frac{3}{x-2} + \frac{4}{x-3} = 1$$

40. *Proposed by Valmir Krasniqi, Department of Mathematics, University of Prishtina, Republic of Kosova .*
Find all functions $f : \mathbb{R} \to \mathbb{R}$, such that $f(f(x) + y)) = x + f(f(y) - x)$, for all $x, y \in \mathbb{R}$.

Solutions

31. *Proposed by Ercole Suppa, Teramo, Italy (Dedicated to Italo D'Ignazio)* Let ΔABC be a triangle and let AX, AY be isogonal conjugate cevians with respect to $\angle BAC$. Let P, R be the orthogonal projections of B on AX, AY and let S, Q be the orthogonal projections of C on AX, AY respectively. Show that P, S, Q, R lie on a circle with center the midpoints of BC.

Solution by Titu Zvonaru, Comănesti, and Neculai Stanciu, Buzău, Romania. We denote $\alpha = \angle BAX = \angle CAY$; assume that R is in the interior of the segment and is in the exterior of the segment AY and Q is in the exterior of the segment AY.

We have

$$AS = b\cos(A-\alpha), \quad AP = c\cos\alpha, \quad AR = c\cos(A-\alpha), \quad AQ = b\cos\alpha$$

We deduce that

$$AS \cdot AP = AR \cdot AQ$$

i.e. the points P, S, Q, R lie on a circle.

Let M be the midpoint of the side BC, and M' be the projection of the point M on the line AY. We obtain immediately that

$$\frac{RM'}{RY} = \frac{BM}{BY} \Longrightarrow RM' = \frac{RY \cdot MB}{BY}$$
$$\frac{YM'}{YQ} = \frac{YM}{YC} \Longrightarrow \frac{M'Q}{YQ} = \frac{CM}{YC} \Longrightarrow M'Q = \frac{YQ \cdot CM}{YC}.$$

Since

$$RY = BY\cos(C+\alpha), \quad YQ = CY\cos(C+\alpha), \quad BM = CM$$

it follows that M' is the midpoint of the segment RQ. We deduce that M is on the perpendicular bisector of the segment PS, so M is the center of the circle which passed through the points P, S, Q, R. The proof is complete

Also solved by Adnan Ali, India and the proposer.

32. *Proposed by Pham–Thanh Hung, Department of Mathematics, Can Tho University, Can Tho City, Viet Nam* Let x, y, z be nonnegative real numbers satisfying $xyz = 1$. Prove that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} + \frac{12}{x+y+z+3} \geq 5.$$

Solution by Corneliu Mănescu–Avram, Transportation High School, Ploieşti, Romania (slightly modified by the editor). Defining $p = x+y+z$, $q = xy+yz+zx$, $r = xyz$, we have to prove the inequality

$$\frac{q}{r} + \frac{12}{p+3} \geq 5 \tag{1}$$

From the well-known inequality $x^2+y^2+z^2 \geq xy+yz+zy$ we get

$$(xy+yz+zx)^2 = (xy)^2+(yz)^2+(zx)^2+2xyz(x+y+z) \geq 3xyz(x+y+z)$$

whence it follows

$$q^2 \geq 3pr \tag{2}$$

Taking into account the condition $r = 1$, the inequality (1) is equivalent to

$$q \geq 5 - \frac{12}{p+3} = \frac{5p+3}{p+3} \tag{3}$$

From (2) we have $q^2 \geq 3p$ and therefore for proving (3) it suffices to show that

$$3p \geq \left(\frac{5p+3}{p+3}\right)^2$$

which is equivalent to

$$(p-3)(3p^2+2p+3) \geq 0$$

which holds true because by the AGM

$$p = x + y + z \geq 3(xyz)^{\frac{1}{3}} = 3$$

and $3p^2 + 2p + 3 > 0$. The proof is complete.

Also solved by Nicuşor Zlota, "Traian Vuia"Technical College, Focşani, Romania, Adnan Ali, Student in A.E.C.S-4, Mumbai, India, Titu Zvonaru, Comăneşti, Romania and Neculai Stanciu, Buzău, Romania; and the proposer

33. *Proposed by Arber Igrishta, Vushtrri, Republic of Kosova* Let $p > 2$ be a prime number. A little boy wrote the numbers $1, 2, \cdots, p-1$ on a blackboard. He picks any two numbers a, b, erases them with a sponge and writes the number ab instead. This process continues until only one number is left. Prove that the number left is not divisible by p.

Solution by Adnan Ali, Student in A.E.C.S -4, Mumbai, India We prove that the number left on the board in the end is $(p-1)!$. Firstly, let $a_i = i$ where $1 \leq i \leq p-1$. Now, when the boy picks up any two numbers a_k, a_ℓ, and writes their product $a_k a_\ell$, there are $p-3$ numbers left on the blackboard. And continuing the same way by keeping on picking up any two numbers from the blackboard, erasing them and writing their product instead, he, obviously will eventually reach a stage where there will be only two numbers left on the blackboard. So, now one number out of the two will be the number which the boy did not choose yet and the other will be the product of the rest if the numbers on the blackboard. So, now he must choose the two numbers and write their product instead, giving him the final number $a_1 \cdots a_{p-1} = 1 \cdots (p-1) = (p-1)!$. From Wilson's Theorem, we know that

$$(p-1)! \equiv -1 \pmod p$$

and so the number left on the black board in the end is not divisible by p, precisely what we had set out to prove.

Also solved by Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania, Titu Zvonaru, Comăneşti, Romania and Neculai Stanciu, Buzău, Romania, and the proposer

34. *Proposed by Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest and Neculai Stanciu, "George Emil Palade" School Buzău, Romania.* If $x, y, z > 0$,then prove that in any triangle ABC with area S and side lengths a, b, c holds the Inequality

$$\frac{y+z}{x}a^2 + \frac{z+x}{y}b^2 + \frac{x+y}{z}c^2 \geq 8\sqrt{3} \cdot S$$

Solution by Ioan Viorel Codreanu, Satulung, Maramureş, Romania (expanded by the editor)
Using the AM–GM inequality we have

$$\frac{y+z}{x}a^2 + \frac{z+x}{y}b^2 + \frac{x+y}{z}c^2 \geq 3\sqrt[3]{\frac{(y+z)(z+x)(x+y)}{xyz}(abc)^2}$$

Also by the AM–GM we have

$$(y+z)(z+x)(x+y) \geq 2\sqrt{yz} \cdot 2\sqrt{zx} \cdot 2\sqrt{xy} = 8xyz$$

Then we get

$$\frac{y+z}{x}a^2 + \frac{z+x}{y}b^2 + \frac{x+y}{z}c^2 \geq 3\sqrt[3]{\frac{(y+z)(z+x)(x+y)}{xyz}(abc)^2} \geq 3\sqrt[3]{8(abc)^2}$$

and it suffices to show that

$$6\sqrt[3]{(abc)^2} \geq 8\sqrt{3} \cdot S$$

We use the well known identities

$$S = rp, \qquad rR = \frac{abc}{4p}$$

where as usual r is the incircle, R is the excircle and p
the semiperimeter. The inequality becomes

$$3\sqrt{3}R^2 \geq 4rp$$

and the proof concludes with the two well known inequalities

$$2p \leq 3\sqrt{3}R, \qquad r \leq R/2$$

Also solved by the proposers. One incorrect solution has been received

35. *Proposed by Proposed by Armend Sh. Shabani, University of Prishtina, Republic of Kosova.* If $a + b + c = 0$ show that

$$\frac{a^7 + b^7 + c^7}{7} = \frac{a^4 + b^4 + c^4}{2} \cdot \frac{a^3 + b^3 + c^3}{3}.$$

Solution by Neculai Stanciu, Buzău, Romania and Titu Zvonaru, Comăneşti, Romania
We denote $ab + bc + ac = m$ and $abc = n$. Then a, b and c are the roots of the equation $x^3 + mx - n = 0$ and we also denote $S_k = a^k + b^k + c^k$.

Using Viète' s relations we deduce

$$S_1 = a + b + c = 0, S_2 = -2m, S_3 = +mS_1 - 3n = 0, S_4 + mS_2 - nS_1 = 0,$$

$$S_5 + mS_3 - nS_2 = 0, S_7 + mS_5 - nS_4 = 0$$

So, we obtain that

$$S_3 = 3n, S_4 = 2m^2, S_5 = -5mn, S_7 = 7m^2n.$$

Hence

$$\frac{S_7}{7} = m^2 \cdot n = \frac{S_4}{2} \cdot \frac{S_3}{3}.$$

Also solved by Adnan Ali, Student in A.E.C.S-4, Mumbai, India, Nicuşor Zlota, "Traian Vuia" Technical College, Focşani, Romania, Corneliu Mănescu-Avram, Transportation High School, Ploieşti, Romania and the proposer