Mathproblems
ISSN: 2217-446X, url: http://www.mathproblems-ks.org
Volume 6, Issue 2 (2016), Pages 559-595

PROBLEMS AND SOLUTIONS

Proposals and solutions must be legible and should appear on separate sheets, each indicating the name of the sender. Drawings must be suitable for reproduction. Proposals should be accompanied by solutions. An asterisk (*) indicates that neither the proposer nor the editors have supplied a solution. The editors encourage undergraduate and pre-college students to submit solutions. Teachers can help by assisting their students in submitting solutions. Student solutions should include the class and school name. Solutions will be evaluated for publication by a committee of professors according to a combination of criteria. Questions concerning proposals and/or solutions can be sent by e-mail to: *mathproblems-ks@hotmail.com*

Solutions to the problems stated in this issue should arrive before October 15, 2017

Problems

152. *Proposed by Marian Dinca, Bucharest, Romania and Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Rumania.* Let a, b, c and d be the lenghts of the sides of a convex quadrilateral inscribed in a circle with radius R. Prove the inequality

$$\frac{a^2}{b+c+d-a}+\frac{b^2}{a+c+d-b}+\frac{c^2}{a+b+d-c}+\frac{d^2}{a+b+c-d}\geq 2\sqrt{2}R.$$

153. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let $\alpha > 1$ and let $a, b \in \mathbb{R}$, $b \neq 0$. Calculate

$$\lim_{n\to\infty}\begin{pmatrix} 1-\frac{a}{n^\alpha} & -\frac{b}{n} \\ \frac{b}{n} & 1+\frac{a}{n^\alpha}\end{pmatrix}^n.$$

154. *Proposed by Anastasios Kotronis, Athens, Greece and Haroun Meghaichi, student, the University of Science and Technology, Houari Boumediene, Algiers, Algeria.* Let m be a positive integer and

$$S_{n,m}=\sum_{k=1}^{n}(-1)^k\binom{n}{k}k^{-m}.$$

Show that:
1) $S_{n,1}=-\ln n-\gamma-\frac{1}{2n}+\mathcal{O}(n^{-2})(n\to+\infty)$,

2) $S_{n,2} = -\frac{\ln^2 n}{2} - \gamma \ln n - \frac{\gamma^2}{2} - \frac{\pi^2}{12} - \frac{\ln n}{2n} + \frac{1-\gamma}{2n} + \mathcal{O}\left(n^{-2}\ln n\right)\,(n \to +\infty)$,
3) There exist real numbers $a_m, \ldots, a_0$ and $b_{m-1}, \ldots, b_0$ such that

$$S_{n,m} = \sum_{k=0}^{m} a_{m-k} \ln^{m-k} n + \sum_{k=0}^{m-1} b_{m-k-1} \frac{\ln^{m-k-1} n}{n} + \mathcal{O}\left(n^{-2}\ln^{m-1} n\right)\,(n \to +\infty).$$

and determine them.

155. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Let $a \in \mathbb{R}_+^*$ and let $(L_n)_{n\geq 0}$ be Lucas sequence and a positive real sequence such that $\lim_{n\to\infty} \frac{a_{n+1}}{n^2 a_n} = a$. Find

$$\lim_{n\to\infty}\left(\sqrt[n+1]{\frac{a_{n+1}L_{n+1}}{(2n+1)!!}} - \sqrt[n]{\frac{a_n L_n}{(2n-1)!!}}\right).$$

156. *Proposed by Dorlir Ahmeti, University of Prishtina, Republic of Kosovo and Alexander Gunning, Australia.* With $(2n-1)^3$ same cubes we build a cube. We say any cube which is still in the cube is if at least three faces of that cube are not shared with any other cube. We begin by removing a cube (by doing this we will cause other cubes to become and we may repeat this procedure, removing further cubes. What is the minimum number of moves required to remove the cube which is in the centre of cube.

157. *Proposed by Cornel Ioan Vălean, Timiş, Rumania.* Prove that

$$2\sum_{n=1}^{\infty}\left(\zeta(3)\zeta(6) - H_n^{(3)}H_n^{(6)}\right) + 7\sum_{n=1}^{\infty}\frac{H_n^{(2)}}{n^6} = 10\zeta(3)\zeta(5) - 2\zeta(3)\zeta(6) - \frac{23}{12}\zeta(8).$$

where $H_n^{(m)} = 1 + \dfrac{1}{2^m} + \cdots + \dfrac{1}{n^m}$ denotes the nth harmonic number.

158. *Proposed by Sava Grozdev, VUZF University of Finance, Business and Entrepreneurship, Bulgaria, Hiroshi Okumura, Department of Mathematics, Yamato University, Osaka, Japan and Deko Dekov, Stara Zagora, Bulgaria .* Given triangle ABC with side lengths $BC = a, CA = b$ and $AB = c$. Prove that the pedal triangle of the inverse of the orthocenter of triangle ABC in the Circumcircle of triangle ABC is similar to the orthic triangle of triangle ABC. Find the similitude ratio as function of a, b, c.

Solutions

No problem is ever permanently closed. We will be very pleased considering for publication new solutions or comments on the past problems.

---

145. *Proposed by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy.* Let $0 \le x \le 1$. Prove that $x^x \le x^2 - x + 1 - x^2(1-x)^4$.

Solution 1 by Moti Levy, Rehovot, Israel.

Let $y = 1 - x$. Then our original inequality is equivalent to

$$(1-y)^{1-y} \le 1 - y + y^2 - y^4 + 2y^5 - y^6,\ 0 \le y \le 1.$$

By taking the first three terms of the binomial expansion (the inequality is justified since the absolute values of the terms are decreasing),

$$(1-y)^{1-y} \le 1 - (1-y)\,y + \frac{(1-y)\,(-y)}{2}y^2 = 1 - y + y^2 - \frac{1}{2}y^3 + \frac{1}{2}y^4.$$

Hence, it is enough to show that

$$-\frac{1}{2}y^3 + \frac{1}{2}y^4 \le 2y^5 - y^6$$

which is equivalent to

$$y^2\,(2-y) + \frac{1}{2}\,(1-y) \ge 0, \quad for \quad 0 \le y \le 1.$$

Solution 2 by the proposer. This is a refinement of $x^x \le x^2 - x + 1$ (Crux Mathematicorum, problem 3815, Vol.39(2)). The inequality is equivalent to $x \ln x \le \ln(x^2 - x + 1 - x^2(1-x^4))$ and by using $\ln x \le (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$ which is proved at the end, we come to

$$f(x) = x\left((x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3\right) - \ln(x^2 - x + 1 - x^2(1-x^4)) \le 0$$

$$\begin{aligned} f'(x) &= \frac{1}{6}\frac{(8x^7 - 43x^6 + 98x^5 - 110x^4 + 14x^3 + 56x^2 - 37x + 5)(x-1)^2}{x - 1 + x^6 - 4x^5 + 6x^4 - 4x^3} \\ &\doteq \frac{(x-1)^2}{6}\frac{g(x)}{h(x)} \end{aligned}$$

Now we prove that $h(x) < 0$ while $g(x)$ is strictly decreasing passing from positive values to negative values. It follows that $f(x)$ starts decreasing and then increases. Since $f(0) = f(1) = 0$, this proves the statement.

$h(x) \le 0$ is equivalent to

$$x^6 + 6x^4 + x \le 4x^5 + 4x^3 + 1, \quad 0 \le x \le 1$$

and this is in turn implied by

$$6x^4 \le 3x^5 + 4x^3$$

which is easy AGM. Indeed

$$3x^5 + 4x^3 \ge 2\sqrt{12}x^4 = 4\sqrt{3}x^4 > 6x^4$$

Moreover

$$g'(x) = 56x^6 - 258x^5 + 490x^4 - 440x^3 + 42x^2 + 112x - 37$$

and we want to show that $g'(x) < 0$ so that $g(x)$ decreases. Since $56x^6 - 56x^5 \leq 0$,

$$g'(x) < 0 \iff -202x^5 + 490x^4 - 440x^3 + 42x^2 + 112x - 37 \doteq G(x) < 0$$

The tangent to $G(x)$ at $x = \frac{1}{2} + \frac{1}{40}$ is

$$r(x) = -\frac{8781149}{12800000} - \frac{213941}{256000}x, \quad r(0) < 0, \; r(1) < 0.$$

so $r(x) < 0$ for any $0 \leq x \leq 1$. Moreover

$$G(x) - r(x) = -\frac{(1616000x^3 - 2223200x^2 + 740230x + 1054011)(40x - 21)^2}{12800000}$$

and

$$(1616000x^3 - 2223200x^2 + 740230x + 1054011) \geq$$
$$\geq (2\sqrt{1616000 \cdot 740230} - 2223200)x^2 + 1054011 > -36000x^2 + 1054011 > 0$$

so

$$G(x) - r(x) \leq 0 \text{ that is } G(x) \leq r(x) < 0$$

and this concludes the proof of the inequality. The last step is

$$\ln x \leq (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3$$

that is

$$h(x) = \ln x - (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{3}(x-1)^3 \leq 0.$$

Since $h'(x) = -(x-1)^3/x$, $h(x)$ is monotonic increasing and by $\lim_{x \to 0^+} h(x) = -\infty$, $h(1) = 0$, it follows $h(x) \leq 0$ for any $0 \leq x \leq 1$.

Also solved by Moubinool Omarjee, Paris, France.

146. *Proposed by Ovidiu Furdui, Technical University of Cluj-Napoca, Cluj-Napoca, Romania.* Let $n \geq 1$ be an integer. Solve in $\mathcal{M}_2(\mathbb{Z})$ the equation $X^{2n+1} - X = I_2$.

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria.

Consider am matrix $X \in \mathcal{M}_2(\mathbb{Z})$ satisfying $X^{2n+1} - X = I_2$. from

$$I_2 = X(X - I_2)(X^{2n-1} + \cdots + X + I_2)$$

we conclude that $\det(X)$ and $\det(X - I_2)$ are integer divisors of 1. But $\det(X - I_2) = \det(X) - \operatorname{tr}(X) + 1$. It follows that there exists $\epsilon, \epsilon' \in \{-1, 1|\}$ such that $\det(X) = \epsilon$ and $\operatorname{tr}(X) = 1 + \epsilon - \epsilon'$. Thus

$$(\operatorname{tr}(X), \det(X)) \in \{(1,1), (3,1), (1,-1), (-1,-1)\}$$

and the possible characteristic polynomials of X are

	Characteristic Polynomial	roots	
(a)	$\lambda^2 - \lambda + 1$	$e^{i\pi/3}$	$e^{-i\pi/3}$
(b)	$\lambda^2 - 3\lambda + 1$	$\frac{3-\sqrt{5}}{2}$	$\frac{3+\sqrt{5}}{2}$
(c)	$\lambda^2 - \lambda - 1$	$\frac{1-\sqrt{5}}{2}$	$\frac{1+\sqrt{5}}{2}$
(d)	$\lambda^2 + \lambda - 1$	$-\frac{1+\sqrt{5}}{2}$	$\frac{-1+\sqrt{5}}{2}$

On the other hand, any complex eigenvalue λ of X must satisfy $\lambda^{2n+1} = 1 + \lambda$ and consequently if $|\lambda| > 1$ then $|\lambda|^{2n+1} < 2|\lambda|$ so $|\lambda| < 2^{1/(2n)} \leq \sqrt{2}$. It follows that all the eigenvalues of X must belong to the open disk $D(0, \sqrt{2})$. This excludes the cases (b), (c) and (d) because $(3 + \sqrt{5})/2 > \sqrt{2}$ and $(1 + \sqrt{5})/2 > \sqrt{2}$.
For (a) the eigenvalues of X are $\{e^{i\pi/3}, e^{-i\pi/3}\}$, but if λ is one of these eigenvalues then

$$\frac{3}{2} = \Re(\lambda + 1) = \Re(\lambda^{2n+1}) \leq |\lambda|^{2n+1} = 1$$

which contradicts the fact that $\lambda^{2n+1} = 1 + \lambda$. It follows that the proposed equation has no solutions in $\mathcal{M}_2(\mathbb{Z})$.

Solution 2 by by Michel Bataille, Rouen, France.
We show that there are no solutions. To this aim, we assume that for some $X \in \mathcal{M}_2(\mathbb{Z})$ we have $X^{2n+1} - X = I_2$. First, we observe that X cannot be of the form aI_2 where $a \in \mathbb{Z}$. Otherwise we would have $a^{2n+1} = a + 1$, which cannot occur (either if $a \in \{-1, 0, 1\}$ or if $|a| > 1$ since then a prime divisor of a divides a^{2n+1} but not $a + 1$). It follows that the minimal polynomial of X is not of degree 1. Thus, this minimal polynomial is the characteristic polynomial $x^2 - tx + \det(X)$ (where t is the trace of X). Another remark is that $\det(X) \in \{-1, 1\}$. This is readily deduced from $\det(X) \cdot \det(X^{2n} - I_2) = 1$ (since $X(X^{2n} - I_2) = I_2$) with $\det(X)$ and $\det(X^{2n} - I_2)$ in $\mathbb{Z}$ (note that $X^{2n} - I_2 \in \mathcal{M}_2(\mathbb{Z})$). Now, we show that we are led to a contradiction in the two cases $\det(X) = 1$ and $\det(X) = -1$.
If $\det(X) = 1$, then the polynomial $x^{2n+1} - x - 1$ is a multiple of the minimal polynomial $x^2 - tx + 1$. Thus, the complex roots $r, \frac{1}{r}$ of the latter are also roots of $x^{2n+1} - x - 1$. This provides the relations $r^{2n+1} = r + 1$ and $r^{2n+1} = 1 - r^{2n}$, hence $r^{2n} = -r$ and so $r + 1 = r \cdot r^{2n} = -r^2$. Therefore $r^2 + r + 1 = 0$ and r must be a cubic root of unity different from 1. However, from $r^{2n+1} = -r^2$ we obtain $r^{2n-1} = -1$, which is wrong when r is a cubic root of unity. We have reached our contradiction.
Similarly, if $\det(X) = -1$, the roots $s, -\frac{1}{s}$ of the minimal polynomial $x^2 - tx - 1$ must be roots of $x^{2n+1} - x - 1$. This time, we get the relations $s^{2n+1} = s + 1$ and $s^{2n+1} = s^{2n} - 1$. Thus, $s^{2n} = s + 2$ and so $s(s + 2) = s + 1$, that is, $s^2 - s - 1 = 0$ and $s \in \{\frac{1+\sqrt{5}}{2}, \frac{1-\sqrt{5}}{2}\}$. However, we would have $s^{2n+1} = s + 1 = s^2$, hence $s^{2n-1} = 1$, which is obviously wrong. Again, we obtain a contradiction and the proof is complete.

Also solved by Moti Levy, Rehovot, Israel; Moubinool Omarjee, Paris, France and the proposer.

147. *Proposed by Anastasios Kotronis, Athens, Greece.* Let a_n be the sequence defined by the relations

$$a_{n+3} - \left(1 + \frac{b - p - 1}{n + 3}\right) a_{n+2} + \frac{b - 2a}{n + 3} a_{n+1} + \frac{2a}{n + 3} a_n = 0$$

and

$$a_0 = 1, a_1 = b - p, a_2 = a + \frac{(b - p)^2 - p}{2},$$

where $a, b \in \mathbb{R} \setminus \{-2, -2, 0, 1, \ldots\}$.

(1) Show that $\lim_{n\to\infty} n^{p+1} a_n = \frac{e^{a+b}}{\Gamma(-p)}$.

(2) Find $\lim_{n\to\infty} n \left(n^{p+1} a_n - \frac{e^{a+b}}{\Gamma(-p)}\right)$ if it exists.

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria. Note that the recurrence relation defining the a_n's can be written in the following way

$$(n+3)a_{n+3}-(n+2)a_{n+2}-(b-p)a_{n+2}+(b-2a)a_{n+1}+2a\,a_n=0 \qquad (1)$$

So, if we define $G(x)=\sum_{n=0}^{\infty}a_nx^n$, then it follows from (1) and the initial conditions imposed on a_0, a_1 and a_2, that

$$(1-x)G'(x)+(p-(1-x)(b+2ax))G(x)=0$$

This proves that $x\mapsto(1-x)^{-p}e^{-bx-ax^2}G(x)$ is constant (just calculate the derivative), and since it is equal to 1 for $x=0$ we conclude that

$$G(x)=(1-x)^pe^{bx+ax^2}$$

So, let Log be the principal branch of the logarithm, and define the analytic function

$$G(z)=(1-z)^pe^{bz+az^2}=e^{p\mathrm{Log}(1-z)}\,e^{bz+az^2}$$

for $z\in\mathbb{C}\setminus[1,+\infty)$. The above discussion shows that $G(z)=\sum_{n=0}^{\infty}a_nz^n$ for z in the open unit disc $D(0,1)$. If fact G is analytic in the interior of any set of the form

$$\Delta_\phi=\{z\in\mathbb{C}:|\mathrm{Arg}(z-1)|\geq\phi\}$$

where $0<\phi<\pi/2$. Moreover, since $k:z\mapsto e^{bz+az^2}$ is an entire function (*i.e.* analytic in the hole complex plane), we have

$$k(z)=k(1)-k'(1)(1-z)+\mathcal{O}(1-z)^2)$$

as $z\to1$, and consequently:

$$G(z)=k(1)(1-z)^p-k'(1)(1-z)^{p+1}+\mathcal{O}(1-z)^{p+2})$$

as $z\to1$ in $\Delta_\phi\setminus\{1\}$. Using Corollary 3. from ["*Singularity analysis of generating functions*", Siam J. Disc. Math., Vol. 3(2),pp. 216–240 (1990) by Flajolet and Odlyzko], we conclude that

$$a_n=k(1)\binom{n-p-1}{n}-k'(1)\binom{n-p-2}{n}+\mathcal{O}(n^{-p-3})$$

Now, noting that

$$\begin{aligned}
k(1)&=e^{a+b},\\
k'(1)&=e^{a+b}(b+2a)\\
\binom{n-p-1}{n}&=\frac{1}{\Gamma(-p)}\left(\frac{1}{n^{p+1}}+\frac{p(p+1)}{2n^{p+2}}+\mathcal{O}\left(\frac{1}{n^{p+3}}\right)\right)\\
\binom{n-p-2}{n}&=-\frac{p+1}{\Gamma(-p)n^{p+2}}+\mathcal{O}\left(\frac{1}{n^{p+3}}\right)
\end{aligned}$$

Thus

$$n^{p+1}a_n=\frac{e^{a+b}}{\Gamma(-p)}-\frac{(p+2b+4a)e^{a+b}}{2\Gamma(-p-1)}\cdot\frac{1}{n}+\mathcal{O}\left(\frac{1}{n^2}\right)$$

In particular,

$$\lim_{n\to\infty}n^{p+1}a_n=\frac{e^{a+b}}{\Gamma(-p)},\quad\text{and}\quad\lim_{n\to\infty}n\left(n^{p+1}a_n-\frac{e^{a+b}}{\Gamma(-p)}\right)=-\frac{(p+2b+4a)e^{a+b}}{2\Gamma(-p-1)}.$$

Solution 2 by by Moti Levy, Rehovot, Israel. Let $F(z) := \sum_{n=0}^{\infty} a_n z^n$ be the generating function of the sequence $(a_n)_{n\geq 0}$. We rewrite the recurrence formula and multiply by z^n.

$$(n+3)\,a_{n+3} - (n+3)\,a_{n+2} - (b-p-1)\,a_{n+2} + (b-2a)\,a_{n+1} + 2a * a_n = 0,$$

$$4\sum_{n=0}^{\infty}(n+3)\,a_{n+3}z^n - \sum_{n=0}^{\infty}(n+3)\,a_{n+2}z^n - (b-p-1)\sum_{n=0}^{\infty}a_{n+2}z^n$$
$$+ (b-2a)\sum_{n=0}^{\infty}a_{n+1}z^n + 2a*\sum_{n=0}^{\infty}a_n z^n = 0.$$

$$z^{-2}\left(\sum_{n=0}^{\infty}a_{n+3}z^{n+3}\right)' - z^{-1}\left(\sum_{n=0}^{\infty}a_{n+2}z^{n+2}\right)'$$
$$-(b-p)\,z^{-2}\sum_{n=0}^{\infty}a_{n+2}z^{n+2} + (b-2a)\,z^{-1}\sum_{n=0}^{\infty}a_{n+1}z^{n+1} + 2a*\sum_{n=0}^{\infty}a_n z^n$$
$$=0,$$

$$z^{-2}\left(F(z) - a_0 - a_1 z - a_2 z^2\right)' - z^{-1}\left(F(z) - a_0 - a_1 z\right)'$$
$$-(b-p)\,z^{-2}\left(F(z) - a_0 - a_1 z\right) + (b-2a)\,z^{-1}\left(F(z) - a_0\right) + 2aF(z)$$
$$= 0,$$

$$\left(F(z) - a_0 - a_1 z - a_2 z^2\right)' - z\left(F(z) - a_0 - a_1 z\right)'$$
$$-(b-p)\left(F(z) - a_0 - a_1 z\right) + (b-2a)\,z\left(F(z) - a_0\right) + 2az^2F(z)$$
$$= 0,$$

$$F'(z)\,(1-z) + F(z)\left(-(b-p) + (b-2a)\,z + 2az^2\right) = 0$$
$$F'(z)\,(1-z) = F(z)\left((b-p) - (b-2a)\,z - 2az^2\right)$$

Finally we obtain the differential equation for $F(z)$,

$$\frac{F'(z)}{F(z)} = b + 2az + \frac{p}{z-1}, \quad F(0) = 1.$$

$$\ln F(z) = bz + az^2 + p\ln(z-1) + c$$

$$F(z) = ke^{az^2+bz}\,(z-1)^p$$

$$1 = F(0) = k\,(-1)^p$$

$$k = \frac{1}{(-1)^p} = (-1)^p$$

The generating function is

$$F(z) = e^{az^2+bz}\cdot(1-z)^p\,.$$

The Darboux's lemma (see [1] , Theorem 5.3.1 on page 179) is now used to evaluate the asymptotic growth of the generating function coefficients a_n,

Theorem. *("Darboux's lemma")*
Let $v(z)$ be analytic in some disk $|z|<1+\eta$, $\eta>0$ and suppose that in a neighborhood of $z=1$ it has the expansion

$$v\left(z\right)=\sum_{j\geq 0}v_{j}\left(1-z\right)^{j}.$$

Let $\beta\notin\{\ldots,-2-1,0,1,2,...\}$. Then

$$[z^{n}]\left\{\left(1-z\right)^{\beta}v\left(z\right)\right\}=[z^{n}]\left\{\sum_{j=0}^{m}v_{j}\left(1-z\right)^{j}\right\}+O\left(n^{-m-\beta-2}\right)$$
$$=\sum_{j=0}^{m}v_{j}\binom{n-\beta-j-1}{n}+O\left(n^{-m-\beta-2}\right).$$

The function $e^{az^{2}+bz}$ is entire and the first terms of its power series are,

$$e^{az^{2}+bz}=e^{a+b}\left(1-\left(2a+b\right)\left(1-z\right)+\cdots\right).$$

(1) Set $\mathbf{m=0}$, $\beta=p$, $\mathbb{R}\ni p\notin\{...-2,-1,0,1,...\}$, $v\left(z\right)=e^{az^{2}+bz}$, $a,b\in\mathbb{R}$; and apply the Darboux's lemma above to obtain,

$$a_{n}=e^{a+b}\binom{n-p-1}{n}+O\left(n^{-p-2}\right).$$

Using the well-known asymptotic expansion of the binomial coefficient:

$$\binom{n-p-1}{n}\sim\frac{n^{-p-1}}{\Gamma\left(-p\right)}\left[1+\frac{p\left(p+1\right)}{2n}+\frac{p\left(p+1\right)\left(p+2\right)\left(3p+1\right)}{24n^{2}}+\cdots\right],$$

$$a_{n}=e^{a+b}\frac{n^{-p-1}}{\Gamma\left(-p\right)}\left[1+\frac{p\left(p+1\right)}{2n}+\frac{p\left(p+1\right)\left(p+2\right)\left(3p+1\right)}{24n^{2}}+\cdots\right]+O\left(n^{-p-2}\right),$$

which implies

$$a_{n}=e^{a+b}\frac{n^{-p-1}}{\Gamma\left(-p\right)}+O\left(n^{-p-2}\right),$$

and

$$\lim_{n\to\infty}n^{p+1}a_{n}=\frac{e^{a+b}}{\Gamma\left(-p\right)}.$$

(2) Now we need an extra term.
Set $\mathbf{m=1}$, $\beta=p$, $\mathbb{R}\ni p\notin\{...-2,-1,0,1,...\}$, $v\left(z\right)=e^{az^{2}+bz}$, $a,b\in\mathbb{R}$; and apply the Darboux's lemma above to obtain

$$a_{n}=\sum_{j=0}^{m}v_{j}\binom{n-\beta-j-1}{n}+O\left(n^{-m-\beta-2}\right)$$
$$=e^{a+b}\left[\binom{n-p-1}{n}-\left(2a+b\right)\binom{n-p-2}{n}\right]+O\left(n^{-p-3}\right).$$

$$a_{n}=e^{a+b}\frac{n^{-p-1}}{\Gamma\left(-p\right)}\left[1+\frac{p\left(p+1\right)}{2n}+\frac{p\left(p+1\right)\left(p+2\right)\left(3p+1\right)}{24n^{2}}+\cdots\right]$$
$$-\left(2a+b\right)e^{a+b}\frac{n^{-p-2}}{\Gamma\left(-p-1\right)}\left[1+\frac{\left(p+1\right)\left(p+2\right)}{2n}+\frac{\left(p+1\right)\left(p+2\right)\left(3p+4\right)}{24n^{2}}+\cdots\right]+O\left(n^{-p-3}\right).$$

$$n\left(n^{p+1}a_n-\frac{e^{a+b}}{\Gamma\left(-p\right)}\right)=e^{a+b}\left(\frac{p\left(p+1\right)}{2\Gamma\left(-p\right)}-\frac{2a+b}{\Gamma\left(-p-1\right)}\right)+O\left(n^{-1}\right).$$

$$\lim_{n\to\infty}n\left(n^{p+1}a_n-\frac{e^{a+b}}{\Gamma\left(-p\right)}\right)=e^{a+b}\left(\frac{p\left(p+1\right)}{2\Gamma\left(-p\right)}-\frac{2a+b}{\Gamma\left(-p-1\right)}\right)$$
$$=\frac{e^{a+b}}{\Gamma\left(-p\right)}\left(p+1\right)\left(\frac{p}{2}+2a+b\right).$$

Reference:

[1] Wilf, Herbert S., "*Generatingfunctionlogy*", 2^{nd} edition, Academic Press, 1992.

Also solved by the proposer.

148. *Proposed by D.M. Bătineţu-Giurgiu, "Matei Basarab" National College, Bucharest, Romania, and Neculai Stanciu, "George Emil Palade" School, Buzău, Romania.* Find

$$\lim_{x\to\infty}\left(x^{\cosh^2 t}\left((\Gamma(x+1))^{-(\sinh^2 t)/x}-(\Gamma(x+2))^{-(\sinh^2 t)/(x+1)}\right)\right).$$

where $t\in\mathbb{R}$ and Γ is the Gamma function.

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria.

The answer is $(\sinh^2 t)e^{\sinh^2 t}$.

Using the well known expansion, in the neighborhood of $+\infty$:

$$\log\Gamma(x+1)=x\log x-x+\frac{1}{2}\log x+\frac{1}{2}\log(2\pi)+\mathcal{O}\left(\frac{1}{x}\right)$$

we see immediately that

$$(\Gamma(x+1))^{1/x}=\exp\left(\log x-1+\frac{1}{2}\frac{\log x}{x}+\frac{1}{2x}\log(2\pi)+\mathcal{O}\left(\frac{1}{x^2}\right)\right)$$
$$=\frac{x}{e}\left(1+\frac{1}{2}\frac{\log x}{x}+\frac{1}{2x}\log(2\pi)+\mathcal{O}\left(\frac{\log^2 x}{x^2}\right)\right)$$

and consequently, for a nonzero α we have in the neighborhood of $+\infty$:

$$(\Gamma(x+1))^{\alpha/x}=\frac{x^\alpha}{e^\alpha}\left(1+\frac{\alpha}{2}\frac{\log x}{x}+\frac{\alpha}{2x}\log(2\pi)+\mathcal{O}\left(\frac{\log^2 x}{x^2}\right)\right)$$

But

$$\frac{\log(1+x)}{1+x}=\frac{\log x}{x}\left(1+\frac{\log(1+1/x)}{\log x}\right)\left(1+\frac{1}{x}\right)^{-1}$$
$$=\frac{\log x}{x}+\mathcal{O}\left(\frac{\log x}{x^2}\right)$$
$$(1+x)^\alpha=x^\alpha\left(1+\frac{\alpha}{x}+\mathcal{O}\left(\frac{1}{x^2}\right)\right)$$

Thus

$$(\Gamma(x+2))^{\alpha/(x+1)}=\frac{x^\alpha}{e^\alpha}\left(1+\frac{\alpha}{x}\right)\left(1+\frac{\alpha}{2}\frac{\log x}{x}+\frac{\alpha}{2x}\log(2\pi)\right)+\mathcal{O}\left(\frac{\log^2 x}{x^{2-\alpha}}\right)$$
$$=\frac{x^\alpha}{e^\alpha}\left(1+\frac{\alpha}{2}\frac{\log x}{x}+\left(1+\log\sqrt{2\pi}\right)\frac{\alpha}{x}\right)+\mathcal{O}\left(\frac{\log^2 x}{x^{2-\alpha}}\right)$$

Hence

$$x^{1-\alpha}\left((\Gamma(x+1))^{\alpha/(x)}-(\Gamma(x+2))^{\alpha/(x+1)}\right)=-\alpha e^{-\alpha}+\mathcal{O}\left(\frac{\log^2 x}{x}\right)$$

Therefore,

$$\lim_{x\to\infty}\left(x^{1-\alpha}\left((\Gamma(x+1))^{\alpha/(x)}-(\Gamma(x+2))^{\alpha/(x+1)}\right)\right)=-\alpha e^{-\alpha}.$$

which is also valid when $\alpha = 0$. Choosing $\alpha = -\sinh^2 t$ we get the announced answer.

Solution 2 by by Soumitra Mandal, Chandar Nagore, India.
Since,

$$\lim_{x\to\infty}\frac{\sqrt[x]{\Gamma(x+1)}}{x}=\lim_{\substack{n\to\infty\\ n\in\mathbb{N}}}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\overbrace{=}^{D'ALEMBERT}=$$

$$=\lim_{x\to\infty}\left(\frac{(n+1)!}{n!}\cdot\frac{1}{\left(1+\frac{1}{n}\right)^n}\cdot\frac{1}{n+1}\right)=\frac{1}{e}$$

Now, we have

$$\lim_{x\to\infty}\left(\frac{\sqrt[x+1]{\Gamma(x+2)}}{\sqrt[x]{\Gamma(x+1)}}\right)^{-\sinh^2(t)}=\lim_{x\to\infty}\left(\frac{\sqrt[x+1]{\Gamma(x+2)}}{x+1}\cdot\frac{1}{\frac{\sqrt[x]{\Gamma(x+1)}}{x}}\cdot\frac{x+1}{x}\right)^{-\sinh^2(t)}=1$$

$$\therefore\lim_{x\to\infty}\frac{u(x)-1}{\ln u(x)}=1\ \textit{where}\ u(x)=\left(\frac{\sqrt[x+1]{\Gamma(x+2)}}{\sqrt[x]{(x+1)}}\right)^{-\sinh^2(t)}$$

again,

$$\lim_{x\to\infty}\Big(u(x)\Big)^x=\lim_{x\to\infty}\left(\frac{\sqrt[x+1]{\Gamma(x+2)}}{\sqrt[x]{\Gamma(x+1)}}\right)^{-x\sinh^2(t)}=\lim_{x\to\infty}\left(\frac{\Gamma(x+2)}{\Gamma(x+1)}\cdot\frac{1}{\sqrt[x+1]{\Gamma(x+2)}}\right)^{-\sinh^2(t)}$$

$$=\lim_{x\to\infty}\left(\frac{1}{\frac{\sqrt[x+1]{\Gamma(x+2)}}{x+1}}\right)^{-\sinh^2(t)}=e^{-\sinh^2(t)}$$

$$\therefore\lim_{x\to\infty}\left(x^{\cosh^2(t)}\left(\Big(\Gamma(x+1)\Big)^{-\frac{\sinh^2(t)}{x}}-\Big(\Gamma(x+2)\Big)^{-\frac{\sinh^2(t)}{x+1}}\right)\right)$$

$$=\lim_{x\to\infty}\left(-x^{\cosh^2(t)}\Big(\Gamma(x+1)\Big)^{\frac{\sinh^2(t)}{x}}\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln u(x)\right)$$

$$=\lim_{x\to\infty}\left(-\left(\frac{\sqrt[x]{\Gamma(x+1)}}{x}\right)^{-\sinh^2(t)}\cdot\frac{u(x)-1}{\ln u(x)}\cdot\ln\Big(u(x)\Big)^x\right)=\sinh^2(t)e^{\sinh^2(t)}$$

Solution 3 by Moti Levy, Rehovot, Israel. Let $a=\cosh^2 t$ then our limit becomes

$$L=\lim_{x\to\infty} x^a\left(\left(\Gamma(x+1)^{\frac{1}{x}}\right)^{1-a}-\left(\Gamma(x+2)^{\frac{1}{x+1}}\right)^{1-a}\right)$$
$$=\lim_{x\to\infty} x^a\left(\Gamma(x+1)^{\frac{1}{x}}\right)^{1-a}\left(1-\left(\frac{\Gamma(x+2)^{\frac{1}{x+1}}}{\Gamma(x+1)^{\frac{1}{x}}}\right)^{1-a}\right).$$

Using the asymptotic expression for $\Gamma(x+1)$,

$$\Gamma(x+1)\sim\sqrt{2\pi x}\left(\frac{x}{e}\right)^x,$$

we obtain

$$\Gamma(x+1)^{\frac{1}{x}}\sim\frac{x}{e},$$
$$\Gamma(x+2)^{\frac{1}{x+1}}\sim\frac{x+1}{e}.$$

Hence

$$L=\lim_{x\to\infty} x^a\left(\frac{x}{e}\right)^{1-a}\left(1-\left(\frac{x+1}{x}\right)^{1-a}\right)=e^{a-1}\lim_{x\to\infty} x\left(1-\left(\frac{x+1}{x}\right)^{1-a}\right)$$
$$=e^{a-1}\lim_{x\to\infty}\frac{1-\left(\frac{x+1}{x}\right)^{1-a}}{x^{-1}}.$$

Applying L'Hopital's rule

$$L=e^{a-1}\lim_{x\to\infty}\frac{-(1-a)\left(\frac{x+1}{x}\right)^{-a}\left(-x^{-2}\right)}{\left(-x^{-2}\right)}=(a-1)\,e^{a-1}.$$

$$\lim_{x\to\infty} x^{\cosh^2 t}\left(\left(\Gamma(x+1)^{\frac{1}{x}}\right)^{-\sinh^2 t}-\left(\Gamma(x+2)^{\frac{1}{x+1}}\right)^{-\sinh^2 t}\right)$$

$$=(\cosh$$

$^2t-1e^{\left(\cosh^2 t-1\right)}=e^{\sinh^2 t}\sinh^2 t.$

Also solved by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy; Moubinool Omarjee, Paris, France and the proposers.

149. *Proposed by Arkady Alt, San Jose, California, USA.* Let D be the set of strictly decreasing sequences of positive real numbers with first term equal to 1. For given positive p, r and any $x_{\mathbb{N}}=(x_1,x_2,\ldots)\in D$, let $S(x_{\mathbb{N}})=\sum_{n=1}^{\infty}\frac{x_n^{p+r}}{x_{n+1}^p}$ if this series converges and define $S(x_{\mathbb{N}})=\infty$ otherwise. Find $\inf\{S(x_{\mathbb{N}})|x_{\mathbb{N}}\in D\}$.

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria.

The answer is $A(p,r)\overset{\text{def}}{=}\left(\frac{(p+r)^{p+r}}{r^r p^p}\right)^{1/r}$.

Consider a sequence $x_{(\mathbb{N})} \in D$. For $n > 2$ we have, using Hölder's inequality:

$$\begin{aligned}
\sum_{k=1}^{n-1} x_k^r &= \sum_{k=1}^{n-1} \frac{x_k^r}{x_{k+1}^{pr/(p+r)}} \cdot x_{k+1}^{pr/(p+r)} \\
&\leq \left(\sum_{k=1}^{n-1} \left(\frac{x_k^r}{x_{k+1}^{pr/(p+r)}}\right)^{\frac{p+r}{r}}\right)^{\frac{r}{p+r}} \left(\sum_{k=1}^{n-1} \left(x_{k+1}^{pr/(p+r)}\right)^{\frac{p+r}{p}}\right)^{\frac{p}{p+r}} \\
&\leq \left(\sum_{k=1}^{n-1} \frac{x_k^{p+r}}{x_{k+1}^p}\right)^{\frac{r}{p+r}} \left(\sum_{k=1}^{n-1} x_{k+1}^r\right)^{\frac{p}{p+r}}
\end{aligned}$$

So we have proved that

$$\left(\sum_{k=1}^{n-1} x_k^r\right)^{1+p/r} \leq \left(\sum_{k=1}^{n-1} \frac{x_k^{p+r}}{x_{k+1}^p}\right) \left(\sum_{k=2}^{n} x_k^r\right)^{p/r} \tag{1}$$

On the other hand, using the arithmetic mean-geometric mean inequality, we have for $x, t > 0$ that

$$\frac{1+x}{1+t} = \frac{1+t(x/t)}{1+t} \geq \left(\frac{x}{t}\right)^{t/(1+t)}$$

Applying this with $x = \sum_{k=2}^{n-1} x_k^r$ and $t = p/r$ we see that

$$\left(\sum_{k=1}^{n-1} x_k^r\right)^{1+p/r} \geq \frac{(1+t)^{1+t}}{t^t} \left(\sum_{k=2}^{n-1} x_k^r\right)^{p/r}$$

Or equivalently

$$\left(\sum_{k=1}^{n-1} x_k^r\right)^{1+p/r} \geq A(p,r) \left(-x_n^r + \sum_{k=2}^{n} x_k^r\right)^{p/r} \tag{2}$$

Combining (1) and (2) we get

$$\sum_{k=1}^{n-1} \frac{x_k^{p+r}}{x_{k+1}^p} \geq A(p,r) \left(1 - \frac{x_n^r}{\sum_{k=2}^n x_k^r}\right)^{p/r} \tag{3}$$

and this is also valid for $n = 2$. Now, let us consider two cases:

- If $\sum_{k=1}^{\infty} x_k^r = +\infty$ then from the inequality

  $$0 \leq \frac{x_n^r}{\sum_{k=2}^n x_k^r} \leq \frac{x_1^r}{\sum_{k=2}^n x_k^r}$$

  we conclude that

  $$\lim_{n\to\infty} \frac{x_n^r}{\sum_{k=2}^n x_k^r} = 0$$

- If $\sum_{k=1}^{\infty} x_k^r = \ell < +\infty$, then clearly $\lim_{n\to\infty} x_n^r = 0$ and again

  $$\lim_{n\to\infty} \frac{x_n^r}{\sum_{k=2}^n x_k^r} = 0$$

Combining the above results and letting n tend to infinity in (3) we conclude that $S(x_{\mathbb{N}}) \geq A(p,r)$, and consequently

$$\inf\{S(x_{\mathbb{N}})|x_{\mathbb{N}} \in D\} \geq A(p,r) \tag{4}$$

Conversely, consider the sequence $a_{\mathbb{N}} = (a_n)_{n\geq 1}$ defined by $a_n = \alpha^{n-1}$ with $\alpha = \left(\frac{p}{p+r}\right)^{1/r} < 1$. Clearly we have

$$S(a_{\mathbb{N}}) = \sum_{n=1}^{\infty} \frac{\alpha^{(r+p)(n-1)}}{\alpha^{pn}} = \frac{1}{\alpha^p} \cdot \frac{1}{1-\alpha^r} = \left(\frac{p+r}{p}\right)^{p/r} \cdot \frac{p+r}{r} = A(p,r).$$

Hence,$\inf\{S(x_{\mathbb{N}})|x_{\mathbb{N}} \in D\} = A(p,r)$ and the lower bound is in fact attained on a geometric sequence.

Also solved by the proposer.

150. *Proposed by Cornel Ioan Vălean, Timiş, Romania.* Find

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{k+n}\frac{H_{k+n}^3}{k+n},$$

where $H_n = \sum_{j=1}^{n} 1/j$ denotes the nth harmonic number.

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria. We will use a general principle. Consider a an analytic f in the unit disk $D(0,1)$, and suppose that its power series expansion is given by $f(z) = \sum_{n=1}^{\infty} a_n z^n$. Now, using the integral form of the remainder we may write for $|z| < 1$ the following

$$\begin{aligned}\sum_{n=1}^{\infty} a_{n+k} z^{n+k} &= f(z) - \sum_{n=1}^{k} \frac{f^{(n)}(0)}{n!} z^n \\ &= \frac{z^{k+1}}{k!}\int_0^1 (1-t)^k f^{(k+1)}(tz)dt\end{aligned}$$

It follows that for $|w| < 1$ we have

$$\begin{aligned}\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty} a_{n+k} z^{n+k}\right) w^k &= z\int_0^1 \left(\sum_{k=1}^{\infty} \frac{(f')^{(k)}(tz)}{k!}(zw(1-t))^k\right) dt \\ &= z\int_0^1 \big(f'(tz + zw(1-t)) - f'(tz)\big)dt \\ &= \left[\frac{1}{1-w} f(zw + tz(1-w)) - f(tz)\right]_{t=0}^{t=1} \\ &= \frac{f(z) - f(zw)}{1-w} - f(z)\end{aligned} \tag{1}$$

Now in our case we have $a_n = H_n^3/n$ and $f(z) = \sum_{n=1}^{\infty} \frac{H_n^3}{n} z^n$. Since the series defining $f(-1)$ does converge by the alternating series test (this is not straightforward but it can be proved that the coefficients decrease to 0 starting from a certain index), it is easy to show that uniformly in $z \in (-1,0)$ we have $\sum_{n=1}^{\infty} a_{n+k} z^{n+k} = \mathcal{O}(\log^3 k)$

and the series $\sum_{k=1}^{\infty}(\log^3 k)\,|w|^k$ is convergent. Hence, we may take the limit as $z \to (-1)^+$ in (1) to obtain

$$\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+k}H_{n+k}^3}{n+k}\right)w^k = \frac{f(-1)-f(-w)}{1-w} - f(-1)$$

Now, the series

$$\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+k}H_{n+k}^3}{n+k}\right)$$

is convergent (another non trivial statement that I will leave to the reader). So, using Abel's theorem we conclude that

$$\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+k}H_{n+k}^3}{n+k}\right) = \lim_{w\to 1^-}\frac{f(-1)-f(-w)}{1-w} - f(-1) \tag{2}$$

Now, we are saved by our calculations in the solution to problem 140 in the previous issue, where we have shown that

$$\begin{aligned} xf'(x) &= \sum_{n=1}^{\infty} H_n^3 x^n \\ &= -\frac{\log^3(1-x)}{1-x} - \frac{\pi^2}{2}\frac{\log(1-x)}{1-x} + 3\frac{\mathrm{Li}_3(1-x)-\mathrm{Li}_3(1)}{1-x} \\ &\quad + \frac{3}{2}\frac{\log(x)\log^2(1-x)}{1-x} + \frac{\mathrm{Li}_3(x)}{1-x} \end{aligned}$$

and consequently

$$\lim_{x\to(-1)^+} xf'(x) = -\frac{\log^3(2)}{2} - \frac{\pi^2}{4}\log(2) + \frac{3}{2}\left(\mathrm{Li}_3(2) + \frac{i\pi}{2}\log^2(2)\right) - \frac{3}{2}\mathrm{Li}_3(1) + \frac{1}{2}\mathrm{Li}_3(-1)$$

Recalling that

$$\begin{aligned} \mathrm{Li}_3(1) &= \zeta(3) \\ \mathrm{Li}_3(-1) &= -\frac{3}{4}\zeta(3) \\ \mathrm{Li}_3(2) &= \frac{7}{8}\zeta(3) + \frac{\pi^2}{4} - \frac{i\pi}{2}\log^2(2) \end{aligned}$$

where for the last one we used the formula 6.7 from "Lewin, L. (1981). Polylogarithms and Associated Functions. New York: North-Holland" to express $\mathrm{Li}_3(2)$ in terms of $\mathrm{Li}_3(1/2)$ and we used formula 6.12 of the same book to evaluate $\mathrm{Li}_3(1/2)$. It follows that

$$\lim_{x\to 1^-}(-f'(-x)) = -\frac{9}{16}\zeta(3) - \frac{1}{2}\log^3(2) + \frac{\pi^2}{8}\log(2)$$

So, by the Hospital rule we see that

$$\lim_{w\to 1^-}\frac{f(-1)-f(-w)}{1-w} = -\frac{9}{16}\zeta(3) - \frac{1}{2}\log^3(2) + \frac{\pi^2}{8}\log(2) \tag{3}$$

On the other hand $f(-1)$ is given in the article of I. Mezö, "Nonlinear Euler sums" in Pacific Journal of Mathematics (Vol. 272, No 1,2014), with a sign error in the last term:

$$-f(-1)=\sum_{n=1}^{\infty}\frac{(-1)^nH_n^3}{n}=\frac{\pi^4}{144}+\frac{\pi^2}{8}\log^2 2-\frac{\log^4 2}{4}-\frac{9}{8}\log 2\,\zeta(3), \tag{4}$$

Combining (2), (3) and (4) we conclude that

$$\sum_{k=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{(-1)^{n+k}H_{n+k}^3}{n+k}\right)=\frac{\pi^4}{144}+(\log 2+\log^2 2)\frac{\pi^2}{8}-\frac{\log^3 2}{2}-\frac{\log^4 2}{4}-\frac{9(1+\log 4)}{16}\zeta(3)$$

which is the desired conclusion.

Solution 2 by Moti Levy, Rehovot, Israel.

Let $F(z):=\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\frac{H_{k+n}^3}{k+n}z^{k+n}, \quad |z|<1$. Setting $m=k+n$ and rearranging the order of summation give

$$F(z)=\sum_{m=2}^{\infty}\left(H_m^3-\frac{H_m^3}{m}\right)z^m, \quad |z|<1. \tag{1}$$

Let $f(z)$ be the generating function of the sequence $\left(H_m^3\right)_{m\geq 1}$. Expression of $f(z)$ appeared in [1],

$$f(z)=\frac{1}{1-z}\left(-\frac{\pi^2}{2}\ln(1-z)-\ln^3(1-z)+\frac{3}{2}\ln^2(1-z)\ln z+3\mathrm{Li}_3(1-z)+\mathrm{Li}_3(z)-3\zeta(3)\right). \tag{2}$$

The function $f(z)$ is the analytic continuation of $\sum_{m=1}^{\infty}H_m^3z^m, \ |z|<1$ to $\mathbb{C}/[1,\infty)$.
The second term in (1) can be obtained by integration term by term of $\sum_{m=1}^{\infty}H_m^3z^m, \ |z|<1$,

$$\sum_{m=2}^{\infty}\frac{H_m^3}{m}z^m=\int_0^z\sum_{m=1}^{\infty}H_m^3t^{m-1}dt-z, \ |z|<1.$$

Let

$$G(z):=(f(z)-z)-\left(\int_0^z\frac{f(t)}{t}dt-z\right)=f(z)-\int_0^z\frac{f(t)}{t}dt. \tag{3}$$

The function $G(z)$ is the analytic continuation of $F(z)$ to $\mathbb{C}/[1,\infty)$.
Thus $G(-1)=f(-1)-\int_0^{-1}\frac{f(t)}{t}dt$ is finite real number, and it is equal to $\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{k+n}\frac{H_{k+n}^3}{k+n}$.
Relation (4) can be found in [2], (6.10) page 155,

$$\mathrm{Li}_3(1-x)=-\mathrm{Li}_3\left(\frac{x}{x-1}\right)-\mathrm{Li}_3(x)+\mathrm{Li}_3(1)+\frac{\pi^2}{6}\ln(1-x)-\frac{1}{2}\ln^2(1-x)\ln x+\frac{1}{6}\ln^3(1-x). \tag{4}$$

Using (4), $f(z)$ in (2) is simplified as follows:

$$f(z)=\frac{1}{1-z}\left(-\frac{1}{2}\ln^3(1-z)-3\mathrm{Li}_3\left(\frac{z}{z-1}\right)-2\mathrm{Li}_3(z)\right).$$

$$f(-1)=\frac{1}{2}\left(-\frac{1}{2}\ln^3 2-3\mathrm{Li}_3\left(\frac{1}{2}\right)-2\mathrm{Li}_3(-1)\right). \tag{5}$$

These special values are known,

$$\mathrm{Li}_3\left(\frac{1}{2}\right)=\frac{1}{24}\left(4\ln^3 2+21\zeta(3)-2\pi^2\ln 2\right), \tag{6}$$

$$\mathrm{Li}_3(-1)=-\frac{3}{4}\zeta(3). \tag{7}$$

$$f(-1) = \frac{1}{2}\left(-\frac{1}{2}\ln^3 2 - \frac{3}{24}\left(4\ln^3 2 + 21\zeta(3) - 2\pi^2\ln 2\right) - 2\left(-\frac{3}{4}\zeta(3)\right)\right) \quad (8)$$

$$= \frac{1}{8}\pi^2\ln 2 - \frac{9}{16}\zeta(3) - \frac{1}{2}\ln^3 2. \quad (9)$$

$$\int_0^{-1}\frac{f(u)}{u}du = \int_0^1\frac{f(-u)}{u}du \quad (10)$$

$$= \int_0^1\frac{1}{(1+u)u}\left(-\frac{1}{2}\ln^3(1+u) - 3\mathrm{Li}_3\left(\frac{u}{1+u}\right) - 2\mathrm{Li}_3(-u)\right)du$$

$$= -\frac{1}{2}\int_0^1\frac{\ln^3(1+u)}{(1+u)u}du - 3\int_0^{\frac{1}{2}}\frac{\mathrm{Li}_3(u)}{u}du - 2\int_0^1\frac{\mathrm{Li}_3(-u)}{(1+u)u}du$$

$$\int_0^1\frac{\ln^3(1+u)}{(1+u)u}du = \int_0^1\frac{\ln^3(1+u)}{u}du - \int_0^1\frac{\ln^3(1+u)}{1+u}du$$

$$= \left(\frac{1}{15}\pi^4 + \frac{1}{4}\pi^2\ln^2 2 - \frac{1}{4}\ln^4 2 - 6\mathrm{Li}_4\left(\frac{1}{2}\right) - \frac{21}{4}\zeta(3)\ln 2\right) - \frac{1}{4}\ln^4 2.$$

$$\int_0^{\frac{1}{2}}\frac{\mathrm{Li}_3(u)}{u}du = \mathrm{Li}_4\left(\frac{1}{2}\right).$$

$$\int_0^1\frac{\mathrm{Li}_3(-u)}{(1+u)u}du = \int_0^1\frac{\mathrm{Li}_3(-u)}{u}du - \int_0^1\frac{\mathrm{Li}_3(-u)}{1+u}du = -\frac{7}{720}\pi^4 - \left(\frac{\pi^4}{288} - \frac{3}{4}\zeta(3)\ln 2\right)$$

$$= -\frac{19}{1440}\pi^4 + \frac{3}{4}\zeta(3)\ln 2.$$

$$\int_0^{-1}\frac{f(u)}{u}du = \frac{-1}{144}\pi^4 - \frac{1}{8}\pi^2\ln^2 2 + \frac{1}{4}\ln^4 2 + \frac{9}{8}\zeta(3)\ln 2.$$

$$\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{k+n}\frac{H_{k+n}^3}{k+n} = \frac{1}{144}\pi^4 + \frac{1}{8}\pi^2(1+\ln 2)\ln 2 - \frac{1}{4}\left(\ln^3 2\right)(\ln 2 + 2) - \frac{9}{8}\zeta(3)\ln 2 - \frac{9}{16}\zeta(3).$$

References:

[1] István Mezõ, "*Nonlinear Euler Sums*", The Pacific Journal of Mathematics, Vol.272, No. 1, 2014.

[2] Leonard Lewin, "Polylogarithms and Associated Function", North Holland, 1981.

Also solved by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy and the proposer.

151. *Proposed by Albert Stadler, Herrliberg, Switzerland.* Prove that

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=n}^{\infty}\frac{1}{1+k^2} = \left(\frac{3}{2} + \frac{\pi}{2}\coth\pi\right)\sum_{k=1}^{\infty}\frac{1}{k(1+k^2)} - \sum_{k=1}^{\infty}\frac{1}{k(1+k^2)^2}.$$

Solution 1 by Omran Kouba, Higher Institute for Applied Sciences and Technology Damascus, Syria. First recall that for $x \notin i\mathbb{Z}$ we have

$$\pi\coth(\pi x) = \frac{1}{x} + \sum_{k=1}^{\infty}\frac{2x}{x^2+k^2}$$

So, taking $x = 1$ and rearranging we get

$$\frac{1}{2} + \frac{\pi}{2}\coth\pi = \sum_{k=0}^{\infty}\frac{1}{1+k^2} \tag{1}$$

Now, define

$$A = \sum_{k=1}^{\infty}\frac{1}{k(1+k^2)} \qquad B = \sum_{k=1}^{\infty}\frac{1}{k(1+k^2)^2}$$

$$D = \sum_{n=1}^{\infty}\frac{1}{n}\left(\sum_{k=n}^{\infty}\frac{1}{1+k^2}\right) \qquad C = \left(\frac{1}{2} + \frac{\pi}{2}\coth\pi\right)A$$

Using (1) we see that

$$C = \left(\sum_{j=0}\frac{1}{1+j^2}\right)\left(\sum_{k=1}^{\infty}\frac{1}{k(1+k^2)}\right)$$

and because, clearly we have

$$\sum_{k,j\geq 1}\frac{1}{k(1+k^2)(1+j^2)} = \sum_{k,j\geq 1}\frac{1}{j(1+j^2)(1+k^2)}$$

we see immediately that

$$\begin{aligned} C &= A + \sum_{k,j\geq 1}\frac{1}{k(1+k^2)(1+j^2)} \\ &= A + \frac{1}{2}\sum_{k,j\geq 1}\frac{1}{(1+k^2)(1+j^2)}\left(\frac{1}{k}+\frac{1}{j}\right) \\ &= A + \frac{1}{2}\left(\sum_{k=j\geq 1}\frac{k+j}{kj(1+k^2)(1+j^2)} + 2\sum_{1\leq j<k}\frac{k+j}{kj(1+k^2)(1+j^2)}\right) \end{aligned}$$

That is

$$C = A + B + \underbrace{\sum_{1\leq j<k}\frac{k+j}{kj(1+k^2)(1+j^2)}}_{E} \tag{2}$$

Now, note that for $k > j$ we have

$$\begin{aligned} \frac{k+j}{kj(1+k^2)(1+j^2)} &= \frac{k^2-j^2}{kj(k-j)(1+k^2)(1+j^2)} \\ &= \frac{(k^2+1)-(j^2+1)}{kj(k-j)(1+k^2)(1+j^2)} \\ &= \frac{1}{kj(k-j)(1+j^2)} - \frac{1}{kj(k-j)(1+k^2)} \\ &= \frac{1}{j^2(1+j^2)}\left(\frac{1}{k-j}-\frac{1}{k}\right) - \frac{1}{k(1+k^2)}\left(\frac{1}{k-j}+\frac{1}{j}\right) \end{aligned}$$

Thus

$$\begin{aligned} E &= \sum_{j=1}^{\infty} \frac{1}{j^2(1+j^2)} \sum_{k=j+1}^{\infty} \left(\frac{1}{k-j} - \frac{1}{k} \right) - \sum_{k=2}^{\infty} \frac{1}{k^2(1+k^2)} \sum_{j=1}^{k-1} \left(\frac{1}{k-j} + \frac{1}{j} \right) \\ &= \sum_{j=1}^{\infty} \frac{H_j}{j^2(1+j^2)} - \sum_{k=2}^{\infty} \frac{2H_{k-1}}{k^2(1+k^2)} \end{aligned}$$

where $H_n = \sum_{k=1}^{n} 1/k$ is the nth harmonic number. Noting that $H_{k-1} = H_k - 1/k$ we conclude that

$$E = 2\sum_{k=1}^{\infty} \frac{1}{k^3(1+k^2)} - \sum_{k=1}^{\infty} \frac{H_k}{k^2(1+k^2)}$$

But

$$\frac{1}{k^2(1+k^2)} = \frac{1}{k^2} - \frac{1}{1+k^2}$$

So

$$E = 2\sum_{k=1}^{\infty} \frac{1}{k^3} - 2\sum_{k=1}^{\infty} \frac{1}{k(1+k^2)} - \sum_{k=1}^{\infty} \frac{H_k}{k^2} + \sum_{k=1}^{\infty} \frac{H_k}{1+k^2}$$

That is

$$E = 2\sum_{k=1}^{\infty} \frac{1}{k^3} - \sum_{k=1}^{\infty} \frac{H_k}{k^2} - 2A + D$$

where we used the straightforward fact that

$$\sum_{k=1}^{\infty} \frac{H_k}{1+k^2} = \sum_{1 \le n \le k} \frac{1}{n(1+k^2)} = \sum_{n=1}^{\infty} \frac{1}{n} \sum_{k=n}^{\infty} \frac{1}{1+k^2} = D$$

Replacing the expression for E in (2) we get

$$C + A - B = D + 2\sum_{k=1}^{\infty} \frac{1}{k^3} - \sum_{k=1}^{\infty} \frac{H_k}{k^2} \tag{3}$$

The final step is to note that

$$2\sum_{k=1}^{\infty} \frac{1}{k^3} = \sum_{k=1}^{\infty} \frac{H_k}{k^2} \tag{4}$$

to conclude that

$$D = C + A - B = \left(\frac{3}{2} + \frac{\pi}{2} \coth \pi \right) A - B$$

which is the desired conclusion.

The only remaining step is to prove the well-known result (4). This can be done as follows: First, note that

$$H_k = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+k} \right) = \sum_{n=1}^{\infty} \frac{k}{n(n+k)}.$$

Hence

$$\begin{aligned}\sum_{k=1}^{\infty}\frac{H_k}{k^2} &= \sum_{k,n\geq 1}\frac{1}{kn(k+n)} = \sum_{k,n\geq 1}\frac{k+n}{kn(k+n)^2}\\ &= 2\sum_{k,n\geq 1}\frac{1}{k(k+n)^2} = 2\sum_{k=1}^{\infty}\frac{1}{k}\sum_{j=k+1}\frac{1}{j^2} = 2\sum_{1\leq k<j}\frac{1}{kj^2}\\ &= 2\sum_{j=2}^{\infty}\frac{1}{j^2}\sum_{k=1}^{j-1}\frac{1}{k} = 2\sum_{j=2}^{\infty}\frac{1}{j^2}\left(H_j-\frac{1}{j}\right)\\ &= 2\sum_{j=1}^{\infty}\frac{H_j}{j^2} - 2\sum_{j=1}^{\infty}\frac{1}{j^3}\end{aligned}$$

and (4) follows immediately. This concludes the solution of the problem

Solution 2 by Moti Levy, Rehovot, Israel.

By changing order of summation,

$$\sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=n}^{\infty}\frac{1}{1+k^2} = \sum_{k=1}^{\infty}\frac{1}{1+k^2}\sum_{n=1}^{k}\frac{1}{n} = \sum_{k=1}^{\infty}\frac{H_k}{1+k^2}.$$

One can recognize that

$$\frac{3}{2}+\frac{\pi}{2}\coth\pi = 2+\sum_{k=1}^{\infty}\frac{1}{1+k^2},$$

so the original problem can be restated as follows: Show that,

$$\sum_{k=1}^{\infty}\frac{H_k}{1+k^2} = \left(2+\sum_{k=1}^{\infty}\frac{1}{1+k^2}\right)\left(\sum_{k=1}^{\infty}\frac{1}{k\left(1+k^2\right)}\right) - \sum_{k=1}^{\infty}\frac{1}{k\left(1+k^2\right)^2}. \qquad (11)$$

The left hand side of (11) is a reminiscent of Euler's sum, which can be evaluated using complex summation method. See the classical article by Flajolet and Salvy [1].

$$2\sum_{k=1}^{\infty}r(k)\,H_k + \sum_{k=1}^{\infty}r^{'}(k) + \sum_{c\in\{0\}\cup\{poles\ of\ r(s)\}}\text{Residue}\left[r(s)\left(\gamma+\psi(-s)\right)^2, c\right] = 0 \qquad (12)$$

where $r(s)$ is rational function satisfying the two conditions: 1) $r(s)$ is $O\left(s^{-2}\right)$ at infinity, 2) $r(s)$ has no pole in $\mathbb{Z}\setminus\{0\}$.

The function $\psi(s)$ is the Digamma function which satisfies,

$$\gamma+\psi(-s) = \frac{1}{s}+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{-s+k}\right). \qquad (13)$$

Our rational function $r(s)=\frac{1}{1+s^2}$, clearly meets the two conditions, and

$$\sum_{k=1}^{\infty}r^{'}(k) = 2\sum_{k=1}^{\infty}\frac{k}{\left(1+k^2\right)^2}. \qquad (14)$$

The residues are:

$$\text{Residue}\left[r\left(s\right)\left(\gamma+\psi\left(-s\right)\right)^{2},0\right]=0,$$
$$\text{Residue}\left[r\left(s\right)\left(\gamma+\psi\left(-s\right)\right)^{2},i\right]=-\frac{1}{2}i\left(\gamma+\psi\left(-i\right)\right)^{2},$$
$$\text{Residue}\left[r\left(s\right)\left(\gamma+\psi\left(-s\right)\right)^{2},-i\right]=\frac{1}{2}i\left(\gamma+\psi\left(i\right)\right)^{2}. \tag{15}$$

We substitute (14) and (15) in (12) and get,

$$\sum_{k=1}^{\infty}\frac{H_k}{1+k^2}=\sum_{k=1}^{\infty}\frac{k}{\left(1+k^2\right)^2}-\frac{i}{4}\left(\left(\gamma+\psi\left(i\right)\right)^2-\left(\gamma+\psi\left(-i\right)\right)^2\right) \tag{16}$$
$$=\sum_{k=1}^{\infty}\frac{k}{\left(1+k^2\right)^2}-\frac{i}{4}\left(\left(\gamma+\psi\left(i\right)\right)+\left(\gamma+\psi\left(-i\right)\right)\right)*\left(\left(\gamma+\psi\left(i\right)\right)-\left(\gamma+\psi\left(-i\right)\right)\right).$$

Now we use equation (13) to express the residues by infinite series,

$$-\frac{i}{4}\left(\left(\gamma+\psi\left(i\right)\right)+\left(\gamma+\psi\left(-i\right)\right)\right)*\left(\left(\gamma+\psi\left(i\right)\right)-\left(\gamma+\psi\left(-i\right)\right)\right)$$
$$=-\frac{i}{4}\left(\frac{1}{i}+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{-i+k}\right)+-\frac{1}{i}+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{i+k}\right)\right)*$$
$$*\left(\frac{1}{i}+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{-i+k}\right)+\frac{1}{i}-\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{i+k}\right)\right)$$
$$=-\frac{i}{4}\left(\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{-i+k}+\frac{1}{k}-\frac{1}{i+k}\right)\right)\left(\frac{2}{i}+\sum_{k=1}^{\infty}\left(\frac{1}{k}-\frac{1}{-i+k}-\frac{1}{k}+\frac{1}{i+k}\right)\right)$$
$$=-\frac{i}{4}\left(2\sum_{k=1}^{\infty}\frac{1}{k\left(1+k^2\right)}\right)\left(\frac{2}{i}-2i\sum_{k=1}^{\infty}\frac{1}{1+k^2}\right)=\left(\sum_{k=1}^{\infty}\frac{1}{k\left(1+k^2\right)}\right)\left(1+\sum_{k=1}^{\infty}\frac{1}{1+k^2}\right). \tag{17}$$

Since $\frac{k}{(k^2+1)^2}=\frac{1}{k(1+k^2)}-\frac{1}{k(1+k^2)^2}$, we can rewrite (17),

$$\sum_{k=1}^{\infty}\frac{H_k}{1+k^2}=\sum_{k=1}^{\infty}\frac{1}{k\left(1+k^2\right)}+\left(\sum_{k=1}^{\infty}\frac{1}{k\left(1+k^2\right)}\right)\left(1+\sum_{k=1}^{\infty}\frac{1}{1+k^2}\right)-\sum_{k=1}^{\infty}\frac{1}{k\left(1+k^2\right)^2}$$
$$=\left(2+\sum_{k=1}^{\infty}\frac{1}{1+k^2}\right)\left(\sum_{k=1}^{\infty}\frac{1}{k\left(1+k^2\right)}\right)-\sum_{k=1}^{\infty}\frac{1}{k\left(1+k^2\right)^2}.$$

Reference:

[1] Philippe Flajolet and Bruno Salvy, "*Euler Sums and Contour Integral Representations*", Experimental Mathematics, Vol. 7 (1998), No. 1.

Also solved by the proposer.

MATHCONTEST SECTION

This section of the Journal offers readers an opportunity to solve interesting and elegant mathematical problems mainly appeared in Math Contest around the world and most appropriate for training Math Olympiads. Proposals are always welcomed. The source of the proposals will appear when the solutions be published.

Proposals

105. Let $f : [0; +\infty) \to \mathbb{R}$ be a continuous function such that $\lim\limits_{x\to+\infty} f(x) = L$ exists (it may be finite or infinite). Prove that

$$\lim_{n\to\infty} \int_0^1 f(nx)\,\mathrm{d}x = L.$$

106. Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). Suppose that f has infinitely many zeros, but there is no $x \in (a, b)$ with $f(x) = f'(x) = 0$. (a) Prove that $f(a)f(b) = 0$. (b) Give an example of such a function on $[0, 1]$.

107. Let A be a $n \times n$ complex matrix whose eigenvalues have absolute value at most 1. Prove that

$$\|A^n\| \le \frac{n}{\ln 2}\|A\|^{n-1}.$$

(Here $\|B\| = \sup\limits_{\|x\|\le 1} \|Bx\|$ for every $n \times n$ matrix B and $\|x\| = \sqrt{\sum\limits_{i=1}^{n} |x_i|^2}$ for every complex vector $x \in \mathbb{C}^n$.)

108. Let k and n be positive integers with $n \ge k^2 - 3k + 4$, and let

$$f(z) = z^{n-1} + c_{n-2}z^{n-2} + \cdots + c_0$$

be a polynomial with complex coefficients such that

$$c_0c_{n-2} = c_1c_{n-3} = \cdots = c_{n-2}c_0 = 0$$

Prove that $f(z)$ and $z^n - 1$ have at most $n - k$ common roots.

109. Let n be a positive integer, and let $p(x)$ be a polynomial of degree n with integer coefficients. Prove that

$$\max_{0\le x\le 1} \left|p(x)\right| \ge \frac{1}{e^n}.$$

Solutions

100. Let $(a_n)_{n\in N}$ be a sequence of real numbers such that $\lim\limits_{n\to\infty} n\,(a_n-1)=l\in(-\infty,\infty)$ and let $p\geq 1$ be a natural number. Calculate $\lim\limits_{n\to\infty}\prod\limits_{k=1}^{n}\left(a_n+\frac{1}{\sqrt[p]{kn}}\right)$.

(Jozsef Wildt IMC 2016)

Solution 1 by Michel Bataille, Rouen, France. We show that the required limit is e^{ℓ} if $p=1$, $e^{\ell+2}$ if $p=2$ and ∞ if $p\geq 3$.

From the hypothesis, we have $a_n=1+\frac{\ell}{n}+o(1/n)$ as $n\to\infty$, hence, in the calculations that follow, we may suppose that n is large enough to ensure that $a_n>0$.Let

$$P_n=\prod_{k=1}^{n}\left(a_n+\frac{1}{\sqrt[p]{kn}}\right)=a_n^n\prod_{k=1}^{n}\left(1+\frac{b_n}{k^{1/p}}\right)$$

where $b_n=\frac{1}{n^{1/p}a_n}$. Note that $b_n\sim\frac{1}{n^{1/p}}$ as $n\to\infty$ (since $\lim\limits_{n\to\infty}a_n=1$).

We set $\sigma_n=\sum\limits_{k=1}^{n}\ln\left(1+\frac{b_n}{k^{1/p}}\right)$ so that $\ln(P_n)=n\ln(a_n)+\sigma_n$.

Since

$$n\ln(a_n)=n\ln\left(1+\frac{\ell}{n}+o(1/n)\right)=n\left(\frac{\ell}{n}+o(1/n)\right)=\ell+o(1)$$

as $n\to\infty$, we see that $\lim\limits_{n\to\infty}n\ln(a_n)=\ell$. To study σ_n, we first recall some well-known results: $x-\frac{x^2}{2}\leq\ln(1+x)\leq x$ for positive x, $\sum\limits_{k=1}^{n}\frac{1}{n}\sim\ln(n)$ and, if $\alpha<1$, $\sum\limits_{k=1}^{n}\frac{1}{n^{\alpha}}\sim\frac{n^{1-\alpha}}{1-\alpha}$ as $n\to\infty$. From the first of these results, we readily obtain

$$b_nu_n-\frac{b_n^2}{2}v_n\leq\sigma_n\leq b_nu_n \tag{1}$$

with $u_n=\sum\limits_{k=1}^{n}\frac{1}{k^{1/p}}$ and $v_n=\sum\limits_{k=1}^{n}\frac{1}{k^{2/p}}$. We now distinguish the cases $p=1,p=2$ and $p\geq 3$. If $p=1$, then $b_n\sim\frac{1}{n}$, $u_n\sim\ln(n)$ and $\lim\limits_{n\to\infty}v_n=\frac{\pi^2}{6}$ so that $\lim\limits_{n\to\infty}b_nu_n=\lim\limits_{n\to\infty}b_n^2v_n=0$ and from (1), $\lim\limits_{n\to\infty}\sigma_n=0$. Thus, $\lim\limits_{n\to\infty}\ln(P_n)=\ell$ and $\lim\limits_{n\to\infty}P_n=e^{\ell}$. If $p=2$, then $b_n\sim\frac{1}{\sqrt{n}}$, $u_n\sim 2\sqrt{n}$ and $v_n\sim\ln(n)$ as $n\to\infty$. It follows that $\lim\limits_{n\to\infty}b_nu_n=2$ and $\lim\limits_{n\to\infty}b_n^2v_n=0$. With the help of (1), we deduce that $\lim\limits_{n\to\infty}\sigma_n=2$ and then that $\lim\limits_{n\to\infty}P_n=e^{\ell+2}$. Lastly, if $p\geq 3$, then $u_n\sim\frac{p}{p-1}\cdot n^{1-\frac{1}{p}}$, $\quad v_n\sim\frac{p}{p-2}\cdot n^{1-\frac{2}{p}}$ and

$$1-\frac{b_nv_n}{2u_n}\leq\frac{\sigma_n}{b_nu_n}\leq 1$$

with $\frac{b_n v_n}{2u_n} \sim \frac{n^{-\frac{1}{p}}}{2} \cdot \frac{p}{p-2} n^{1-\frac{2}{p}} \cdot \frac{p-1}{p} n^{\frac{1}{p}-1} = \frac{p-1}{2(p-2)} \cdot n^{-2/p}$. Thus, $\lim_{n\to\infty} \frac{b_n v_n}{2u_n} = 0$ and therefore $\sigma_n \sim b_n u_n$ as $n \to \infty$. As a result, $\lim_{n\to\infty} \sigma_n = \infty$ (since $b_n u_n \sim \frac{p}{p-1} \cdot n^{1-\frac{2}{p}}$ and $p > 2$) and $\lim_{n\to\infty} P_n = \infty$ follows.

Solution 2 by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy. Let $L(p)$ be the limit we are searching. Clearly $a_n = 1 + \frac{l}{n} + o\left(\frac{1}{n}\right)$

$$\int_1^{n+1} \ln\left(a_n + \frac{1}{\sqrt[p]{xn}}\right) dx \le \int_1^n \ln\left(a_n + \frac{1}{\sqrt[p]{xn}}\right) dx \le$$
$$\le \ln\left[\prod_{k=1}^n \left(a_n + \frac{1}{\sqrt[p]{kn}}\right)\right] = \sum_{k=2}^n \ln\left(a_n + \frac{1}{\sqrt[p]{kn}}\right) + \ln\left(a_n + \frac{1}{\sqrt[p]{n}}\right) \le$$
$$\le \int_1^n \ln\left(a_n + \frac{1}{\sqrt[p]{xn}}\right) dx + \underbrace{\ln\left(a_n + \frac{1}{\sqrt[p]{n}}\right)}_{\to 0}$$

Since $a_n \to 1$, the last term tends to zero. So the result is

$$L \doteq \lim_{n\to\infty} \exp\left\{\int_1^n \ln\left(a_n + \frac{1}{\sqrt[p]{xn}}\right) dx\right\}$$

Moreover

$$\int_1^n \ln\left(a_n + \frac{1}{\sqrt[p]{xn}}\right) dx \underbrace{=}_{x=\frac{1}{ny^p}} \frac{1}{n} \int_{n^{\frac{-2}{p}}}^{n^{\frac{-1}{p}}} \frac{p}{y^{p+1}} \ln(a_n + y) dy =$$
$$= \frac{y^{-p} \ln(a_n + y)}{n}\Big|_{n^{-1/p}}^{n^{-2/p}} + \frac{1}{n} \int_{n^{\frac{-2}{p}}}^{n^{\frac{-1}{p}}} \frac{dy}{y^p(a_n + y)} =$$
$$= \frac{n^2 \ln(a_n + n^{\frac{-2}{p}})}{n} - \underbrace{\frac{n \ln(a_n + n^{\frac{-1}{p}})}{n}}_{\to 0} + \frac{1}{n} \int_{n^{\frac{-2}{p}}}^{n^{\frac{-1}{p}}} \frac{dy}{y^p(a_n + y)}$$

Let $p > 2$.

$$\ln(a_n + n^{\frac{-2}{p}}) = 1 + n^{\frac{-2}{p}} + \frac{l}{n} + o\left(\frac{1}{n}\right)$$

whence

$$\frac{n^2 \ln(a_n + n^{\frac{-2}{p}})}{n} \sim n \ln\left(1 + n^{\frac{-2}{p}} + O\left(\frac{1}{n}\right)\right) \sim n^{1-\frac{2}{p}} \to +\infty$$

Since

$$\frac{1}{n} \int_{n^{\frac{-2}{p}}}^{n^{\frac{-1}{p}}} \frac{dy}{y^p(a_n + y)} > 0$$

it follows

$$\int_1^n \ln\left(a_n + \frac{1}{\sqrt[p]{xn}}\right) dx \to +\infty \implies L = +\infty$$

Let $p = 2$.

$$\frac{n^2 \ln(a_n + n^{\frac{-2}{p}})}{n} \sim n \ln\left(1 + \frac{l+1}{n} + o\left(\frac{1}{n}\right)\right) \sim l + 1 + o(1)$$

Moreover

$$\frac{1}{n}\int_{n^{-1}}^{n^{\frac{-1}{2}}} \frac{dy}{y^2(a_n+y)} = \frac{1}{n}\left(\frac{1}{a_n^2}\ln\frac{a_n+y}{y} - \frac{1}{a_n y}\Big|_{n^{-1}}^{n^{\frac{-1}{2}}}\right) =$$
$$= \frac{1}{na_n^2}\ln\frac{a_n+n^{-1/2}}{n^{-1/2}} - \frac{1}{na_n^2}\ln\frac{a_n+\frac{1}{n}}{\frac{1}{n}} - \frac{\sqrt{n}}{na_n} + \frac{n}{na_n} \to 1$$

It follows

$$\int_1^n \ln\left(a_n + \frac{1}{\sqrt{xn}}\right) dx \to l + 2 \implies L = e^{l+2}$$

Let $p = 1$.

$$\ln(a_n + n^{\frac{-2}{p}}) = 1 + \frac{l}{n} + o\left(\frac{1}{n}\right)$$

$$\frac{1}{n}\int_{n^{\frac{-2}{p}}}^{n^{\frac{-1}{p}}} \frac{dy}{y^p(a_n+y)} = \frac{1}{n}\int_{n^{-2}}^{n^{-1}} \frac{dy}{y(a_n+y)} = \frac{1}{n}\frac{1}{a_n}\ln\frac{y}{a_n+y}\Big|_{n^{-2}}^{n^{-1}} \to 0$$

so

$$\int_1^n \ln\left(a_n + \frac{1}{xn}\right) dx \to l + 1 \implies L = e^l$$

101. Let $f, g : [a, b] \to R$ be two nonnegative continuous functions. Assume that f attains its maximum at a unique point on $[a, b]$ and g attains its maximum at the same point as f and possibly at other points.

1) Prove that $\lim\limits_{n\to\infty} \frac{\int_a^b f^{n+1}(x)g(x)dx}{\int_a^b f^n(x)dx} = \|f\|_\infty \|g\|_\infty$.

2) Does the result hold under no assumption on f and g?

(Jozsef Wildt IMC 2016)

Solution by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy. Let p be the unique point such that $f(p) = ||f||_\infty$. We have also $g(p) = ||g||_\infty$.

$$f^{n+1}g = f^n \cdot f \cdot g \le f^n ||f||_\infty ||g||_\infty$$

so we get

$$\frac{\int_a^b f^{n+1}(x)g(x)dx}{\int_a^b f^n(x)dx} \le ||f||_\infty ||g||_\infty \frac{\int_a^b f^n(x)dx}{\int_a^b f^n(x)dx} = ||f||_\infty ||g||_\infty$$

We know by the continuity of f, the uniqueness of p and the compactness of $[a, b]$ that for any ε small enough there exist δ_ε such that ($I = [p - \delta_\varepsilon, p + \delta_\varepsilon]$)

$$x \in I \implies f(x) \ge ||f||_\infty - \varepsilon,\ g(x) \ge ||g||_\infty - \varepsilon$$

(this follows by the continuity alone) and

$$x \in [a, b]\backslash I \implies f(x) \le ||f||_\infty - \varepsilon$$

(here the uniqueness of p and compactness of $[a,b]$ are used). Indeed let's suppose the above false. This means that: *for any* ε, *for any* $I_\delta \doteq (p-\delta, p+\delta)$, $\max_{x\in[a,b]\setminus I_\delta} f(x) > ||f||_\infty - \varepsilon$. Let's take $\varepsilon_k \to 0$ and I_δ fixed independent of ε_k. This would imply the existence of a sequence $\{x_k\} \in [a,b]\setminus I_\delta$, such that $f(x_k) \to ||f||$. From $\{x_k\}$, via the compactness of $[a,b]$, we can extract a subsequence $\{x_{k_n}\} \to p' \in [a,b]\setminus I_\delta$. The continuity of f yields $f(p') = ||f||$ which is a contradiction with the uniqueness of p.

$$\frac{\int_a^b f^{n+1}(x)g(x)dx}{\int_a^b f^n(x)dx} \geq \frac{\int_I f^n \cdot f(x)\cdot g(x)dx}{\int_I f^n(x)dx\left(1+\frac{\int_{[a,b]\setminus I} f^n dx}{\int_I f^n dx}\right)} \geq$$
$$\geq (||f||_\infty - \varepsilon)(||g||_\infty - \varepsilon)\frac{\int_I f^n \cdot dx}{\int_I f^n(x)dx\left(1+\frac{\int_{[a,b]\setminus I} f^n dx}{\int_I f^n dx}\right)} =$$
$$= (||f||_\infty - \varepsilon)(||g||_\infty - \varepsilon)\frac{1}{\left(1+\frac{\int_{[a,b]\setminus I} f^n dx}{\int_I f^n dx}\right)}$$

Now we show that

$$\lim_{n\to\infty}\frac{\int_{[a,b]\setminus I} f^n dx}{\int_I f^n dx} = 0$$

Indeed also by the continuity of f we know that there exists an interval $I' \subset I$ such that $f \geq ||f||_\infty - \varepsilon/2$ and then

$$\frac{\int_{[a,b]\setminus I} f^n dx}{\int_I f^n dx} \leq \frac{\int_{[a,b]\setminus I} f^n dx}{\int_{I'} f^n dx} \leq \frac{(b-a-|I|)\cdot(||f||_\infty - \varepsilon)^n}{|I'|\cdot(||f||_\infty - \varepsilon/2)^n} \to 0$$

We have proven that for any $\varepsilon > 0$ we have

$$(||f||_\infty - \varepsilon)(||g||_\infty - \varepsilon) \leq \frac{\int_a^b f^{n+1}(x)g(x)dx}{\int_a^b f^n(x)dx} \leq ||f||_\infty ||g||_\infty$$

that is the result.

b) If f and g attains their maximum at different points, the result needs not to be true. Let $f(x) = e^{-nx}$, $g(x) = x$.

$$\int_0^1 e^{-(n+1)x}xdx = x\frac{e^{-(n+1)x}}{-(n+1)}\Big|_0^1 + \frac{1}{n+1}\int_0^1 e^{-(n+1)x}dx =$$
$$= \frac{e^{-(n+1)}}{-(n+1)} + \frac{1}{(n+1)^2}(1-e^{-n-1})$$

so

$$\lim_{n\to\infty}\frac{\int_0^1 e^{-(n+1)x}xdx}{\int_0^1 e^{-nx}dx} = \lim_{n\to\infty}\frac{\frac{1}{(n+1)^2} + e^{-n-1}\left(\frac{-1}{n+1} - \frac{-1}{(n+1)^2}\right)}{\frac{1}{n}(1-e^{-n})} = 0 \neq ||e^{-x}||_\infty ||x||_\infty = 1$$

102. Let $f \in C^3(R^n, R)$ with $f(0) = f'(0) = 0$. Prove that there exist $h \in C^3(R^m, S_n(R))$, such that $f(x) = x^t h(x) x$, when $S_n(R)$, is the set of symmetric matrix, and x^t is the transpose of x.

(Jozsef Wildt IMC 2016)

Solution by Moubinool Omarjee, Paris, France. We have $f(x) = x^t h(x) x$, where $h(x) = \int_0^1 (1-u)H(ux)du$ with $H(v) = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}(v)\right)$ the Hessian of f, h is of class C^1 with theorems of derivative under integral sign.

103. Find the nature of the series $\sum\limits_{n\geq 1} \frac{e^{i\ln(p_n)}}{p_n}$ when $(p_n)_{n\geq 1}$ is the prime number increasing order, and i imaginary complex number.

(Jozsef Wildt IMC 2016)

Solution We didnt receive any solution. The solutions for this problem can also be sent during this issue.

104. Let a, b, and c be positive real numbers. Prove that

$$\left(\frac{(6n+1)a-b}{n(b+c)}\right)^2 + \left(\frac{(6n+1)b-c}{n(c+a)}\right)^2 + \left(\frac{(6n+1)c-a}{n(a+b)}\right)^2 \geq 27$$

for any positive integer $n \geq 1$.

(Jozsef Wildt IMC 2016)

Solution 1 by Arkady Alt, San Jose, California, USA.

Since $x^2 + y^2 + z^2 \geq \frac{(x+y+z)^2}{3}$ for any real x, y, z we obtain

$$\sum_{cyc}\left(\frac{(6n+1)a-b}{n(b+c)}\right)^2 \geq \frac{1}{3}\left(\sum_{cyc}\frac{(6n+1)a-b}{n(b+c)}\right)^2.$$

Since $\sum\limits_{cyc}\frac{(6n+1)a-b}{n(b+c)} = 6\sum\limits_{cyc}\frac{a}{b+c} + \frac{1}{n}\sum\limits_{cyc}\frac{a-b}{b+c}$ and by Cauchy Inequality

$$\sum_{cyc}\frac{a}{b+c} = (a+b+c)\sum_{cyc}\frac{1}{b+c} - 3 = \frac{1}{2}\left(\sum_{cyc}(b+c)\cdot\sum_{cyc}\frac{1}{b+c}\right) - 3 \geq \frac{9}{2} - 3 = \frac{3}{2}$$

then $\sum\limits_{cyc}\frac{(6n+1)a-b}{n(b+c)} \geq 6\cdot\frac{3}{2} + \frac{1}{n}\sum\limits_{cyc}\frac{a-b}{b+c} = 9 + \frac{1}{n}\sum\limits_{cyc}\frac{a-b}{b+c}$.

Noting that triples (a, b, c) and $\left(\frac{1}{b+c}, \frac{1}{c+a}, \frac{1}{a+b}\right)$ agreed in order

($(a-b)\left(\frac{1}{b+c} - \frac{1}{c+a}\right) = \frac{(a-b)^2}{(b+c)(c+a)} \geq 0$) by Rearrangement Inequality we have

$\sum\limits_{cyc}\frac{a}{b+c} \geq \sum\limits_{cyc}\frac{b}{b+c}$ and, therefore, $\sum\limits_{cyc}\frac{a-b}{b+c} \geq 0$. Hence $\sum\limits_{cyc}\frac{(6n+1)a-b}{n(b+c)} \geq 9$ and we finally obtain

$$\sum_{cyc}\left(\frac{(6n+1)a-b}{n(b+c)}\right)^2 \geq \frac{1}{3}\left(\sum_{cyc}\frac{(6n+1)a-b}{n(b+c)}\right)^2 \geq \frac{1}{3}\cdot 81 = 27.$$

Solution 2 by Paolo Perfetti, Department of Mathematics, University Tor Vergata, Rome, Italiy. From $a^2 + b^2 + c^2 \geq (a+b+c)^2/3$ which holds true regardless the sign of a, b, c, we get

$$\sum_{\text{cyc}}\left(\frac{(6n+1)a-b}{n(b+c)}\right)^2 \geq \frac{1}{3}\left(\sum_{\text{cyc}}\frac{(6n+1)a-b}{n(b+c)}\right)^2$$

so we come to prove

$$\left(6\sum_{\text{cyc}}\frac{a}{b+c}+\frac{1}{n}\sum_{\text{cyc}}\frac{a-b}{b+c}\right)^2 \geq 81$$

Now $\displaystyle\sum_{\text{cyc}}\frac{a}{b+c} \geq \frac{3}{2}$ is the famous Nesbitt's inequality so it suffices to show that

$$\sum_{\text{cyc}}\frac{a-b}{b+c} \geq 0 \iff \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b} \geq \frac{b}{b+c}+\frac{c}{c+a}+\frac{a}{a+b} \qquad (1)$$

Let's suppose $a \geq b \geq c$. It follows that

$$\frac{1}{b+c} \geq \frac{1}{a+c} \geq \frac{1}{a+b}$$

The Rearrangement–inequality yields (1) being (a, b, c) and $\left(\frac{1}{b+c}, \frac{1}{a+c}, \frac{1}{a+b}\right)$ equally sorted.

t Let $a \geq c \geq b$. It follows that

$$\frac{1}{b+c} \geq \frac{1}{a+b} \geq \frac{1}{a+c}$$

Again the Rearrangement–inequality yields (1) being (a, c, b) and $\left(\frac{1}{b+c}, \frac{1}{a+b}, \frac{1}{a+c}\right)$ equally sorted.

So we have got

$$\left(6\sum_{\text{cyc}}\frac{a}{b+c}+\frac{1}{n}\sum_{\text{cyc}}\frac{a-b}{b+c}\right)^2 \geq \left(6\sum_{\text{cyc}}\frac{a}{b+c}+0\right)^2 \geq 36\frac{9}{4} = 81$$

and this completes the proof.

Also solved by Henry Ricardo, Westchester Area Math Circle, Purchase, NY, USA and Michel Bataille, Rouen, France and Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania.

MATHNOTES SECTION

A note on Mitrinović - Adamović and Lazarević inequalities

Edward Neuman

Abstract. Generalizations and refinements of Mitrinović - Adamović inequality for trigonometric functions and I. Lazarević inequality for hyperbolic functions are established. The main result is obtained using the Schwab-Borchardt mean.
Keywords: Mitrinović - Adamović and Lazarević inequalities, trigonometric and hyperbolic functions, Schwab-Borchardt mean, classical bivariate means.

1. Introduction

In recent years the following inequalities

$$(\cos x)^{1/3} < \frac{\sin x}{x} \qquad (0 < |x| < \pi/2) \tag{18}$$

and

$$(\cosh x)^{1/3} < \frac{\sinh x}{x} \qquad (x \neq 0) \tag{19}$$

have attracted attention of several researchers. Inequality (18) is due to D.S. Mitrinović and A.A. Adamović while the inequality (19) has been discovered by I. Lazarević. For details see [4, p. 238]. Several refinements of the inequalities (18) and (19) appear in mathematical literature (see, e.g., [6]). Generalizations of Mitrinović - Adamović and Lazarević inequalities to the two-parameter generalized trigonometric, hyperbolic and Jacobian elliptic functions have been obtained recently. For details the interested reader is referred to [7] and the references therein. In this note we shall prove a chain of inequalities which in particular cases will provide refinements of inequalities (18) and (19). In Section 2 we provide definitions and notation. A main result of this note is established in Section 3.

2. DEFINITIONS

Let a and b be positive numbers. The Schwab - Borchardt mean of a and b, denoted by $SB(a,b) \equiv SB$, is defined as follows

$$SB(a,b) = \begin{cases} \dfrac{(b^2-a^2)^{1/2}}{\cos^{-1}(a/b)} & \text{if } \ a < b, \\ \dfrac{(a^2-b^2)^{1/2}}{\cosh^{-1}(a/b)} & \text{if } \ b < a, \\ a & \text{if } \ a = b \end{cases} \tag{20}$$

(see, e.g., [1], [3]). It is easy to prove that SB is a nonsymmetric and homogeneous mean of degree 1 in variables a and b. This mean has been studied recently in [5], [9], and [10].

For the later use we record two results involving the Schwab-Borchardt mean. The first one is the invariance property (see [1], [3])

$$SB(a,b) = SB(A, \sqrt{Ab}), \tag{21}$$

where

$$A = \frac{a+b}{2} \tag{22}$$

is the arithmetic mean of a and b. The following bounds for SB

$$(ab^2)^{1/3} < SB(a,b) < \frac{a+2b}{3} \tag{23}$$

have been established in [9].

In what follows we will assume that the numbers a and b are positive and unequal. As usual, the symbols I, L and G will stand, respectively, for the identric, logarithmic and geometric means of a and b:

$$I = e^{-1}\left(\frac{a^a}{b^b}\right)^{\frac{1}{a-b}}, L = \frac{a-b}{\log a - \log b}, G = \sqrt{ab} \tag{24}$$

(see [2]). A classical results states that all four means listed in (22) and (24) are comparable, symmetric and homogeneous of degree 1 in their variables. Moreover, the chain of inequalities

$$\begin{aligned} G < (AG^2)^{1/3} < \left[G\left(\frac{A+G}{2}\right)^2\right]^{1/3} < L < \frac{1}{3}(A+2G) \\ < \frac{A+G}{2} < \frac{2A+G}{3} < I < A \end{aligned} \tag{25}$$

is satisfied for all numbers a and b. For more details see [8, 11, 12, 13, 14]

3. MAIN RESULT

We will need the following:

Proposition 3.1. Let $a > 0$ be such that $a \neq 1$. Further, let

$$\lambda = \frac{a+1}{2} \qquad \text{and} \qquad \mu = \sqrt{a}.$$

Then the following inequalities:

$$\begin{aligned} a^{1/3} < (\lambda a)^{2/9} < \left[\mu\left(\frac{\lambda+\mu}{2}\right)^2\right]^{2/3} < \left(\frac{a-1}{\ln a}\right)^{2/3} < \left[\frac{1}{3}(\lambda+2\mu)\right]^{2/3} \\ < \left(\frac{\lambda+\mu}{2}\right)^{2/3} < \left[\frac{1}{3}(2\lambda+\mu)\right]^{2/3} < (e^{-1}a^{\frac{a}{a-1}})^{2/3} < \lambda^{2/3} < SB(a,1). \end{aligned} \tag{26}$$

hold true.

Proof. In order to establish the desired result we use (21) with $b = 1$ followed by application of the left inequality in (23) to obtain

$$SB(a,1) = SB\left(\lambda, \sqrt{\lambda}\right) > \lambda^{2/3}.$$

This completes proof of the last inequality in (26). The remaining ones follow from the chain of inequalities (25) where now

$$A = \lambda, \qquad G = \mu, \qquad L = \frac{a-1}{\ln a} \qquad \text{and} \qquad I = (e^{-1}a^{\frac{a}{a-1}}).$$

The desired result now follows. □

The main result of this note reads as follows:
Theorem 3.2. Let $0 < |x| < \pi/2$. Then the following inequalities

$$
\begin{aligned}
(\cos x)^{1/3} &< \left(\cos^2 \frac{x}{2} \cos x\right)^{2/9} < \left[\sqrt{\cos x}\left(\frac{\cos^2 \frac{x}{2} + \sqrt{\cos x}}{2}\right)\right]^{2/3} \\
&< \left(\frac{\cos x - 1}{\ln(\cos x)}\right)^{2/3} < \left[\frac{1}{3}\left(\cos^2 \frac{x}{2} + 2\sqrt{\cos x}\right)\right]^{2/3} \\
&< \left(\frac{\cos^2 \frac{x}{2} + \sqrt{\cos x}}{2}\right)^{2/3} < \left[\frac{1}{3}(2\cos^2 \frac{x}{2} + \sqrt{\cos x}\right)^{2/3} \\
&< \left(e^{-1}(\cos x)^{\frac{\cos x}{\cos x - 1}}\right)^{2/3} < \left(\cos^2 \frac{x}{2}\right)^{4/3} < \frac{\sin x}{x}.
\end{aligned}
\tag{27}
$$

Proof. To obtain (27) we let in (20) $a = \cos x$ to obtain

$$SB(\cos x, 1) = \frac{\sin x}{x}$$

and next utilize the chain of inequalities (26) with a as defined above and

$$\lambda = \cos^2 \frac{x}{2} \qquad \text{and} \qquad \mu = \sqrt{\cos x}.$$

We omit further details. □

In a similar fashion one can obtain a refinement of Lazarević inequality (19) by letting in (26) $a = \cosh x$, $x \neq 0$. Using (20) we obtain easily

$$SB(\cosh x, 1) = \frac{\sinh x}{x}.$$

Next we apply Proposition 3.1 to obtain the desired result.

References

[1] J. M. Borwein, P.B. Borwein, *Pi and the AGM - A Study in Analytic Number Theory and Computational Complexity*, Wiley, New York, 1987.

[2] P.S. Bullen, D.S. Mitrinović, and P.M. Vasić, Eds. *Means and Their Inequalities*, D. Reidel Publishing Company, Dodrecht, 1988.

[3] B.C. Carlson, *Algorithms involving arithmetic and geometric means*, Amer. Math. Monthly, **78**, (1971), 496–505.

[4] D. S. Mitrinović *Analytic Inequalities*, Springer-Verlag, Berlin, 1970.

[5] E. Neuman, *Inequalities for the Schwab - Borchardt mean nd their applications*, J. Math. Inequal., **5** (2011), 601–609.

[6] E. Neuman, *Refinements and generalizations of certain inequalities involving trigonometric and hyperbolic functions*, Adv. Inequal. Appl., **1** (2012), 1–11.

[7] E. Neuman, *Inequalities for the generalized trigonometric, hyperbolic and Jacobian elliptic functions*, J. Math. Inequal. **9** (2015), 709–726.

[8] E. Neuman, J. Sándor, *On certain means of two arguments and their extensions*, Int. J. Math. Math. Sci., **16** (2003), 981–993.

[9] E. Neuman, J. Sándor, *On the Schwab-Borchardt mean*, Math. Pannon., **14** (2003), 253–266.

[10] E. Neuman, J. Sándor, *On the Schwab - Borchardt mean II*, Math. Pannon. **17** (2006), No. 1, 49–59.

[11] E. Neuman, J. Sándor, *On some inequalities invplving trigonometric and hyperbolic functions with emphasis on the Cusa-Huygens, Wilker and Hyugens inequalities*, Math. Inequal. Appl., **13** (2010), 715–723.

[12] J. Sándor, *A note on some inequalities for means*, Arch. Math. (Basel)., **56** (1991), 471–473.

[13] J. Sándor, *On certain inequalities for means*, J. Math. Anal. Appl., **189** (1995), 602–606.

[14] J. Sándor, *On certain inequalities for means II*, J. Math. Anal. Appl., **199** (1996), 629–635.

Edward Neuman, Mathematical Research Institute,
144 Hawthorn Hollow, Carbondale, IL 62903, USA
E-mail: edneuman76@gmail.com

JUNIOR PROBLEMS

Solutions to the problems stated in this issue should arrive before October 15, 2017.

Proposals

66. *Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam.*
Let x, y, z be real numbers in the interval $\left[\frac{1}{2}, 2\right]$. Find the minimum and maximum possible value of

$$f(x,y,z) = \frac{x}{yz+1} + \frac{y}{zx+1} + \frac{z}{xy+1}.$$

67. *Proposed by Daniel Sitaru, Mathematics Department, Colegiul National Economic Theodor Costescu, Drobeta Turnu - Severin, Mehedinti, Romania.*
Let $n \in \mathbb{N}$ such that $n \geq 2$. Prove that in any triangle ABC the following relationship holds:

$$\sum \left(\frac{\sqrt[n]{b} + \sqrt[n]{c} - 2\sqrt[n]{a}}{\sqrt[n]{b} + \sqrt[n]{c}} \right)^2 + \frac{3}{\sqrt[n]{abc}} \prod (\sqrt[n]{b} + \sqrt[n]{c} - \sqrt[n]{a}) \leq 3.$$

68. *Proposed by Michael Rozenberg, Tel Aviv, Israel and Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Romania.* Let a, b, c and d be non negative real numbers, none three of which 0 such that $a + b + c + d = 4$. Prove that

$$\frac{a^2 + b^2 + c^2 + d^2}{ab + bc + cd + da + ac + bd} + \frac{12abcd}{(ab + bc + cd + da + ac + bd)^2} \geq 1.$$

When equality occurs?

69. *Proposed by Mohammed Aassila, Strasbourg, France.*
Let N be a positive fixed integer. How many integers $1 \leq n \leq N$ are such that:

$$11 \times 2^{n-1} \equiv 4n + 6 \pmod{13}?$$

70. *Proposed by Dorlir Ahmeti, University of Prishtina, Department of Mathematics, Republic of Kosova.* Let suppose that in a board are written numbers in a line like this

$$1 \qquad \tfrac{1}{2} \qquad \tfrac{1}{3} \quad \ldots \quad \tfrac{1}{n-1} \qquad \tfrac{1}{n}$$

Now we add first number with seconnd and write down to middle of the numbers , same thing we add the $k-th$ number with $(k+1)-th$ number and write down to middle of the numbers and we create a new line with $n-1$ new numbers and we do the same thing with new line and repeated the same proces until we left with only one number. Find the last number it is written on the board. For example if $n = 3$ then

$1 \quad \frac{1}{2} \quad \frac{1}{3}$

$\frac{3}{2} \quad \frac{5}{6}$

$\frac{7}{3}$

Solutions

61. *Proposed by Nguyen Viet Hung, Hanoi University of Science, Vietnam.* Given a tetrahedron $A_1A_2A_3A_4$ with the volume V, let I and r be incenter and inradius, respectively. Denote by S_i the area of triangle opposite to vertex $A_i (i = 1; 2; 3; 4)$. Prove that

$$\sum_{n=1}^{4} S_i IA_i^2 = \frac{2rS_1S_2S_3S_4}{9V^2} \sum_{1\le i<j\le 4} A_iA_j \sin\angle(A_i, A_j),$$

where $\angle(A_i, A_j)$ is the dihedral angle at edge A_iA_j.

Solution by Michel Bataille, Rouen, France. Let $S = S_1 + S_2 + S_3 + S_4$ and

$$\mathcal{K} = \frac{1}{S} \sum_{1\le i<j\le 4} S_iS_j \cdot A_iA_j^2.$$

We show that $\mathcal{K}$ equals either side of the desired equality. First, consider the orthogonal projections A_1' and A_1'' of A_1 onto the opposite face $(A_2A_3A_4)$ and onto the edge A_2A_3, respectively. Then $\angle(A_2, A_3) = \angle A_1A_1''A_1'$ and so

$$\sin\angle(A_2, A_3) = \frac{A_1A_1'}{A_1A_1''} = \frac{3V/S_1}{2S_4/A_2A_3} = \frac{3V}{2} \cdot \frac{A_2A_3}{S_1S_4}.$$

This result immediately generalizes to any $\sin\angle(A_i, A_j)$ and we deduce that the right-hand side $\mathcal{R}$ of the required equality is

$$\mathcal{R} = \frac{2rS_1S_2S_3S_4}{9V^2} \cdot \frac{3V}{2} \sum_{1\le i<j\le 4} \frac{S_iS_j}{S_1S_2S_3S_4} \cdot A_iA_j^2 = \frac{r}{3V} \sum_{1\le i<j\le 4} S_iS_j \cdot A_iA_j^2 = \mathcal{K} \quad (1)$$

where the last equality follows from $V = \frac{1}{3}rS_1 + \frac{1}{3}rS_2 + \frac{1}{3}rS_3 + \frac{1}{3}rS_4 = \frac{rS}{3}$.
Second, the barycentric coordinates of I relatively to (A_1, A_2, A_3, A_4), which are proportional to the volumes of the tetrahedrons $IA_2A_3A_4, IA_1A_3A_4, IA_1A_2A_4, IA_1A_2A_3$, are also proportional to S_1, S_2, S_3, S_4, hence

$$S\mathbf{I} = S_1\mathbf{A_1} + S_2\mathbf{A_2} + S_3\mathbf{A_3} + S_4\mathbf{A_4}$$

and therefore we have for example:

$$S\overrightarrow{A_1I} = S_2\overrightarrow{A_1A_2} + S_3\overrightarrow{A_1A_3} + S_4\overrightarrow{A_1A_4}.$$

The dot product $(S\overrightarrow{A_1I}) \cdot (S\overrightarrow{A_1I})$ gives $S^2IA_1^2$ as

$$S_2^2 \cdot A_1A_2^2 + S_3^2 \cdot A_1A_3^2 + S_4^2 \cdot A_1A_4^2 + 2S_2S_3\overrightarrow{A_1A_2} \cdot \overrightarrow{A_1A_3} + 2S_2S_4\overrightarrow{A_1A_2} \cdot \overrightarrow{A_1A_4} + 2S_3S_4\overrightarrow{A_1A_3} \cdot \overrightarrow{A_1A_4},$$

that is,

$$S^2IA_1^2 = (S_2+S_3+S_4)(S_2A_1A_2^2+S_3A_1A_3^2+S_4A_1A_4^2) - (S_2S_3A_2A_3^2+S_2S_4A_2A_4^2+S_3S_4A_3A_4^2)$$

using $2\overrightarrow{A_1A_2}\cdot\overrightarrow{A_1A_3} = A_1A_2^2 + A_1A_3^2 - A_2A_3^2$ and similar relations for the other dot products. Since $S_2+S_3+S_4 = S - S_1$, we can rewrite the previous result as

$$S_1 IA_1^2 = \frac{S_1}{S}\left(S_2A_1A_2^2 + S_3A_1A_3^2 + S_4A_1A_4^2 - \mathcal{K}\right).$$

Writing $S_2IA_2^2, S_3IA_3^2$ and $S_4IA_4^2$ in a similar way and adding the four equalities, we get

$$\sum_{i=1}^{4} S_i IA_i^2 = 2\mathcal{K} - \mathcal{K} = \mathcal{K}. \tag{2}$$

The desired equality follows from (1) and (2).

Also solved by the proposer.

62. *Proposed by Daniel Sitaru, Mathematics Department, Colegiul National Economic Theodor Costescu, Drobeta Turnu - Severin, Mehedinti, Romania.* Let be $A', A'' \in (BC); B', B'' \in (AC); C', C'' \in (AB)$ in ΔABC such that $AA' \cap BB' \cap CC' \neq \emptyset$ and $AA'' \cap BB'' \cap CC'' \neq \emptyset$. Prove that

$$\frac{27[A'B'C']}{[A''B''C'']} \leq \left(\frac{BA'}{BA''} + \frac{CB'}{CB''} + \frac{AC'}{AC''}\right)^3,$$

where $[ABC]$ is area of triangle ABC.

Solution 1 by Ioan Viorel Codreanu, Satulung, Maramures, Romania. Let $\frac{BA'}{A'C} = x, \frac{CB'}{B'A} = y, \frac{AC'}{C'B} = z$ and $\frac{BA''}{A''C} = x', \frac{CB''}{B''A} = y', \frac{AC''}{C''B} = z'$. Then $AB' = \frac{AC}{y+1}$ and $AC' = \frac{z\cdot AB}{z+1}$. We get

$$[B'AC'] = \frac{AB'\cdot AC'\cdot \sin A}{2} = \frac{z}{(z+1)(y+1)}\cdot [ABC],$$

and the similar relations $[C'BA'] = \frac{x}{(x+1)(z+1)}\cdot [ABC], [B'CA'] = \frac{y}{(y+1)(x+1)}\cdot [ABC]$. We have

$$[A'B'C'] = [ABC] - [B'AC'] - [C'BA'] - [B'CA'] = [ABC]\left(1 - \sum\frac{x}{(x+1)(z+1)}\right)$$

and using the **Ceva Theorem** $\prod x = 1$, we get

$$[A'B'C'] = \frac{2[ABC]}{\prod(x+1)}.$$

Analogously, we prove that

$$[A''B''C''] = \frac{2[ABC]}{\prod(x'+1)},$$

and then

$$\frac{[A'B'C']}{[A''B''C'']} = \frac{\prod(x'+1)}{\prod(x+1)}.$$

We have $BA' = \frac{x\cdot BC}{x+1}, BA'' = \frac{x'\cdot BC}{x'+1}$ it results that $\frac{BA'}{BA''} = \frac{x(x'+1)}{x'(x+1)}$ and the similar relations. Using the **Ceva Theorem** $\prod x = 1, \prod x' = 1$ and the **AM-GM Inequality**, we get

$$\left(\sum\frac{BA'}{BA''}\right)^3 \geq 27\prod\frac{BA'}{BA''} = 27\prod\frac{x(x'+1)}{x'(x+1)} = 27\frac{\prod(x'+1)}{\prod(x+1)} = \frac{27[A'B'C']}{[A''B''C'']}.$$

Solution 2 by Neculai Stanciu, "George Emil Palade" School, Buzău, Romania and Titu Zvonaru, Comăneşti, Romania. We denote: $x' = \frac{BA'}{A'C}$, $y' = \frac{CB'}{B'A}$, $z' = \frac{AC'}{C'B}$, $x'' = \frac{BA''}{A''C}$, $y'' = \frac{CB''}{B''A}$, $z'' = \frac{AC''}{C''B}$.
By Routh' theorem we have

$$[A'B'C'] = \frac{x'y'z'+1}{(x'+1)(y'+1)(z'+1)}[ABC], [A''B''C''] = \frac{x''y''z''+1}{(x''+1)(y''+1)(z''+1)}[ABC]$$

Because we have $x'y'z' = x''y''z'' = 1$, the inequality to prove becomes

$$\frac{27(x''+1)(y''+1)(z''+1)}{(x'+1)(y'+1)(z'+1)} \le \left(\frac{x'(x''+1)}{x''(x'+1)} + \frac{y'(y''+1)}{y''(y'+1)} + \frac{z'(z''+1)}{z''(z'+1)}\right)^3$$

which yields immediately by AM-GM inequality.

Solution 3 by Michel Bataille, Rouen, France. Let P (resp. Q) be the point of concurrency of the cevians AA', BB', CC' (resp. AA'', BB'', CC''). In barycentric coordinates relatively to (A, B, C), we have $P = (x_1 : x_2 : x_3)$ and $Q = (y_1 : y_2 : y_3)$ where $x_1, x_2, x_3, y_1, y_2, y_3$ are positive real numbers and $x_1+x_2+x_3 = y_1+y_2+y_3 = 1$. With these notations, the coordinates of A', B', C' are

$$A' = (0 : x_2 : x_3), \quad B' = (x_1 : 0 : x_3), \quad C' = (x_1 : x_2 : 0) \tag{1}$$

and therefore $\frac{[A'B'C']}{[ABC]} = |\delta|$ where

$$\delta = \begin{vmatrix} 0 & \frac{x_1}{x_1+x_3} & \frac{x_1}{x_1+x_2} \\ \frac{x_2}{x_2+x_3} & 0 & \frac{x_2}{x_1+x_2} \\ \frac{x_3}{x_2+x_3} & \frac{x_3}{x_1+x_3} & 0 \end{vmatrix}.$$

We readily obtain $\frac{[A'B'C']}{[ABC]} = \frac{2x_1x_2x_3}{(x_1+x_2)(x_2+x_3)(x_1+x_3)}$; a similar result holds for $\frac{[A''B''C'']}{[ABC]}$ and it follows that the left-hand side of the inequality is $\mathcal{L}$ with

$$\mathcal{L} = \frac{27x_1x_2x_3(y_1+y_2)(y_2+y_3)(y_1+y_3)}{y_1y_2y_3(x_1+x_2)(x_2+x_3)(x_1+x_3)}.$$

From (1), we have $(x_2+x_3)A' = x_2B + x_3C$, hence $(x_2+x_3)\overrightarrow{BA'} = x_3\overrightarrow{BC}$ and so $BA' = \frac{x_3 \cdot BC}{x_2+x_3}$. Similarly, $BA'' = \frac{y_3 \cdot BC}{y_2+y_3}$ so that $\frac{BA'}{BA''} = \frac{x_3(y_2+y_3)}{y_3(x_2+x_3)}$. In the same way, we arrive at

$$\frac{CB'}{CB''} = \frac{x_1(y_1+y_3)}{y_1(x_1+x_3)}, \qquad \frac{AC'}{AC''} = \frac{x_2(y_1+y_2)}{y_2(x_1+x_2)}$$

and the right-hand side $\mathcal{R}$ writes as

$$\mathcal{R} = \left(\frac{x_3(y_2+y_3)}{y_3(x_2+x_3)} + \frac{x_1(y_1+y_3)}{y_1(x_1+x_3)} + \frac{x_2(y_1+y_2)}{y_2(x_1+x_2)}\right)^3.$$

The desired inequality $\mathcal{R} \ge \mathcal{L}$ now results from $(a_1+a_2+a_3)^3 \ge 27a_1a_2a_3$ (AM-GM) applied to

$$a_1 = \frac{x_3(y_2+y_3)}{y_3(x_2+x_3)}, \ a_2 = \frac{x_1(y_1+y_3)}{y_1(x_1+x_3)}, \ a_3 = \frac{x_2(y_1+y_2)}{y_2(x_1+x_2)}.$$

Also solved by the proposer.

63. *Proposed by Leonard Giugiuc, National College Traian, Drobeta Turnu Severin, Romania.* Let $a, b, c \in \mathbb{R}$. Prove that

$$9\sqrt{2}(ab(a-b) + bc(b-c) + ca(c-a)) \le \sqrt{3}\left((a-b)^2 + (b-c)^2 + (c-a)^2\right)^{\frac{3}{2}}.$$

Solution 1 by Michel Bataille, Rouen, France. The inequality is obvious if $ab(a-b)+bc(b-c)+ca(c-a)\le 0$ and otherwise is equivalent to

$$54\left((a-b)(b-c)(a-c)\right)^2 \le \left((a-b)^2+(b-c)^2+(c-a)^2\right)^3 \qquad (1)$$

(since $ab(a-b)+bc(b-c)+ca(c-a)=(a-b)(b-c)(a-c)$). Let $\mathcal{L}(a,b,c)=54\left((a-b)(b-c)(a-c)\right)^2$ and $\mathcal{R}(a,b,c)=\left((a-b)^2+(b-c)^2+(c-a)^2\right)^3$. If $a_1=a-c,\ b_1=b-c$ and $c_1=0$, then $a_1-b_1=a-b,\ b_1-c_1=b-c,\ a_1-c_1=a-c$ so that $\mathcal{L}(a_1,b_1,c_1)=\mathcal{L}(a,b,c)$ and $\mathcal{R}(a_1,b_1,c_1)=\mathcal{R}(a,b,c)$. It follows that it suffices to prove (1) in the case when $c=0$, that is, to show that $54(a-b)^2a^2b^2\le\left((a-b)^2+b^2+a^2\right)^3$ or equivalently,

$$27a^2b^2(a-b)^2\le 4(a^2+b^2-ab)^3. \qquad (2)$$

Now, it is straightforward to check the identity

$$4(a^2+b^2-ab)^3-27a^2b^2(a-b)^2=(a-2b)^2(2a-b)^2(a+b)^2$$

so that (2) writes as $(a-2b)^2(2a-b)^2(a+b)^2\ge 0$ and clearly holds.

Solution 2 by Arkady Alt, San Jose, California, USA. Due to cyclic symmetry of inequality we may assume that $a=\max\{a,b,c\}$. Since the inequality is obviously holds if $b<c$ (because then
$ab(a-b)+bc(b-c)+ca(c-a)=(a-b)(a-c)(b-c)\le 0$) suffice to consider only case when $b\ge c,$ that is $a\ge b\ge c$. Let $x=b-c, y=a-b, p=x+y, q=xy$. Then $x,y\ge 0, a=c+x+y, b=c+x,$
$ab(a-b)+bc(b-c)+ca(c-a)=(x+y)\,xy=pq, (a-b)^2+(b-c)^2+(c-a)^2=\left(x^2+y^2+(x+y)^2\right)=2\left(x^2+y^2+xy\right)=2\left(p^2-q\right)$ and in the new notation the inequality is

$9\sqrt{2}pq\le\sqrt{3}\left(2\left(p^2-q\right)\right)^{3/2}$, where $q\ge 0$ and $q\le\dfrac{p^2}{4}$(condition of solvability of Vieta's System $\begin{cases} x+y=p\\ xy=q\end{cases}$ in real x,y). We have $\sqrt{3}\left(2\left(p^2-q\right)\right)^{3/2}-9\sqrt{2}pq\ge$ $\sqrt{3}\left(2\left(p^2-\dfrac{p^2}{4}\right)\right)^{3/2}-9\sqrt{2}p\cdot\dfrac{p^2}{4}=\sqrt{3}\left(\dfrac{3p^2}{2}\right)^{3/2}-\dfrac{9p^3}{2\sqrt{2}}=\dfrac{9p^3}{2\sqrt{2}}-\dfrac{9p^3}{2\sqrt{2}}=0.$

Also solved by Kevin Soto Palacios, Huarmey,Peru; Ravi Prakash, New Delhi, India; Nicuşor Zlota, Traian Vuia Technical College, Focşani, Romania and the proposer.

64. *Problem proposed by Arkady Alt, San Jose, California, USA.* Let $\Delta(x,y,z):=2(xy+yz+xz)-(x^2+y^2+z^2)$ and let a,b,c be sidelengths of a triangle with area F. Prove that

$$\Delta\left(a^3,b^3,c^3\right)\le\frac{64F^3}{\sqrt{3}}.$$

Solution by Michel Bataille, Rouen, France. In the featured solution of problem 1973 in *Mathematics Magazine*, Vol. 89, No 4, October 2016, p. 297, it is proved that

$$\Delta(a,b,c)\cdot\Delta(a^3,b^3,c^3)\le\left(\Delta(a^2,b^2,c^2)\right)^2 \qquad (1)$$

whenever a,b,c are positive real numbers. Taking for a,b,c the sidelengths of the triangle, we calculate

$$\Delta(a,b,c)=2(ab+bc+ca)-(a^2+b^2+c^2)=2(s^2+r^2+4rR)-(2s^2-2r^2-8rR)=4r(r+4R)>0$$

where s, r, R are the semi-perimeter, the inradius, the circumradius of the triangle, respectively, and

$$\Delta(a^2, b^2, c^2) = 2(a^2b^2 + b^2c^2 + c^2a^2) - (a^4 + b^4 + c^4) = 16F^2$$

(from Heron's formula). Applying (1), we deduce

$$\Delta(a^3, b^3, c^3) \leq \frac{64F^4}{r(r+4R)}$$

and see that it is sufficient to show that $\sqrt{3}F \leq r(r+4R)$ or, since $F = rs$,

$$\sqrt{3}s \leq r + 4R.$$

We are done since the latter is a known inequality, proved in O. Bottema *et al.*, *Geometric Inequalities*, Wolters-Noordhoff Publishing, 1968, **5.5**, p. 49.

Also solved by the proposer.

65. *Proposed by Dorlir Ahmeti, University of Prishtina, Department of Mathematics, Republic of Kosova.* Find all function $f : \mathbb{N} \to \mathbb{N}$ such that $mf(n) + f(m)$ is divisible by $f(m)(f(n)+1)$ for all $m, n \in \mathbb{N}$.

Solution by Michel Bataille, Rouen, France. The identity function $\mathrm{id}_{\mathbb{N}}$, defined by $\mathrm{id}_{\mathbb{N}}(n) = n$ for all $n \in \mathbb{N}$, is clearly a solution. We show that there are no other solutions. To this end, we consider an arbitrary solution f and prove that we must have $f(m) = m$ for all $m \in \mathbb{N}$. For each pair $(m, n) \in \mathbb{N} \times \mathbb{N}$, we have

$$mf(n) + f(m) = g(m,n)f(m)(f(n)+1) \tag{1}$$

for some positive integer $g(m, n)$.
Let $a = f(1)$. With $(m, n) = (1, 1)$, (1) yields $2a = g(1,1)a(a+1)$, hence $2 = (a+1)g(1,1)$ and so $a + 1 = 2$, that is, $f(1) = 1$. From (1), we then deduce that

$$(2g(m,1) - 1)f(m) = m \tag{2}$$

for any positive integer m. Consider any $m > 1$; such an integer can be written as $m = 2^r \cdot s$ for a unique pair (r, s) where r is a nonnegative integer and s is a positive odd integer. Using (2), we obtain $(2g(m,1)-1)f(2^rs) = 2^rs$ or, setting $f(2^rs) = 2^{r'}s'$ ($r' \geq 0, s'$ odd), $(2g(m,1)-1)2^{r'}s' = 2^rs$. This demands $r' = r$ and $s' = d$, some divisor of s, so that $f(m) = f(2^rs) = 2^rd$ where $s = dd'$ for integers d, d'. Note that in particular $f(2^r) = 2^r$.
Now, equality (1) with $m = 2^rs$ and $n = 2^u$ ($u \in \mathbb{N}$) gives $2^ud' + 1 = g(m,n)(2^u + 1)$. As a result, the integer $2^u + 1$ divides $2^ud' + 1 = (2^u + 1)d' + 1 - d'$, hence also divides $d' - 1$. Since u is arbitrary, $d' - 1$ has infinitely many divisors. The only possibility is $d' = 1$ and so $f(2^rs) = 2^rs$. The desired result $f(m) = m$ follows and the proof is complete.

Also solved by the proposer.

Late Acknowledgment

An-Anduud Problem solving group from Mongolia should have been credited for having solved problems 142 and 143.