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PROBLEMS AND SOLUTIONS

Proposals and solutions must be legible and should appear on separate sheets, each indicating the name of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editors have supplied a solution. The editors encourage undergraduate and pre-college students to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the class and school name. Solutions will be evaluated for publication by a committee of professors according to a combination of criteria. Questions concerning proposals and/or solutions can be sent by e-mail to: *mathproblems-ks@hotmail.com*

Solutions to the problems stated in this issue should arrive before February 15, 2013

Problems

50. *Proposed by Anastasios Kotronis, Athens, Greece.* Show that

$$\sum_{k=0}^{+\infty} \frac{(-1)^k}{\ln(n+k)} = \frac{1}{2\ln n} + \mathcal{O}(n^{-1}\ln^{-2} n) \qquad n \to +\infty.$$

51. *Proposed by D.M. Bătinețu-Giurgiu, Matei Basarab National Colege, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania (Jointly).* Let $a \in (1,\infty)$ and $b \in (0,\infty)$. Calculate

$$\lim_{n\to\infty} n\left(2 - \exp\left(\sum_{k=1}^{n} \frac{(n+k)^{a-1}}{(n+k)^a + b}\right)\right)$$

52. *Proposed by Yuanzhe Zhou, The School of Physics and Technology (SPT) at Wuhan University.* Let $a, b, c > 0$, prove that,

$$\cos\left(\frac{a}{b+c}\right)^2 + \cos\left(\frac{b}{a+c}\right)^2 + \cos\left(\frac{c}{a+b}\right)^2 > \frac{3}{2}$$

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53. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* The Stirling numbers of the first kind denoted by $s(n,k)$, are the special numbers defined by the generating function $z(z-1)(z-2)\cdots(z-n+1) = \sum_{k=0}^{n} s(n,k)z^k$. Let n and m be nonnegative integers with $n > m-1$. Prove that

$$\int_0^1 \frac{\ln^n x}{(1-x)^m}dx = \begin{cases} (-1)^n \cdot n! \cdot \zeta(n+1), & m=1, \\ (-1)^{n+m-1} \cdot \frac{n!}{(m-1)!} \cdot \sum_{i=1}^{m-1}(-1)^i \cdot s(m-1,i) \cdot \zeta(n+1-i), & m \geq 2, \end{cases}$$

where ζ denotes the Riemann zeta function.

54. *Proposed by Moubinool Omarjee, Paris, France.* Let $f : [0,+\infty) \to \mathbb{R}$ be a measurable function such that $g(t) = e^t f(t) \in L^1(\mathbb{R}_+)$; the space of Lebesgue integrable functions. Find

$$\lim_{n\to+\infty} \int_{\mathbb{R}_+} f(t) \left(4\sinh\left(\frac{nt+1}{2}\right)\sinh\left(\frac{nt-1}{2}\right)\right)^{\frac{1}{n}} dt,$$

where $\sinh(x) = \frac{e^x - e^{-x}}{2}$

55. *Proposed by Enkel Hysnelaj, University of Technology, Sydney, Australia.* Let x, y, z, α be real positive numbers. Show that if

$$\sum_{cycl} \frac{nx^3 + (n+1)x}{x^2+1} = \alpha$$

then

$$\sum_{cycl} \frac{1}{x} > \frac{2\alpha}{3} + \frac{27n^3}{9n^2\alpha + \alpha^3}$$

where n is a natural number.

56. *Proposed byJosé Luis Díaz–Barrero, BARCELONA TECH, Barcelona, Spain.* Let $n \geq 2$ be a positive integer. Find all possible values of the number k such that

$$\frac{F_n^2(1+F_{n+1}^2F_{n+2}^2)}{F_{n-1}F_{n+1}} + \frac{F_{n+2}^2(1+F_n^2F_{n+1}^2)}{F_nF_{n+1}} = k + \frac{F_{n+1}^2(1+F_{n+2}^2F_n^2)}{F_{n-1}F_n}$$

57.**Proposed by Naim L. Braha, Department of Mathematics and Computer Sciences, University of Prishtina, Republic of Kosova.* If f_0 and f_1 are constants, then prove for which values of λ, the following series converges:

$$\begin{aligned} f_n = f_1 &\sum_{j=0}^{\lfloor (n-1)/2 \rfloor} \left(2^{n-1-2j} \sum_{1 \ll i_1 \ll \cdots \ll i_j \leq n-2} \prod_{m=1}^{j} \left(\frac{1}{\lambda(i_m+1)^2} - 1 \right) \right) \\ + f_0 &\sum_{j=0}^{\lfloor (n-2)/2 \rfloor} \left(2^{n-2-2j} \sum_{2 \ll i_1 \ll \cdots \ll i_j \leq n-1} \prod_{m=1}^{j} \left(\frac{1}{\lambda(i_m+1)^2} - 1 \right) \right), \end{aligned}$$

where $a \ll b$ if and only if $a+1 < b$.

Solutions

No problem is ever permanently closed. We will be very pleased considering for publication new solutions or comments on the past problems.

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43. *Proposed by D.M. Bătineţu-Giurgiu, Matei Basarab National Colege, Bucharest, Romania and Neculai Stanciu, George Emil Palade Secondary School, Buzău, Romania (Jointly).* Let a be a positive real number and $\Gamma(x)$ be the Gamma function (or Euler's second integral). Calculate

$$\lim_{x\to\infty}\left((x+a)\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}\sin\left(\frac{1}{x+a}\right)-x\left(\Gamma(x+1)\right)^{\frac{1}{x}}\sin\left(\frac{1}{x}\right)\right)$$

Solution by Anastasios Kotronis, Athens, Greece. From Stirling's formula, we have

$$\Gamma(x+1)=\frac{\sqrt{2\pi}x^{x+1/2}}{e^x}\left(1+\mathcal{O}(x^{-1})\right)\qquad x\to+\infty,$$

so

$$\ln\left(\Gamma(x+1)\right)=x\ln x-x+\frac{\ln x}{2}+\frac{\ln 2\pi}{2}+\mathcal{O}(x^{-1})$$

and

$$\begin{aligned}\left(\Gamma(x+1)\right)^{\frac{1}{x}}&=\exp\left(\ln x-1+\frac{\ln x}{2x}+\frac{\ln 2\pi}{2x}+\mathcal{O}(x^{-2})\right)\\&=\frac{x}{e}+\frac{\ln x}{2e}+\frac{\ln 2\pi}{2e}+\mathcal{O}\left(x^{-1}\ln^2 x\right).\end{aligned}$$

Putting $x+1$ in the preceding expression instead of x, we get

$$\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}=\frac{x}{e}+\frac{\ln x}{2e}+\frac{\ln 2\pi}{2e}+\frac{1}{e}+\mathcal{O}\left(x^{-1}\ln^2 x\right).$$

Furthermore,

$$x\sin\left(\frac{1}{x}\right)=1+\mathcal{O}(x^{-2}),$$

$$(x+a)\sin\left(\frac{1}{x+a}\right)=1+\mathcal{O}(x^{-2})$$

and therefore

$$x\left(\Gamma(x+1)\right)^{\frac{1}{x}}\sin\left(\frac{1}{x}\right)=\frac{x}{e}+\frac{\ln x}{2e}+\frac{\ln 2\pi}{2e}+\mathcal{O}\left(x^{-1}\ln^2 x\right)\qquad\text{and}$$

$$(x+a)\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}\sin\left(\frac{1}{x+a}\right)=\frac{x}{e}+\frac{\ln x}{2e}+\frac{\ln 2\pi}{2e}+\frac{1}{e}+\mathcal{O}\left(x^{-1}\ln^2 x\right).$$

On account of the above

$$(x+a)\left(\Gamma(x+2)\right)^{\frac{1}{x+1}}\sin\left(\frac{1}{x+a}\right)-x\left(\Gamma(x+1)\right)^{\frac{1}{x}}\sin\left(\frac{1}{x}\right)=\frac{1}{e}+\mathcal{O}\left(x^{-1}\ln^2 x\right)\to\frac{1}{e}.$$

Also solved by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Konstantinos Tsouvalas, University of Athens, Athens, Greece; and the proposer

44. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let A denote the Glaisher–Kinkelin constant defined by

$$A = \lim_{n\to\infty} n^{-n^2/2-n/2-1/12} e^{n^2/4} \prod_{k=1}^{n} k^k = 1.282427130\ldots.$$

Prove that

$$\sum_{p=1}^{\infty} \frac{\zeta(2p+1)-1}{p+2} = -\frac{\gamma}{2} - 6\ln A + \ln 2 + \frac{7}{6},$$

where ζ is the Riemann zeta function defined by $\zeta(s) = \sum_{k=1}^{\infty} 1/k^s$ for $\Re(s) > 1$.

Solution 1 by Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy. First, we have

$$\sum_{p=1}^{\infty} \frac{\zeta(2p+1)-1}{p+2} = \sum_{k=2}^{\infty}\sum_{p=1}^{\infty} \frac{1}{k^{2p+1}} \frac{1}{p+2}$$

Since

$$\sum_{p=1}^{\infty} \frac{z^{2p+1}}{p+2} = \frac{1}{z^3}\left(\sum_{r=1}^{\infty} \frac{z^{2r}}{r} - z^2 - \frac{z^4}{2}\right) = -\frac{1}{z^3}\left(\ln(1-z^2) + z^2 + \frac{z^4}{2}\right)$$

then

$$\sum_{p=1}^{\infty} \frac{\zeta(2p+1)-1}{p+2} = \sum_{k=2}^{\infty}\sum_{p=1}^{\infty} \frac{1}{k^{2p+2}} \frac{1}{p+2} = -\sum_{k=2}^{\infty} k^3 \left(\ln\left(1-\frac{1}{k^2}\right) + \frac{1}{k^2} + \frac{1}{2k^4}\right)$$

Now, we have

$$\begin{aligned}\sum_{k=2}^{n} k^3 \ln\left(1-\frac{1}{k^2}\right) &= \sum_{k=2}^{n} k^3 \ln(k+1) + \sum_{k=2}^{n} k^3 \ln(k-1) - 2\sum_{k=2}^{n} k^3 \ln k \\ &= \sum_{k=2}^{n} (k+1)^3 \ln(k+1) + \sum_{k=2}^{n} (k-1)^3 \ln(k-1) - 2\sum_{k=2}^{n} k^3 \ln k \\ &+ \sum_{k=2}^{n} \left(3(k-1)\ln(k-1) + 3(k+1)\ln(k+1)\right) \\ &+ \sum_{k=2}^{n} \left(3(k-1)^2\ln(k-1) - 3(k+1)^2\ln(k+1)\right) \\ &+ \sum_{k=2}^{n} \left(\ln(k-1) - \ln(k+1)\right)\end{aligned}$$

and

$$\begin{aligned}\sum_{k=2}^{n} k^3 \ln\left(1-\frac{1}{k^2}\right) &= \Big((n+1)^3\ln(n+1)-n^3\ln n-8\ln 2\Big)\\ &\quad +6\sum_{k=1}^{n} k\ln k-3n\ln n+3(n+1)\ln(n+1)-6\ln 2\\ &\quad +\Big(-3n^2\ln n-3(n+1)^2\ln(n+1)+12\ln 2\Big)-\ln n-\ln(n+1)+\ln 2\\ &= 6\sum_{k=1}^{n} k\ln k-3n^2\ln n+n^2-3n\ln n-\frac{n}{2}-\ln n+\frac{1}{3}\end{aligned}$$

Moreover

$$\sum_{k=2}^{n} k=\frac{n(n+1)}{2}-1, \qquad \sum_{k=2}^{n}\frac{1}{2k}=\frac{1}{2}\ln n+\frac{\gamma}{2}-\frac{1}{2}+o(1)$$

Adding up the three preceding contributions, we get

$$6\sum_{k=1}^{n} k\ln k-3n^2\ln n-3n\ln n-\frac{\ln n}{2}+\frac{3}{2}n^2-\ln 2+\frac{\gamma}{2}-\frac{7}{6}$$

whose limit when $n\to\infty$ is $6\ln A-\ln 2+\frac{\gamma}{2}-\frac{7}{6}$, and the statement follows.

Solution 2 by Anastasios Kotronis, Athens, Greece.
First we observe that for $|x|<1$ is

$$\begin{aligned}\sum_{p\geq 1}\frac{x^{2p+1}}{p+2} &= \frac{1}{x^3}\sum_{p\geq 3}\frac{\left(x^2\right)^p}{p}=-\frac{1}{x^3}\left(\ln(1-x^2)+x^2+\frac{x^4}{2}\right)\\ &= -\frac{x}{2}-\frac{1}{x}-\frac{\ln(1-x^2)}{x^3}\end{aligned}$$

Now, we have

$$\sum_{p=1}^{+\infty}\frac{\zeta(2p+1)-1}{p+2}=\sum_{p\geq 1}\frac{1}{p+2}\left(\sum_{n\geq 1}\frac{1}{n^{2p+1}}-1\right)=\sum_{p\geq 1}\sum_{n\geq 2}\frac{\left(n^{-1}\right)^{2p+1}}{p+2}$$

$$=\sum_{n\geq 2}\sum_{p\geq 1}\frac{\left(n^{-1}\right)^{2p+1}}{p+2}-\sum_{n\geq 2}\left(\frac{1}{2n}+n+n^3\ln\left(1-\frac{1}{n^2}\right)\right)$$

$$=-\lim_{N\to+\infty}\sum_{n=2}^{N}\left(\frac{1}{2n}+n+n^3\ln\left(1-\frac{1}{n^2}\right)\right)=-\lim_{N\to+\infty}\sum_{n=2}^{N}\left(\frac{1}{2n}+n+n^3\ln\left(1-\frac{1}{n^2}\right)\right)$$

$$=-\lim_{N\to+\infty}\left(\frac{H_N-1}{2}+\frac{(N+2)(N-1)}{2}+\ln\left(\prod_{n=2}^{N}\left(\frac{n^2-1}{n^2}\right)^{n^3}\right)\right)$$

$$=-\lim_{N\to+\infty}\left(\frac{H_N}{2}+\frac{N^2}{2}+\frac{N}{2}-\frac{3}{2}+\ln A_N\right)$$

But

$$\begin{aligned}
A_N &= \prod_{n=2}^{N} \frac{(n-1)^{n^3}}{n^{n^3}} \prod_{n=2}^{N} \frac{(n+1)^{n^3}}{n^{n^3}} \\
&= \frac{1}{N^{N^3}} \prod_{k=2}^{N-1} k^{(k+1)^3-k^3} \cdot \frac{(N+1)^{N^3}}{2^{2^3}} \prod_{k=3}^{N} k^{(k-1)^3-k^3} \\
&= \frac{1}{2} \cdot \frac{(N+1)^{N^3}}{N^{(N+1)^3}} \prod_{k=1}^{N} k^{6k} \\
&= \left(N^{-N^2/2-N/2-1/12} e^{N^2/4} \prod_{k=1}^{N} k^k \right)^6 \cdot \frac{(N+1)^{N^3} N^{3N^2+3N+1/2}}{2N^{(N+1)^3} e^{3N^2/2}},
\end{aligned}$$

Since $H_N = \ln N + \gamma + o(1)$ and

$$\ln \frac{(N+1)^{N^3} N^{3N^2+3N+1/2}}{2N^{(N+1)^3} e^{3N^2/2}} = -\frac{N^2}{2} - \frac{N}{2} - \frac{\ln N}{2} + \frac{1}{3} - \ln 2 + o(1),$$

then on account of the preceding, we get

$$\sum_{p=1}^{+\infty} \frac{\zeta(2p+1)-1}{p+2} = \lim_{N\to+\infty} \left(-\frac{\gamma}{2} - 6\ln A + \ln 2 + \frac{7}{6} + o(1) \right)$$

as desired.

Also solved by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; Konstantinos Tsouvalas, University of Athens, Athens, Greece; and the proposer.

45. *Proposed by Moubinool Omarjee, Paris, France.* Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$. Prove that there exist $(A, B, C) \in M_2(\mathbb{Z}) \times M_2(\mathbb{Z}) \times M_2(\mathbb{Z})$ such that $M = A^2 + B^2 + C^2$.

Solution by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia. (a) Let $a - d = 2k+1,\ k \in \mathbb{Z}$ or $a - d = 4k,\ k \in \mathbb{Z}$. Since $2k+1 = (k+1)^2 - k^2$ and $4k = (k+1)^2 - (k-1)^2$, there exists $x, t \in \mathbb{Z}$ such that $a - d = x^2 - t^2$. Then we have $a - 1 - x^2 = d - 1 - t^2$ and let us denote $a - 1 - x^2 = d - 1 - t^2 = yz,\ y, z \in \mathbb{Z}$. Then, we have

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} x^2+yz+1 & b \\ c & t^2+yz+1 \end{pmatrix} \\
&= \begin{pmatrix} x^2+yz & xy+yt \\ xz+zt & t^2+yz \end{pmatrix} + \begin{pmatrix} 1 & b-xy-yt \\ c-xy-yz & 1 \end{pmatrix} \\
&= \begin{pmatrix} x & y \\ z & t \end{pmatrix}^2 + \begin{pmatrix} 1 & 0 \\ c-xy-yz & 0 \end{pmatrix} + \begin{pmatrix} 0 & b-xy-yt \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} x & y \\ z & t \end{pmatrix}^2 + \begin{pmatrix} 1 & 0 \\ c-xy-yz & 0 \end{pmatrix}^2 + \begin{pmatrix} 0 & b-xy-yt \\ 0 & 1 \end{pmatrix}^2.
\end{aligned}$$

(b) Let $a-d=4k+2,\ k\in\mathbb{Z}$. Also we have

$$a-d-2=4k=(k+1)^2-(k-1)^2 \Leftrightarrow a-2-(k+1)^2=d-(k-1)^2.$$

Let $a-2-(k+1)^2=d-(k-1)^2=yz,\ x,y\in\mathbb{Z}$. Then we also can show

$$\begin{aligned}
\begin{pmatrix} a & b \\ c & d \end{pmatrix} &= \begin{pmatrix} 2+(k+1)^2+yz & b \\ c & (k-1)^2+yz \end{pmatrix} \\
&= \begin{pmatrix} (k+1)^2+yz & 2ky \\ 2kz & (k-1)^2+yz \end{pmatrix} \\
&\quad+\begin{pmatrix} 2 & b-2ky \\ c-2kz & 0 \end{pmatrix} \\
&= \begin{pmatrix} k+1 & y \\ z & k-1 \end{pmatrix}^2+\begin{pmatrix} 1 & 0 \\ c-2kz & 0 \end{pmatrix}^2 \\
&\quad+\begin{pmatrix} 1 & b-2ky \\ 0 & 0 \end{pmatrix}^2.
\end{aligned}$$

Also solved by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; and the proposer.

46. *Proposed by Anastasios Kotronis, Athens, Greece, and Serafeim Tsipelis, Ionnina, Greece (Jointly).* Evaluate

$$\sum_{n=1}^{+\infty}(-1)^{n-1}\frac{\zeta(2n)}{n},$$

where ζ is the Riemann zeta function.

Solution 1 by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania. More generally it is known that (cf., [1, Formula 11, p. 160])

$$\sum_{k=1}^{\infty}\zeta(2k)\frac{t^{2k}}{k}=\ln\Gamma(1+t)+\ln\Gamma(1-t),\quad |t|<1.$$

Letting $t\to i$, in the preceding formula, we get

$$\begin{aligned}
\sum_{k=1}^{\infty}(-1)^{k-1}\frac{\zeta(2k)}{k} &= -\ln\left(\Gamma(1+i)\cdot\Gamma(1-i)\right) \\
&= -\ln\left(i\cdot\Gamma(i)\cdot\Gamma(1-i)\right) \\
&= -\ln\left(\frac{\pi\cdot i}{\sin(\pi\cdot i)}\right) \\
&= \ln\left(\frac{e^{\pi}-e^{-\pi}}{2\pi}\right),
\end{aligned}$$

where the last equality follows on account of Euler's formula, $\sin z=\dfrac{e^{iz}-e^{-iz}}{2i}$.

References

[1] H. M. Srivastava and J. Choi, *Series Associated with the Zeta and Related Functions*, Kluwer Academic Publishers, 2001.

Solution 2 by Robinson Higuita and Joel Restrepo(Jointly), University of Antioquia, Colombia. We know that

$$\zeta(2n)=\sum_{k=1}^{\infty}\frac{1}{(k^2)^n},$$

therefore

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\zeta(2n)}{n}=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\sum_{k=1}^{\infty}\frac{1}{(k^2)^n}=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{1}{(k^2)^n}.$$

But

$$\ln\left(1+\frac{1}{k^2}\right)=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}\frac{1}{(k^2)^n}.$$

Thus

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{\zeta(2n)}{n}=\sum_{k=1}^{\infty}\ln\left(1+\frac{1}{k^2}\right)=\ln\left(\prod_{k=1}^{\infty}\left(\frac{1+k^2}{k^2}\right)\right).$$

We see that

$$\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right)=\frac{\sinh\pi}{\pi}$$

All zeros of the entire function $\sinh z$ are first order and they are the points $0,-\pi i,\pi i,\cdots,-n\pi i,n\pi i,\cdots$, the convergence exponent of this sequence obviously is $\tau=1$. Moreover,

$$\sum_{n=1}^{\infty}\frac{1}{|n\pi i|}=\sum_{n=1}^{\infty}\frac{1}{n\pi}=+\infty,\ \text{ while }\ ,\sum_{n=1}^{\infty}\frac{1}{(n\pi)^2}<+\infty$$

i.e. $\kappa=1$. Hence, by Theorem (Hadamard's Factorization)

$$\sinh z=e^{g(z)}z\prod_{n=1}^{\infty}\left(1+\frac{z}{n\pi i}\right)e^{-z/n\pi i}\left(1-\frac{z}{n\pi i}\right)e^{z/n\pi i}=e^{g(z)}z\prod_{n=1}^{\infty}\left(1+\frac{z^2}{(n\pi)^2}\right).$$

Further, the order of the entire function $\sinh z$ obviously is $\rho=1$, and hence $g(z)=c_0+c_1z$. The easiest way of determining the coefficients c_0 and c_1 is to observe that

$$e^{g(z)}=\frac{\sinh z}{z\prod_{n=1}^{\infty}\left(1+\frac{z^2}{(n\pi)^2}\right)}$$

is an even function, and hence $e^{c_0+c_1z}=e^{c_0-c_1z}$. Hence $c_1=0$. Putting $z\to 0$ in the above expression we conclude that $e^{c_0}=1$, and hence $g(z)=0$. Thus,

$$\sinh z=z\prod_{n=1}^{\infty}\left(1+\frac{z^2}{(n\pi)^2}\right).$$

Finally,

$$\frac{\sinh\pi}{\pi}=\prod_{n=1}^{\infty}\left(1+\frac{1}{n^2}\right).$$

Also solved by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Konstantinos Tsouvalas, University of

Athens, Athens, Greece; Joaquin Rivero Rodriguez, Spain; and the proposer.

47. *Proposed by David R. Stone, Georgia Southern University, Statesboro, GA, USA.* Identify the graph of the equation

$$y^4 - 10x^2y^3 + 25x^4y^2 - 50x^6y + 24x^8 = 0$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria. Let

$$G = \{(x, y) \in \mathbb{R}^2 \,|\, y^4 - 10x^2y^3 + 25x^4y^2 - 50x^6y + 24x^8 = 0\}.$$

Clearly,

$$G = \{(0,0)\} \bigcup \left\{(x, y) \in \mathbb{R}^* \times \mathbb{R} : f\left(\frac{y}{x^2}\right) = 0\right\},$$

where $f(t) = t^4 - 10t^3 + 25t^2 - 50t + 24$. Now, a simple study of the variations of f shows that the equation $f(t) = 0$ has exactly two real zeros $\lambda_1 \in (0, 1)$ and $\lambda_2 \in (7, 8)$. A numeric evaluation shows that $\lambda_1 \approx 0.63270759$ and $\lambda_2 \approx 7.49826796$. Cosequently, G is the union of the two parabolas P_1 and P_2 of equations $y = \lambda_1 x^2$ and $y = \lambda_2 x^2$ respectively.

Also solved by the proposer.

48. *Proposed by Florin Stanescu, School Cioculescu Serban, Gaesti, jud. Dambovita, Romania.* Let $f : (0, \infty) \to (0, \infty)$ be a differentiable function for which there exist real numbers a and b, $0 < a < b$ such that

(i): $\int_a^b \frac{f(x)}{x^2}dx = \frac{f(b)}{b} - \frac{f(a)}{a}$

(ii): $\frac{f'(a)}{f(a)} = \frac{2}{a}$

(a) Find an example of a function that satisfies the preceding conditions
(b) Show that there exists a $c \in (a, b)$ such that

$$\frac{f(c)}{f(a)} = \left(\frac{c}{a}\right)^2 e^{(c-a)\left(\frac{f'(c)}{f(c)} - \frac{2}{c}\right)}$$

Solution by Adrian Naco, Polytechnic University, Tirana, Albania.
(a) The function $f(x) = kx^2$, where k is a constant number, satisfy the condition (ii), $\forall x \in (0, \infty)$.

(b) The function F is defined by

$$F(x) = \begin{cases} \dfrac{\ln\left(\dfrac{f(x)}{x^2}\right) - \ln\left(\dfrac{f(a)}{a^2}\right)}{x-a}, & x \in (a, \infty) \\ 0 & x = a \end{cases}$$

which is everywhere continuous and differentiable in $(0, \infty)$ and even continuous in the point a. Indeed from the definition

$$\lim_{x \to a^+} F(x) = \left[\ln\left(\frac{f(x)}{x^2}\right)\right]'_{x=a} = \frac{f^{'}(a)}{f(a)} - \frac{2}{a} = 0 = F(0)$$

On account of (i) and applying the integration by part method we have

$$\int_a^b x\left[\frac{f(x)}{x^2}\right]' \mathrm{d}x = \left[x\frac{f(x)}{x^2}\right]_a^b - \int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x$$

$$= \left[\frac{f(b)}{b} - \frac{f(a)}{a}\right] - \int_a^b \frac{f(x)}{x^2}\,\mathrm{d}x = 0$$

Case 1. If $\left[\frac{f(x)}{x^2}\right]' \geq 0$ or $\left[\frac{f(x)}{x^2}\right]' \leq 0$ for all $x \in (0, \infty)$. Then, considering the above result, we get $\left[\frac{f(x)}{x^2}\right]' = 0$ for all $x \in (0, \infty) \Rightarrow f(x) = kx^2$, with $k \in \mathbb{R}$ in which case, the condition (b) holds for every $c \in (0, \infty)$.

Case 2. There are at least two points $\alpha, \beta \in (a, \infty)$ such that

$$\left[\frac{f(x)}{x^2}\right]'_{x=\alpha} > 0 \qquad \text{and} \qquad \left[\frac{f(x)}{x^2}\right]'_{x=\beta} < 0$$

Since the function $\frac{f(x)}{x^2}$ is continuous and differentiable in $[\alpha, \beta]$, based on Darboux Theorem it implies that there exists a point $\gamma \in (\alpha, \beta)$ such that $\left[\frac{f(x)}{x^2}\right]'_{x=\gamma} = 0$. Thus

$$\lim_{x \to \gamma} F(x) = \left[\ln\left(\frac{f(x)}{x^2}\right)\right]'_{x=\gamma} = \frac{f^{'}(\gamma)}{f(\gamma)} - \frac{2}{\gamma} = 0$$

The function F is continuous in $[\alpha, \gamma]$ and differentiable in (α, γ), thus, considering the fact that $F(\alpha) = F(\gamma) = 0$ and based on Rolle's Theorem, we have that there exists $c \in (\alpha, \gamma)$ such that $F'(c) = 0$ or $\left[\frac{\ln\frac{f(x)}{x^2} - \ln\frac{f(a)}{a^2}}{x-a}\right]'_{x=c} = 0$. That is,

$$\frac{(c-a)\left(\frac{f^{'}(c)}{f(c)} - \frac{2}{c}\right) - \left[ln\frac{f(c)}{c^2} - \ln\frac{f(a)}{a^2}\right]}{c-a} = 0 \Rightarrow (c-a)\left(\frac{f^{'}(c)}{f(c)} - \frac{2}{c}\right) - \left[ln\frac{f(c)}{c^2} - \ln\frac{f(a)}{a^2}\right] = 0$$

or

$$\ln\frac{f(c)}{c^2} = \ln\frac{f(a)}{a^2} + \ln e^{(c-a)\left(\frac{f^{'}(c)}{f(c)} - \frac{2}{c}\right)} \Leftrightarrow \ln\frac{f(c)}{c^2} = \ln\frac{f(a)}{a^2} + \ln e^{(c-a)\left(\frac{f^{'}(c)}{f(c)} - \frac{2}{c}\right)}$$

and

$$\frac{f(c)}{c^2} = \frac{f(a)}{a^2}\, e^{(c-a)\left(\frac{f^{'}(a)}{f(c)} - \frac{2}{c}\right)} \Leftrightarrow \frac{f(c)}{f(a)} = \left(\frac{c}{a}\right)^2 e^{(c-a)\left(\frac{f^{'}(a)}{f(c)} - \frac{2}{c}\right)}$$

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; and the proposer.

49. *Proposed by Mihály Bencze, Braşov, Romania.* Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a} \geq 3+\frac{\sqrt[3]{(a-b)^2(b-c)^2(c-a)^2}}{ab+bc+ca}$$

Solution 1 by Adrian Naco, Polytechnic University, Tirana, Albania. The generalization of the problem 49. Let $x_1, x_2, \ldots x_n$, $(n \geq 3)$, be positive real numbers. Prove that

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1}} \geq 3+\frac{n}{2}\frac{\sqrt[n]{\prod_{i=1}^{n}(x_i-x_{i+1})^2}}{\sum_{i=1}^{n} x_i x_{i+1}},$$

where $x_{n+1}=x_1$. Based on Chauchy-Schwarz inequality we have

$$\left(\sum_{i=1}^{n} x_i\right)^2=\left[\sum_{i=1}^{n} \frac{x_i}{\sqrt{x_i x_{i+1}}}\sqrt{x_i x_{i+1}}\right]^2 \leq \sum_{i=1}^{n}\left(\frac{x_i}{\sqrt{x_i x_{i+1}}}\right)^2 \sum_{i=1}^{n}\left(\sqrt{x_i x_{i+1}}\right)^2$$

$$=\left(\sum_{i=1}^{n} \frac{{x_i}^2}{x_i x_{i+1}}\right)\left(\sum_{i=1}^{n} x_i x_{i+1}\right) \Rightarrow \sum_{i=1}^{n} \frac{x_i}{x_{i+1}}=\left(\sum_{i=1}^{n} \frac{{x_i}^2}{x_i x_{i+1}}\right) \geq \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{\sum_{i=1}^{n} x_i x_{i+1}}$$

Furthermore

$$\sum_{i=1}^{n} \frac{x_i}{x_{i+1}}-3=\left(\sum_{i=1}^{n} \frac{{x_i}^2}{x_i x_{i+1}}\right)-3 \geq \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{\sum_{i=1}^{n} x_i x_{i+1}}-3$$

$$=\frac{\sum_{i=1}^{n} {x_i}^2+2\sum_{1 \leq i<j \leq n}^{n} x_i x_j}{\sum_{i=1}^{n} x_i x_{i+1}}-\frac{3\sum_{i=1}^{n} x_i x_{i+1}}{\sum_{i=1}^{n} x_i x_{i+1}}$$

$$\geq \frac{\sum_{i=1}^{n} {x_i}^2+2\sum_{i=1}^{n} x_i x_{i+1}}{\sum_{i=1}^{n} x_i x_{i+1}}-\frac{3\sum_{i=1}^{n} x_i x_{i+1}}{\sum_{i=1}^{n} x_i x_{i+1}} \geq \frac{\sum_{i=1}^{n} {x_i}^2-\sum_{i=1}^{n} x_i x_{i+1}}{\sum_{i=1}^{n} x_i x_{i+1}}=\frac{\frac{1}{2}\left[\sum_{i=1}^{n}(x_i-x_{i+1})^2\right]}{\sum_{i=1}^{n} x_i x_{i+1}}$$

$$\geq \frac{1}{2}\frac{n\sqrt[n]{\prod_{i=1}^{n}(x_i-x_{i+1})^2}}{\sum_{i=1}^{n}x_ix_{i+1}} = \frac{n}{2}\frac{\sqrt[n]{\prod_{i=1}^{n}(x_i-x_{i+1})^2}}{\sum_{i=1}^{n}x_ix_{i+1}}$$

The last inequality is based on the well-known AM-GM inequality

$$a_1 + a_2 + \dots a_n \geq n\sqrt[n]{a_1a_2\dots a_n}$$

where $a_1, a_2, \dots, a_n \in R^+$, and $a_i = (x_i - x_{i+1})^2$, $\forall i \in \{1, 2, \dots, n\}$.
The equality holds only for $n = 3$ and $x_i = x_2 = \dots = x_n$. For $n \geq 4$ the equality is not true because

$$\sum_{1\leq i<j\leq n}^{n} x_ix_j > \sum_{i=1}^{n} x_ix_{i+1}$$

Solution 2 by Eric Milesi Vidal, BARCELONA TECH, Barcelona, Spain.
First we subtract 3 from each side of the inequality

$$\frac{a-b}{b} + \frac{b-c}{c} + \frac{c-a}{a} \geq \frac{\sqrt[3]{(a-b)^2(b-c)^2(c-a)^2}}{ab+bc+ca}$$

Now we prove that the following inequalities hold

$$\sum_{cyclic}(a-b)ca(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) \geq \frac{(a-b)^2+(b-c)^2+(c-a)^2}{3} \geq \sqrt[3]{(a-b)^2(b-c)^2(c-a)^2}$$

RHS is holds applying $AM-GM$ inequality. Equality holds if and only if $a = b = c$.
To prove LHS inequality, we have see that

$$\sum_{cyclic} a^2 - ba - bc + \frac{ca^2}{b} \geq \frac{(a-b)^2+(b-c)^2+(c-a)^2}{3} \Leftrightarrow \sum_{cyclic} a^2 + \frac{3ca^2}{b} \geq 4\sum_{cyclic} ab,$$

or equivalently,

$$\sum_{cyclic} a^2 \geq \sum_{cyclic} ab \quad \text{and} \quad 3\times\sum_{cyclic}\frac{ca^2}{b} \geq 3\times\sum_{cyclic} ab$$

The first inequality is well-known. For the second, we may assume WLOG that $a \geq b \geq c$ so $\frac{cb}{a} \leq \frac{ca}{b} \leq \frac{ba}{c}$ Since both sequences are sorted in the opposite way by applying the rearrangement inequality, we get

$$ab + bc + ca \leq \frac{ca^2}{b} + \frac{ab^2}{c} + \frac{bc^2}{a}$$

Equality holds if and only if $a = b = c$ and we are done.

Also solved by AN-anduud Problem Solving Group, Ulaanbaatar, Mongolia; Paolo Perfetti, Department of Mathematics, Tor Vergata University, Rome, Italy; Ioan Viorel Codreanu, Satulung, Maramure, Romania and the proposer.

MATHCONTEST SECTION

This section of the Journal offers readers an opportunity to solve interesting and elegant mathematical problems mainly appeared in Math Contest around the world and most appropriate for training Math Olympiads. Proposals are always welcomed. The source of the proposals will appear when the solutions be published.

Proposals

36. Find all positive integers n such that $17^{n-1}+19^{n-1}$ divides 17^n+19^n.

37. Let α, β and γ be three distinct complex numbers. Show that they are collinear if, and only if, $\mathrm{Im}(\alpha\overline{\beta}+\beta\overline{\gamma}+\gamma\overline{\alpha})=0$.

38. Through the midpoint M of a chord PQ of a circle, any other chords AB and CD are drawn; chords AD and BC meet PQ at points X and Y respectively. Prove that M is the midpoint of XY.

39. The students of a University Course in Mathematics take their exams in Calculus, Algebra, Physics and Geometry. It is known that 73% passed Calculus exam, 82% passed Algebra, 77% passed Physics and 89% passed Geometry. At least, how many students have passed the exam of all four subjects?

40. Let α, β, γ be the angles of an acute triangle ABC. Prove that

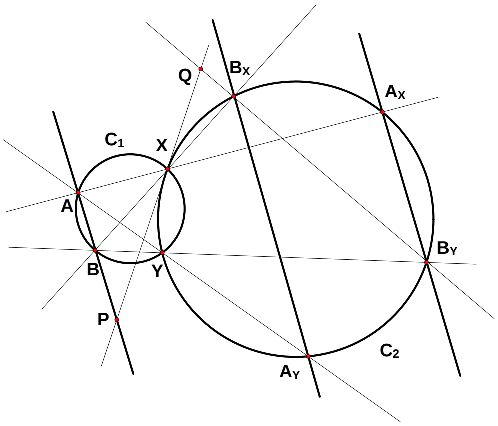
$$\frac{1}{3}\sum_{cyclic}\frac{\tan^2\alpha}{\tan\beta\,\tan\gamma}+3\left(\frac{1}{\tan\alpha+\tan\beta+\tan\gamma}\right)^{2/3}\geq 2$$

Solutions

32. *Two circles $\mathcal{C}_1$ and $\mathcal{C}_2$ have in common two points X and Y. We draw a line that cuts $\mathcal{C}_1$ at points A and B. Next we draw the lines AX, AY which cut $\mathcal{C}_2$ at points A_X and B_Y and lines BX, BY which cut $\mathcal{C}_2$ at points B_X and B_Y respectively. Show that the three lines AB, A_XB_Y and A_YB_X are parallel.*

(Short List of International Mathematical Arhimede Contest 2012)

Solution by Guillem Alsina Oriol, BARCELONA TECH, Barcelona, Spain. In the following figure we observe that $\widehat{BAY} = \widehat{BXY}$ by spanning arc (or capacius arc), and $\widehat{BXY} = \widehat{BXP} + \widehat{PXY}$. Moreover, $\widehat{BXP}$ is equal to $\widehat{QXB_X}$ (opposite by the vertex), and they are equal to $\widehat{XA_YB_X}$ on account of the half inscribed angle. Also by using again the half inscribed angle property, we have $\widehat{PXY} = \widehat{YA_YX}$.



So, we have

$$\widehat{BAY} = X\widehat{A_YB_X} + Y\widehat{A_YX} = Y\widehat{A_YB_X}$$

Since line AA_Y cuts lines AB and A_YB_X and the angles $\widehat{BAY}$ and $Y\widehat{A_YB_X}$ are equal, then lines AB and A_YB_X are parallel.

Now, we only need to see that A_XB_Y is also parallel to these two lines. It will be suffice to see that the angles $A_X\widehat{B_YB_X}$ and $A_Y\widehat{B_XB_Y}$ are equal. We know that $A_X\widehat{B_YB_X}$ is equal to $A_X\widehat{XB_X}$ by spanning arc. Furthermore, the last one is equal to $\widehat{AXB}$ because they are angles opposite by the vertex. Likewise, $A_Y\widehat{B_XB_Y}$ is equal to $\widehat{A_YYB_Y}$, which is equal to $\widehat{AYB}$. Finally, since $\widehat{AXB} = \widehat{AYB}$ by spanning arc, then $A_X\widehat{B_YB_X} = A_Y\widehat{B_XB_Y}$, and the three lines A_XB_Y, A_YB_X and AB are parallel. □

Also solved by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain, Adrian Naco, Polytechnic University, Tirana, Albania, Omran

Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria and José Gibergans-Báguena, BARCELONA TECH, Barcelona, Spain.

33. *Four dice are thrown at the same time on a table. Find the probability that the sum of the points appeared in the upper faces lies between* 14 *and* 18 *points.*

(Training Sessions of Spanish Team for IMO 2012)

Solution by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain. We have to count how many possibilities there are to obtain sums of $15, 16$ and 17 points, respectively. To do it we consider the polynomial

$$z + z^2 + z^3 + z^4 + z^5 + z^6$$

Here, the powers of z keep track of the different faces of the dice and the coefficients of the powers of z show the number of occurrences of each face. The second, third and fourth dies represented by the same polynomial and the outcome of throwing are represented quite naturally by the polynomial

$$(z+z^2+z^3+z^4+z^5+z^6)(z+z^2+z^3+z^4+z^5+z^6)(z+z^2+z^3+z^4+z^5+z^6)(z+z^2+z^3+z^4+z^5+z^6)$$

By expanding this, we get

$$z^4+4\,z^5+10\,z^6+20\,z^7+35\,z^8+56\,z^9+80\,z^{10}+104\,z^{11}+125\,z^{12}+140\,z^{13}+146\,z^{14}$$

$$+140\,z^{15}+125\,z^{16}+104\,z^{17}+80\,z^{18}+56\,z^{19}+35\,z^{20}+20\,z^{21}+10\,z^{22}+4\,z^{23}+z^{24}$$

and we find that there is one way of obtaining a sum score of 4; there are four ways of getting a sum score of 5; ten ways of getting a sum score of 6 and so on. Thus, the number of favorable sums are the coefficients of z^{15}, z^{16}, z^{17}, respectively. That is, $\mathcal{N}_f = 140 + 125 + 104 = 369$ and the number possible sums $\mathcal{N}_p = 1296$. Therefore, the probability required is

$$\mathcal{P}(14 < s < 18) = \frac{\mathcal{N}_f}{\mathcal{N}_p} = \frac{369}{1296} = \frac{41}{144}$$

□

Also solved by José Gibergans-Báguena, BARCELONA TECH, Barcelona, Spain, Eric Milesi Vidal, BARCELONA TECH, Barcelona, Spain, and Oscar Rivero Salgado, BARCELONA TECH, Barcelona, Spain.

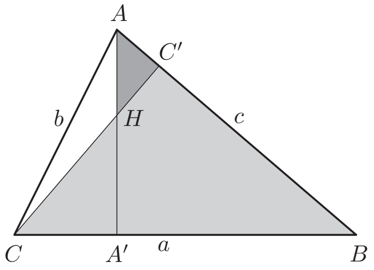
34. *Let P be an interior point of triangle ABC and let H_A, H_B, H_C be the orthocenters of triangles PBC, PAC, and PAB, respectively. Prove that triangles $H_AH_BH_C$ and ABC have the same area.*

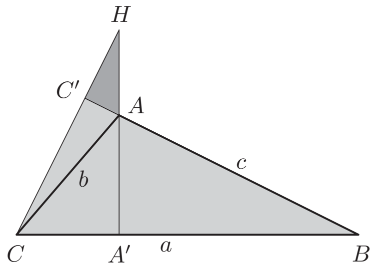
(Catalonian Mathematical Olympiad 2012)

Solution by José Luis Díaz-Barrero BARCELONA TECH, Barcelona, Spain. First, we compute the distance from vertex A to the orthocenter of $\triangle ABC$. That is, $|AH| = \text{dist}(A, H)$. Since $\triangle BCC' \sim \triangle AHC'$, then

$$\frac{AH}{CB} = \frac{AC'}{CC'} \Leftrightarrow \frac{AH}{a} = \frac{AC'}{CC'} \Rightarrow AH = a\,\frac{AC'}{CC'}$$

Furthermore, in triangle ACC' we have $AC' = b\cos A$ and $CC' = b\sin A$. Substi-





tuting in the preceding expression yields

$$\frac{AH}{a} = \frac{b\cos A}{b\sin A} = |\cot A|$$

from which follows $AH = a\,|\cot A|$. Notice that if $\triangle ABC$ is a right triangle with $\widehat{A} = 90°$ the expression is also valid, but in this particular case is $A = H$ and $AH = a\,\cot 90° = 0$. If $\widehat{A} > 90°$, then point H lie on the exterior of $\triangle ABC$ and we have $AC' = b\,\cos(180° - A)$ and $CC' = b\,\sin(180° - A)$. So, $AH = a\,\cot A$ and in this case is $\cot A < 0$. Likewise, we can obtain the distances from H to the acute vertices (angles) of a right or obtuse triangle ABC.

Joining the arbitrary point with the vertices A, B, C we get the triangles PAB, PBC, and PCA. Let $\alpha = \angle BPC, \beta = \angle APB, \gamma = \angle APC$. Obviously $\alpha + \beta + \gamma = 360°$. Al least two of these three angles are obtuse and the other is obtuse, right or acute. So we will examine these three cases separately.

(i) Suppose that the three angles are obtuse. Then, we have $PH_A = -a\cot\alpha$ and $PH_C = -c\cot\gamma$ on account of the preceding. Notice that the angle $y = \angle H_APH_C = 180° - \angle APC = 180° - B$ because the sides H_AP and H_CP are, respectively, perpendicular to the sides BC and AB and one is obtuse and the other acute. The area $\mathcal{A}(PH_AH_C)$ of triangle PH_AH_C is

$$\mathcal{A}(PH_AH_C) = \frac{\overline{PH}_A\,\overline{PH}_C\sin y}{2} = \frac{ac\cot\alpha\cot\gamma\sin B}{2} = \mathcal{A}(ABC)\cot\alpha\cot\gamma$$

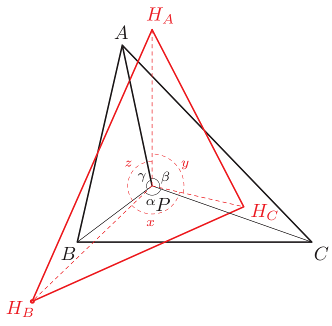


Figura 1

Adding up the areas of the three triangles PH_AH_B, PH_BH_C y PH_CH_A we obtain

$$\mathcal{A}(H_AH_BH_C) = \mathcal{A}(PH_AH_B) + \mathcal{A}(PH_BH_C) + \mathcal{A}(PH_CH_A) \quad \text{and}$$

$$\mathcal{A}(H_A H_B H_C) = \updownarrow A(ABC)\Big(\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha\Big)$$

Since $\alpha + \beta = 360 - \gamma$, then we have $\cot(\alpha+\beta) = -\cot\gamma$ or, equivalently,

$$\cot\gamma = -\cot(\alpha+\beta) = \frac{1 - \cot\alpha\cot\beta}{\cot\alpha + \cot\beta}$$

or

$$\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha = 1 \tag{1}$$

from which follows $\mathcal{A}(H_A H_B H_C) = \mathcal{A}(ABC)$.

(ii) Assume that one of the angles is right, say $\beta = 90°$ (Figura 2). Then $H_B = P$ and

$$\mathcal{A}(H_A H_B H_C) = \mathcal{A}(H_A P H_C) = \frac{\overline{PH_A}\,\overline{PH_C}\sin y}{2} = \frac{ac\cot\alpha\cot\gamma\sin B}{2} = \mathcal{A}(ABC)\cot\alpha\cot\gamma$$

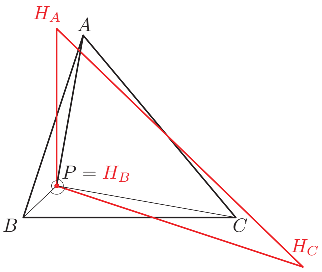


Figura 2

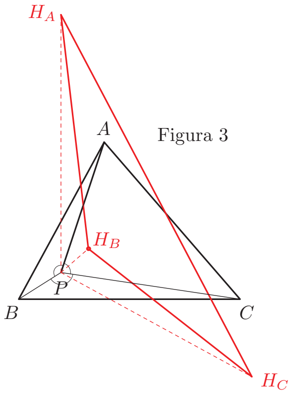


Figura 3

On account of (1), if $\cot\beta = 0$ then $\cot\alpha\cot\gamma = 1$, and the results follows in this case.

(iii) Suppose that one of the angles α, β, γ is acute, say $\widehat{APC} = \beta < 90°$ (Figura 3). Point P is exterior to the triangle $H_A H_B H_C$ and we have $\mathcal{A}(H_A H_B H_C) = \mathcal{A}(PH_A H_C) - \mathcal{A}(PH_A H_B) - \mathcal{A}(PH_C H_B)$. But in this case we have $PH_B = b\cot\beta$, $PH_A = -a\cot\alpha$ and $PH_C = -c\cot\gamma$. So,

$$\begin{aligned}\mathcal{A}(H_A H_B H_C) &= \mathcal{A}(PH_A H_C) - \mathcal{A}(PH_A H_B) - \mathcal{A}(PH_C H_B)\\ &= \big(\cot\alpha\cot\gamma - (-\cot\alpha\cot\beta) - (-\cot\gamma\cot\beta)\big)\mathcal{A}(ABC)\\ &= (\cot\alpha\cot\beta + \cot\beta\cot\gamma + \cot\gamma\cot\alpha)\mathcal{A}(ABC) = \mathcal{A}(ABC)\end{aligned}$$

□

Also solved by José Gibergans-Báguena, BARCELONA TECH, Barcelona, Spain, Eric Milesi Vidal, BARCELONA TECH, Barcelona, Spain, Oscar Rivero Salgado, BARCELONA TECH, Barcelona, Spain, and Ioan Viorel Codreanu, Satulung, Maramure, Romania.

35. *Compute*

$$\lim_{n\to\infty}\frac{1}{n^3}\int_0^n \frac{n^2+x^2}{5^{-x}+7}\,dx$$

(Training Sessions of Spanish Math Team for IMC 2012)

Solution by José Gibergans-Báguena, BARCELONA TECH, Barcelona, Spain. We begin with a Lemma.

Lemma. *Let* $f : (0, +\infty] \to \mathbb{R}$ *and* $g : [0, 1] \to \mathbb{R}$ *be continuous functions. If* $\lim_{x\to\infty} f(x) = \ell$, *then*

$$\lim_{n\to\infty} \frac{1}{n} \int_0^n f(x) g\left(\frac{x}{n}\right)\, dx = \ell \int_0^1 g(x)\, dx$$

Proof. Since the function $h(x) = f(x) - \ell$ tends to zero when $x \to \infty$, then

$$\frac{1}{n} \int_0^n f(x) g\left(\frac{x}{n}\right)\, dx = \frac{1}{n} \int_0^n h(x) g\left(\frac{x}{n}\right)\, dx + \frac{\ell}{n} \int_0^n g\left(\frac{x}{n}\right)\, dx \tag{2}$$

Since $g(x)$ is continuous in $[0, 1]$, then $|g(x)| \leq M$ for some $M \in \mathbb{R}$ and

$$\left| \frac{1}{n} \int_0^n h(x) g\left(\frac{x}{n}\right)\, dx \right| \leq \frac{M}{n} \int_0^n |h(x)|\, dx \tag{3}$$

If $H(x)$ is a primitive of $|h(x)|$ then, applying L'Hospital's rule, we have

$$\lim_{n\to\infty} \frac{H(n)}{n} = \lim_{x\to\infty} |h(x)| = 0$$

and the RHS of (3) tends to zero when $n \to \infty$. Now, setting $x/n = t$, we have

$$\frac{\ell}{n} \int_0^n g\left(\frac{x}{n}\right)\, dx = \ell \int_0^1 g(t)\, dt$$

and from (1) the proof immediately follows.

Setting $f(x) = \dfrac{1}{5^{-x} + 7}$ and $g(x) = 1 + x^2$ into the preceding Lemma and taking into account that $\lim_{x\to\infty} \dfrac{1}{5^{-x} + 7} = \dfrac{1}{7}$, we get

$$\lim_{n\to\infty} \frac{1}{n^3} \int_0^n \frac{n^2 + x^2}{5^{-x} + 7}\, dx \quad = \quad \frac{1}{7} \int_0^1 g(x)\, dx = \frac{1}{7} \left| \frac{x^3}{3} + x \right|_0^1 = \frac{4}{21}$$

and we are done. □

Also solved by José Luis Díaz-Barrero, BARCELONA TECH, Barcelona, Spain, Anastasios Kotronis, Athens, Greece, Ioan Viorel Codreanu, Satulung, Maramure, Romania, Paolo Perfetti, Department of Mathematics, Tor Vergata, University, Rome, Italy and Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

Editors Comment. Problem 31 was retracted by the proposer.

MATHNOTES SECTION

A mean-value formula for the floor function on integers

Wang Xingbo

Abstract. The paper first proves several inequalities related with the floor function, and then deduces and proves a mean-value formula for the floor function with an integer variable. The inequalities and the formulae are useful in some aspects related to analysis and computation of the floor function.

1. Introduction

The floor function and the ceiling function are two special functions that have integer values, and they are widely applied in number theory, discrete mathematics, calculus and computer science. A general introduction to the two functions can be seen in detail in Graham's book [1]. Readers can see that the properties of the two functions are miraculous and fascinating. However, it also can be seen that there still remain quite a lot of problems for us to study. Due to having discrete characteristics of integers, the two functions are still lack of analytic tools like the mean-value theorem in calculus to analyze their intermediary status. Hence it is worth to draw a mean-value theorem for either of the two functions.
In a study on problems of binary trees, I found and proved the following formula

$$\left\lfloor \frac{\alpha+\delta}{2^{z(\alpha)}} \right\rfloor = \left\lfloor \frac{\alpha}{2^{z(\alpha)}} \right\rfloor + \left\lfloor \frac{\delta-1}{2^{z(\alpha)}} + \frac{1}{2} \right\rfloor$$

Since the previous expression is quite similar to the formula $f(x_0+\Delta x) = f(x_0) + f'(\xi)\Delta x$ in calculus, I called it a *mean-value formula of the floor function.* This paper mainly presents the way to obtain and to prove it.

2. Preliminaries

This section presents some necessary preliminaries for later sections.

2.1. Definition and Symbols.

The floor function of a real number x is denoted by $\lfloor x \rfloor$ and it satisfies $\lfloor x \rfloor \le x < \lfloor x \rfloor + 1$; the fractional part of x is denoted by $\{x\}$ and it satisfies $x = \lfloor x \rfloor + \{x\}$; and the ceiling function of x is denoted by $\lceil x \rceil$ verifying $x \le \lceil x \rceil < x+1$. The function $z(\alpha)$, which is defined in [2], represents the position of the first 0-bit that occurs from the least significant bit (lsb) of α's binary representation, e.g., $z(0) = z((00000000)_2) = 1, z(1) = z((00000001)_2) = 2, z(83) = z((01010011)_2) = 3$.

2.2. Lemmas.

The following lemmas are found respectively in Graham's book [1], and Wang's paper[2].

Lemma 1. *For arbitrary real numbers x, y and an integer n, it holds*

(P1): $\lfloor x \rfloor + \lfloor y \rfloor \le \lfloor x+y \rfloor \le \lfloor x \rfloor + \lfloor y \rfloor + 1$

(P2): $\lfloor x\rfloor - \lfloor y\rfloor - 1 \leq \lfloor x-y\rfloor \leq \lfloor x\rfloor - \lfloor y\rfloor < \lfloor x\rfloor - \lfloor y\rfloor + 1$
(P3): $\lfloor n+x\rfloor = n + \lfloor x\rfloor$
(P4): $n\lfloor x\rfloor \leq \lfloor nx\rfloor; n\lfloor x\rfloor = \lfloor nx\rfloor \Leftrightarrow n\{x\} < 1$
(P5): $\lfloor nx\rfloor = \lfloor x\rfloor + \lfloor x+\frac{1}{n}\rfloor + ... + \lfloor x+\frac{n-1}{n}\rfloor$,*particularly,*$\lfloor x\rfloor + \lfloor x+\frac{1}{2}\rfloor = \lfloor 2x\rfloor$
(P6): $\lceil \frac{n}{m}\rceil = \lfloor \frac{n-1}{m}\rfloor + 1$*, where* $m > 0$ *is an integer.*

Lemma 2. *For integer* $\alpha \geq 0, z(\alpha)$ *is the smallest positive solution of the following modulo-inequality with unknown* x*.*

$$0 \leq \alpha \bmod 2^x < 2^{x-1}$$

3. MAIN RESULTS

In the following we present our main results. We begin with (see [4])

Theorem 1. *For arbitrary real numbers* x, y*, it holds*
i. $\lfloor x\rfloor > \lfloor y\rfloor \Rightarrow x > y; \lfloor x\rfloor < \lfloor y\rfloor \Rightarrow x < y$
ii. $x \leq y \Rightarrow \lfloor x\rfloor \leq \lfloor y\rfloor; x \geq y \Rightarrow \lfloor x\rfloor \geq \lfloor y\rfloor$

The following result shows the case $\lfloor x\rfloor = \lfloor y\rfloor$.

Theorem 2. *For a real number* ξ *and a real number* $\delta > 0$*,if* $\lfloor \xi+\delta\rfloor = \lfloor \xi\rfloor$*, then for an arbitrary real number* ω *such that* $0 \leq \omega \leq \delta$*, it holds*

$$\lfloor \xi+\omega\rfloor = \lfloor \xi\rfloor$$

For a real number $\rho < 0$*, if* $\lfloor \xi+\rho\rfloor = \lfloor \xi\rfloor$*, then for an arbitrary real number* η *such that* $\rho \leq \eta \leq 0$*, it holds*

$$\lfloor \xi+\eta\rfloor = \lfloor \xi\rfloor$$

Proof. Since $0 \leq \omega \leq \delta$, then

$$\lfloor \xi\rfloor \leq \xi \leq \xi+\omega \leq \xi+\delta < \lfloor \xi+\delta\rfloor + 1$$

and the condition $\lfloor \xi+\delta\rfloor = \lfloor \xi\rfloor$ becomes

$$\lfloor \xi\rfloor \leq \xi+\omega < \lfloor \xi\rfloor + 1$$

That is the definition of the floor function. Next, we prove the second conclusion. Indeed, by the condition $\rho < 0$ and $\rho \leq \eta \leq 0$, yields

$$\lfloor \xi+\rho\rfloor \leq \xi+\rho \leq \xi+\eta \leq \xi < \lfloor \xi\rfloor + 1$$

Considering $\lfloor \xi+\delta\rfloor = \lfloor \xi\rfloor$, it immediately leads to

$$\lfloor \xi\rfloor \leq \xi+\eta < \lfloor \xi\rfloor + 1$$

Hence the theorem holds.

Theorem 3. *For an arbitrary integer* $\alpha \geq 0$*, it holds*

$$2\left\lfloor \frac{\alpha}{2^{z(\alpha)}}\right\rfloor = \left\lfloor \frac{\alpha}{2^{z(\alpha)-1}}\right\rfloor \tag{4}$$

Proof. For convenience, let $I = z(\alpha)$. By lemma 2, I satisfies $0 \leq \alpha \bmod 2^I < 2^{I-1}$ which is equal to $0 \leq 2(\frac{\alpha \bmod 2^I}{2^I}) < 1$. Note that

$$0 \leq 2(\frac{\alpha \bmod 2^I}{2^I}) < 1 \Leftrightarrow 2\{\frac{\alpha}{2^I}\} < 1$$

By the property (P4) of lemma 1, it yields

$$0 \le \alpha \bmod 2^I < 2^{I-1} \Leftrightarrow 0 \le 2(\frac{\alpha \bmod 2^I}{2^I}) < 1 \Leftrightarrow 2\{\frac{\alpha}{2^I}\} < 1 \Leftrightarrow 2\left\lfloor \frac{\alpha}{2^I} \right\rfloor = \left\lfloor \frac{\alpha}{2^{I-1}} \right\rfloor$$

which proves the statement.

Theorem 4. *For an arbitrary integer $\alpha \ge 0$, it holds*

$$\left\lfloor \frac{\alpha-1}{2^{z(\alpha)}} - \frac{1}{2} \right\rfloor = \left\lfloor \frac{\alpha}{2^{z(\alpha)}} - \frac{1}{2} \right\rfloor = \left\lfloor \frac{\alpha}{2^{z(\alpha)}} \right\rfloor - 1$$

$$\left\lfloor \frac{\alpha+1}{2^{z(\alpha)}} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{\alpha}{2^{z(\alpha)}} \right\rfloor + 1$$

$$\left\lfloor \frac{\alpha+1}{2^{z(\alpha)}} - \frac{1}{2} \right\rfloor = \left\lfloor \frac{\alpha}{2^{z(\alpha)}} \right\rfloor = \left\lfloor \frac{\alpha-1}{2^{z(\alpha)}} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{\alpha+1}{2^{z(\alpha)}} \right\rfloor = \left\lfloor \frac{\alpha}{2^{z(\alpha)}} + \frac{1}{2} \right\rfloor$$

Proof. Also use the symbol $I = z(\alpha)$ for convenience. In the case $I = 1$, it is obvious that

$$\begin{cases} \frac{\alpha-1}{2^I} - \frac{1}{2} < \frac{\alpha}{2^I} - \frac{1}{2} = \frac{\alpha-1}{2^I} < \frac{\alpha+1}{2^I} - \frac{1}{2} = \frac{\alpha}{2^I} \\ \frac{\alpha}{2^I} = \frac{\alpha-1}{2^I} + \frac{1}{2} < \frac{\alpha+1}{2^I} = \frac{\alpha}{2^I} + \frac{1}{2} < \frac{\alpha+1}{2^I} + \frac{1}{2} \end{cases} \tag{5}$$

With a direct computation and by the property (P3) of lemma 1, it holds

$$\left\lfloor \frac{\alpha-1}{2^I} - \frac{1}{2} \right\rfloor = \left\lfloor \frac{\alpha}{2^I} - 1 \right\rfloor = \left\lfloor \frac{\alpha}{2^I} \right\rfloor - 1; \left\lfloor \frac{\alpha+1}{2^I} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{\alpha}{2^I} \right\rfloor + 1$$

By using (4), it yields

$$\left\lfloor \frac{\alpha}{2^I} - \frac{1}{2} \right\rfloor = \left\lfloor \frac{\alpha-1}{2^I} \right\rfloor = \left\lfloor \frac{\alpha}{2^I} + \frac{1}{2} \right\rfloor - 1 = \left\lfloor \frac{\alpha}{2^I} \right\rfloor - 1;$$

$$\left\lfloor \frac{\alpha+1}{2^I} - \frac{1}{2} \right\rfloor = \left\lfloor \frac{\alpha}{2^I} \right\rfloor = \left\lfloor \frac{\alpha-1}{2^I} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{\alpha+1}{2^I} \right\rfloor = \left\lfloor \frac{\alpha}{2^I} + \frac{1}{2} \right\rfloor;$$

Hence the theorem holds in the case $I = 1$.

When $I > 1$, it knows

$$\begin{cases} \frac{\alpha-1}{2^I} - \frac{1}{2} < \frac{\alpha}{2^I} - \frac{1}{2} < \frac{\alpha+1}{2^I} - \frac{1}{2} \le \frac{\alpha-1}{2^I} < \frac{\alpha}{2^I} \\ \frac{\alpha}{2^I} < \frac{\alpha+1}{2^I} \le \frac{\alpha-1}{2^I} + \frac{1}{2} < \frac{\alpha}{2^I} + \frac{1}{2} < \frac{\alpha+1}{2^I} + \frac{1}{2} \end{cases} \tag{6}$$

First by using (4), it yields

$$\left\lfloor \frac{\alpha}{2^I} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{\alpha}{2^{I-1}} \right\rfloor - \left\lfloor \frac{\alpha}{2^I} \right\rfloor = \left\lfloor \frac{\alpha}{2^I} \right\rfloor$$

Referring to the theorem 2 and the formula (6), it immediately obtains

$$\left\lfloor \frac{\alpha}{2^I} \right\rfloor = \left\lfloor \frac{\alpha+1}{2^I} \right\rfloor = \left\lfloor \frac{\alpha-1}{2^I} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{\alpha}{2^I} + \frac{1}{2} \right\rfloor \tag{7}$$

Now, we prove

$$\left\lfloor \frac{\alpha+1}{2^I} - \frac{1}{2} \right\rfloor = \left\lfloor \frac{\alpha}{2^I} \right\rfloor \tag{8}$$

In fact, when $I > 1$, referring to the property (P1) of lemma 1, it yields

$$\left\lfloor \frac{\alpha-1}{2^I} + \frac{1}{2} \right\rfloor = \left\lfloor \frac{\alpha}{2^I} + \frac{1}{2} - \frac{1}{2^I} \right\rfloor \ge \left\lfloor \frac{\alpha}{2^I} \right\rfloor + \left\lfloor \frac{1}{2} - \frac{1}{2^I} \right\rfloor = \left\lfloor \frac{\alpha}{2^I} \right\rfloor$$

By the properties (P2) and (P3) of lemma 1 and referring to (7), it holds

$$\left\lfloor\frac{\alpha-1}{2^I}+\frac{1}{2}\right\rfloor=\left\lfloor\frac{\alpha}{2^I}+\frac{1}{2}-\frac{1}{2^I}\right\rfloor<\left\lfloor\frac{\alpha}{2^I}+\frac{1}{2}\right\rfloor-\left\lfloor\frac{1}{2^I}\right\rfloor+1=\left\lfloor\frac{\alpha}{2^I}+\frac{1}{2}\right\rfloor+1=\left\lfloor\frac{\alpha}{2^I}\right\rfloor+1$$

Thus the formula (8) holds, and it leads to

$$\left\lfloor\frac{\alpha+1}{2^I}+\frac{1}{2}\right\rfloor=\left\lfloor\frac{\alpha+1}{2^I}-\frac{1}{2}+1\right\rfloor=\left\lfloor\frac{\alpha+1}{2^I}-\frac{1}{2}\right\rfloor+1=\left\lfloor\frac{\alpha}{2^I}\right\rfloor+1 \tag{9}$$

Furthermore, it holds

$$\left\lfloor\frac{\alpha+1}{2^I}-\frac{1}{2}\right\rfloor=\left\lfloor\frac{\alpha+1}{2^I}+\frac{1}{2}\right\rfloor-1=\left\lfloor\frac{\alpha}{2^I}\right\rfloor+1-1=\left\lfloor\frac{\alpha}{2^I}\right\rfloor \tag{10}$$

The formulas (7) (8) (9) (10) show that the theorem 4 holds in the case $I>1$. Hence theorem is proven.

By theorem 4, we know that the expression $\lfloor\frac{\alpha-1}{2^{z(\alpha)}}\rfloor$ satisfies

$$\left\lfloor\frac{\alpha-1}{2^{z(\alpha)}}\right\rfloor=\left\lfloor\frac{\alpha}{2^{z(\alpha)}}\right\rfloor+\left\{\begin{array}{c}-1, z(\alpha)=1\\ 0, z(\alpha)>1\end{array}\right.$$

Theorem 5. *Let* $\alpha\geq 0$*,* δ *and* χ *be integers such that* $\chi=...,-3,-2,-1,0,1,2,3,...$*,* $(\chi-\frac{1}{2})\times 2^{z(\alpha)}<\delta\leq(\chi+\frac{1}{2})\times 2^{z(\alpha)}$*; then it holds*

$$\left\lfloor\frac{\alpha+\delta}{2^{z(\alpha)}}\right\rfloor=\left\lfloor\frac{\alpha}{2^{z(\alpha)}}\right\rfloor+\chi \tag{11}$$

Proof. For convenience,we use the symbol $v(\alpha,\delta)$ to denote $\lfloor\frac{\alpha+\delta}{2^{z(\alpha)}}\rfloor$, use the symbol I_χ to denote the interval $(\chi-\frac{1}{2})\times 2^{z(\alpha)}<\delta\leq(\chi+\frac{1}{2})\times 2^{z(\alpha)}$ and keep using $I=z(\alpha)$. Note that the condition of the theorem 5 is the same as that of the theorem 4; hence the conclusions drawn in the theorem 4 can be directly adopted here. According to the theorem 2, it is merely necessary to prove that $v(\alpha,\delta)$ takes an identical value $\lfloor\frac{\alpha}{2^I}\rfloor+\chi$, which merely depends upon α and χ, on the whole interval I_χ. Also by the theorem 2, this is a job to compute the values of $v(\alpha,\delta)$ at five points $\delta=(\chi-\frac{1}{2})\times 2^I, \delta=(\chi-\frac{1}{2})\times 2^I+1, \delta=\chi\times 2^I, \delta=(\chi+\frac{1}{2})\times 2^I$ and $\delta=(\chi+\frac{1}{2})\times 2^I+1$, as follows.

(i). When $\delta=(\chi-\frac{1}{2})\times 2^I$, it yields

$$\begin{aligned}v(\alpha,\delta)&=\lfloor\tfrac{\alpha+\delta}{2^I}\rfloor=\left\lfloor\tfrac{\alpha+(\chi-\frac{1}{2})\times 2^I}{2^I}\right\rfloor\\&=\lfloor\tfrac{\alpha}{2^I}+\chi-\tfrac{1}{2}\rfloor=\chi+\lfloor\tfrac{\alpha}{2^I}-\tfrac{1}{2}\rfloor=\chi-1+\lfloor\tfrac{\alpha}{2^I}\rfloor\end{aligned} \tag{12}$$

(ii). When $\delta=(\chi-\frac{1}{2})\times 2^I+1$, it yields

$$\begin{aligned}v(\alpha,\delta)&=\lfloor\tfrac{\alpha+\delta}{2^I}\rfloor=\left\lfloor\tfrac{\alpha+(\chi-\frac{1}{2})\times 2^I+1}{2^I}\right\rfloor\\&=\lfloor\tfrac{\alpha+1}{2^I}+\chi-\tfrac{1}{2}\rfloor=\chi+\lfloor\tfrac{\alpha+1}{2^I}-\tfrac{1}{2}\rfloor=\chi+\lfloor\tfrac{\alpha}{2^I}\rfloor\end{aligned} \tag{13}$$

(iii). When $\delta=\chi\times 2^I$, it yields

$$v(\alpha,\delta)=\left\lfloor\frac{\alpha+\delta}{2^I}\right\rfloor=\left\lfloor\frac{\alpha+\chi\times 2^I}{2^I}\right\rfloor=\left\lfloor\frac{\alpha}{2^I}+\chi\right\rfloor=\chi+\left\lfloor\frac{\alpha}{2^I}\right\rfloor \tag{14}$$

(iv). When $\delta = (\chi + \frac{1}{2}) \times 2^I$, it yields

$$\begin{aligned} v(\alpha,\delta) &= \left\lfloor \tfrac{\alpha+\delta}{2^I} \right\rfloor = \left\lfloor \tfrac{\alpha+(\chi+\frac{1}{2})\times 2^I}{2^I} \right\rfloor \\ &= \left\lfloor \tfrac{\alpha}{2^I} + \tfrac{1}{2} + \chi \right\rfloor = \chi + \left\lfloor \tfrac{\alpha}{2^I} + \tfrac{1}{2} \right\rfloor = \chi + \left\lfloor \tfrac{\alpha}{2^I} \right\rfloor \end{aligned} \tag{15}$$

(v). When $\delta = (\chi + \frac{1}{2}) \times 2^I + 1$, it yields

$$\begin{aligned} v(\alpha,\delta) &= \left\lfloor \tfrac{\alpha+\delta}{2^I} \right\rfloor = \left\lfloor \tfrac{\alpha+(\chi+\frac{1}{2})\times 2^I+1}{2^I} \right\rfloor \\ &= \chi + 1 + \left\lfloor \tfrac{\alpha+1}{2^I} - \tfrac{1}{2} \right\rfloor = \chi + 1 + \left\lfloor \tfrac{\alpha}{2^I} \right\rfloor \end{aligned} \tag{16}$$

Obviously, the results in (13),(14) and (15) show that $v(\alpha,\delta)$ does take an identical value of $\left\lfloor \frac{\alpha}{2^I} \right\rfloor + \chi$ on the whole interval I_χ, and the value merely depends upon α and χ since $I = z(\alpha)$ depends on α . Note that the results (12) and (16) also imply that, for a fixed χ , $v(\alpha,\delta)$ takes an identical value of $\left\lfloor \frac{\alpha}{2^I} \right\rfloor + \chi - 1$ on the interval left to I_χ, and takes an identical value $\left\lfloor \frac{\alpha}{2^I} \right\rfloor + \chi + 1$ on the interval right to I_χ. Hence, as χ changes, it is true that $v(\alpha,\delta)$ does take the value of $\left\lfloor \frac{\alpha}{2^I} \right\rfloor + \chi$ on every interval I_χ. This ends the proof of the theorem 5.

By the theorem 5, the relationship between the function $v(\alpha,\delta)$ and the variable δ can be illustrated by figure 1. Through the figure, it gets to know that the length of the interval I_χ is 2^I . By expressing the interval I_χ in its equivalent form

$$I_\chi = (-2^{I-1} + \chi \times 2^I, 2^{I-1} + \chi \times 2^I]$$

it shows that the function $v(\alpha,\delta)$ changes with the interval $I_0 = (-2^{I-1}, 2^{I-1}]$ to be a basic unit. We call the interval I_0 a principal interval since the property of the function $v(\alpha,\delta)$ on I_χ can be translated from the property on it.

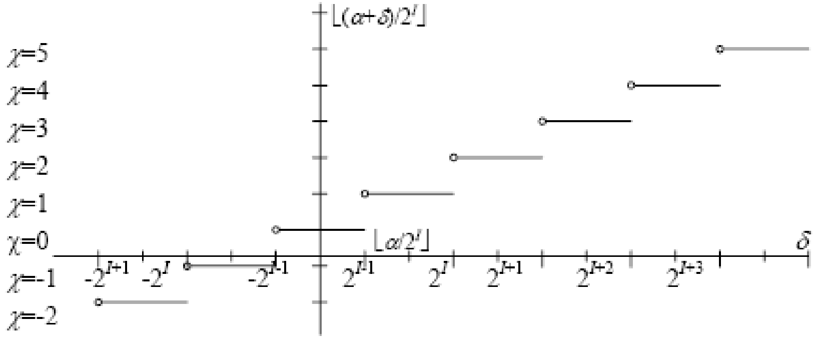


Figure 1. Relationship between the variable δ and the function $v(\alpha,\delta)$.

Finally, we state and prove the following

Theorem 6. *For arbitrary integers* $\alpha \geq 0$ *and* δ *, it holds*

$$\left\lfloor \frac{\alpha+\delta}{2^{z(\alpha)}} \right\rfloor = \left\lfloor \frac{\alpha}{2^{z(\alpha)}} \right\rfloor + \left\lfloor \frac{\delta-1}{2^{z(\alpha)}} + \frac{1}{2} \right\rfloor \tag{17}$$

Proof. Keep using the symbol $I = z(\alpha)$ and suppose $\delta \in I_\chi$ without loss of generality; then by the theorem 5, it yields

$$(\chi - \frac{1}{2}) \times 2^{I} < \delta \leq (\chi + \frac{1}{2}) \times 2^{I} \tag{18}$$

Performing a simple transformation on (18) leads to

$$\frac{\delta}{2^I} - \frac{1}{2} \leq \chi < \frac{\delta}{2^I} + \frac{1}{2}$$

That is

$$\frac{\delta}{2^I} - \frac{1}{2} \leq \chi < \frac{\delta}{2^I} - \frac{1}{2} + 1$$

Referring to definition of the ceiling function yields

$$\chi = \left\lceil \frac{\delta}{2^I} - \frac{1}{2} \right\rceil$$

By the properties (P6) and (P3) of the lemma 1, it follows

$$\left\lceil \frac{\delta}{2^I} - \frac{1}{2} \right\rceil = \left\lceil \frac{\delta - 2^{I-1}}{2^I} \right\rceil = \left\lfloor \frac{\delta - 2^{I-1} - 1}{2^I} \right\rfloor + 1 = \left\lfloor \frac{\delta - 1}{2^I} - \frac{1}{2} \right\rfloor + 1 = \left\lfloor \frac{\delta - 1}{2^I} + \frac{1}{2} \right\rfloor$$

$$\begin{aligned} \left\lceil \tfrac{\delta}{2^I} - \tfrac{1}{2} \right\rceil &= \left\lceil \tfrac{\delta - 2^{I-1}}{2^I} \right\rceil = \left\lfloor \tfrac{\delta - 2^{I-1} + 2^I - 1}{2^I} \right\rfloor \\ &= \left\lfloor \tfrac{\delta - 2^{I-1} - 1}{2^I} \right\rfloor + 1 = \left\lfloor \tfrac{\delta - 1}{2^I} + \tfrac{1}{2} \right\rfloor \end{aligned}$$

Hence the theorem 6 holds.

4. COMPUTER TEST

Computations related to this paper can be tested on personal computers. The C-language program is as follows.

```
\#include"math.h"

int GetI(int a)   /* Find the smallest I that fits $0\leq a mod 2^i < 2^{(i-1)}$ */

\{
    int i,I;

    int \_2i\_1, \_2i;

    for(i=1;;i++)
   \{

     \_2i\_1 = 1$\ll$(i-1);\quad /* Compute $2^{(i-1)}$*/

     \_2i=\_2i\_1$\ll${1}; \quad  /* Compute $2^i$ */

     if(a\%\_2i$<$\_2i\_1)
     \{
```

```
       I=i;             \quad\quad\quad \quad/* I is obtained */

       break;   \}

    \}

return I;
\}

void Test(double a, double delta, int I) /*Perform Test*/

\{
    int U,X,Y;

    int \_2i;

    \_2i=1$\ll$I;   \quad   /* COmpute $2^I$*/

    U=(int)floor((a+delta)/\_2i);

    X=(int)floor(delta/\_2i);

    Y=(int)floor((delta-1)/\_2i+0.5);

    X+=Y;

    if(U!=X)  printf("Err:\%d \%d \%d \%d \%d $\backslash{n}$",a,delta,I,U,X);
\}


void main()

\{
    int I,\_2i,i;

   for(i=10000;i<20000;i++)

   \{I=GetI(i);

     for(int j=-i;j$<=$i;j++)Test(i,j,I);

   \}

    printf("Finished!");

    getchar();
\}
\fi
```

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DEPARTMENT OF MECHATRONIC ENGINEERING, FOSHAN UNIVERSITY, GUANGDONG PROVINCE, PR CHINA,528000
E-mail address: wxbmail@msn.com